



Simple Helices on Fano Threefolds

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Abstract. Building on the work of Nogin, we prove that the braid group B_4 acts transitively on full exceptional collections of vector bundles on Fano threefolds with $b_2 = 1$ and $b_3 = 0$. Equivalently, this group acts transitively on the set of simple helices (considered up to a shift in the derived category) on such a Fano threefold. We also prove that on threefolds with $b_2 = 1$ and very ample anticanonical class, every exceptional coherent sheaf is locally free.

1 Background and Main Results

We refer to [7] for a review of the theory of exceptional bundles and exceptional collections (see also [5, §3.1] for a short account of the basic definitions and some results).

Let X be a (smooth) Fano threefold over \mathbb{C} with $b_2 = 1$ and $b_3 = 0$. By the classification of Fano threefolds (see [8]), it is known that X is either \mathbb{P}^3 , or the 3-dimensional quadric, or V_5 , or V_{22} (in the latter case there are moduli for Fano threefolds of this type). Note that the anticanonical class of X is very ample in this case. It is known that these Fano threefolds can be characterized by the condition $\text{rk } K_0(X) = 4$. Furthermore, in all of these cases the derived category $D^b(X)$ of coherent sheaves on X admits a *full exceptional collection of vector bundles* (E_1, E_2, E_3, E_4) . By definition, this means that $\text{Ext}^n(E_i, E_j) = 0$ for $i > j$, and $n \geq 0$, $\text{Ext}^n(E_i, E_i) = 0$ for $n > 0$, $\text{End}(E_i) = \mathbb{C}$, and the collection (E_1, \dots, E_4) generates $D^b(X)$. The constructions of full exceptional collections in the above four cases are due to Beilinson [1], Kapranov [11], Orlov [20], and Kuznetsov [13], respectively.

There is a natural action of the braid group B_n on the set of full exceptional collections of objects in $D^b(X)$, where X is a smooth projective variety with $\text{rk } K_0(X) = n$ given by *left and right mutations*. Bondal proved that in the case when $\text{rk } K_0(X) = \dim X + 1$, the property of a collection to consist of sheaves (as opposed to complexes) is preserved under mutations (see [3]). Furthermore, Positselski showed in [21] that in this case all full exceptional collections of sheaves actually consist of vector bundles. Thus, in the case when $n = \text{rk } K_0(X) = \dim X + 1$ there is an action of the braid group B_n on the set of full exceptional collections of vector bundles on X . In this paper we will prove transitivity of this action for the case of Fano threefolds of the above type by reducing it to the similar transitivity result on the level of $K_0(X)$ established by Nogin [19].

Theorem 1.1 *Let X be a Fano threefold with $b_2 = 1$ and $b_3 = 0$. Then the action of the braid group on the set of complete exceptional collections of bundles on X is transitive.*

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Note that this result does not establish the conjecture on transitivity of the braid group action on the set of all full exceptional collections (up to a shift of each object) proposed in [4], since we do not know whether every full exceptional collection consists of shifts of sheaves only. Neither does it lead to the classification of exceptional bundles on X , since we do not know whether one can include every exceptional bundle in a full exceptional collection.

One can restate Theorem 1.1 using the notion of a *simple helix*.¹ By definition, a *simple helix of period n* in $D^b(X)$ for a smooth projective variety X is a collection of objects (E_i) numbered by $i \in \mathbb{Z}$, such that for every $m \in \mathbb{Z}$ the sequence $(E_{m+1}, \dots, E_{m+n})$ is full and exceptional, and $\text{Hom}^p(E_i, E_j) = 0$ for $p \neq 0$ and $i \leq j$. Simple helices can exist only if X is a Fano variety with $\text{rk } K_0(X) = \dim X + 1$ (see [4]) and necessarily consist of shifts of vector bundles (see [21]). One can show that in this case $E_{i-n} \simeq E_i(K)$, where K is the canonical class on X (see [3]). Conversely, by [4, Prop. 3.3], starting with any full exceptional collection of vector bundles (E_1, \dots, E_n) on such a variety X , one gets a simple helix by considering

$$(1.1) \quad (\dots, E_1, \dots, E_n, E_1(-K), \dots, E_n(-K), E_1(-2K), \dots).$$

Similarly to the case of exceptional collections one defines an action of the braid group on the set of simple helices. Our theorem can be restated as follows:

The action of the braid group on simple helices in $D^b(X)$, where X is a Fano threefold, is transitive.

The difficult part of the proof of Theorem 1.1 was completed by Nogin in [19] where he proved the transitivity of the action of the braid group on semiorthogonal bases in $K_0(X)$ in the above situation. Our contribution is the following result, perhaps of independent interest.

Theorem 1.2 *Let X be a Fano threefold with very ample anticanonical class and $b_2 = 1$. Let E_1 and E_2 be exceptional bundles on X with the same class in $K_0(X)$. Assume that $\text{Ext}^1(E_1, E_1(-K)) = 0$. Then $E_1 \simeq E_2$.*

The proof of this theorem will be given in the next section. It is based on the trick of considering restrictions to a generic anticanonical K3 surface in X , which was exploited by S. Zube in [23] to prove the stability of an exceptional bundle on \mathbb{P}^3 . Using the same trick we will prove that in the situation of Theorem 1.2 every exceptional sheaf on X is locally free and stable (see Theorem 2.2 below). Now let us show how Theorem 1.2 implies our main result.

Proof of Theorem 1.1 Given a pair of complete exceptional collections of bundles on X , we can mutate one of them to obtain the situation when the two collections will give identical classes in K_0 (by the transitivity of the braid group action on the set of semiorthogonal bases in K_0 proved in our situation by Nogin [19]). It remains to note that every exceptional collection (E_1, \dots, E_n) of vector bundles on X extends to a simple helix (1.1). Hence, $\text{Ext}^1(E_i, E_i(-K)) = 0$ for $i = 1, \dots, n$, and we can apply Theorem 1.2. ■

¹This terminology is due to Bridgeland [5]; the original term from [4] is *geometric helix*.

In the case when X is not of type V_{22} , Theorem 1.1 can be deduced from the transitivity on the level of $K_0(X)$ in a more direct way, as was observed by A. Bondal. Indeed, if X is either \mathbb{P}^3 , or a quadric, or V_5 , then there exists a full exceptional collection of vector bundles on X of the form (L, M, E, F) , where L and M are line bundles. Since line bundles are determined by their classes in $K_0(X)$, it is enough to prove that all exceptional collections of the above form with fixed L and M are in one orbit of the braid group. But in this case (E, F) is a full exceptional collection in the left orthogonal to the subcategory generated by L and M . It is easy to classify all exceptional objects in a triangulated category generated by an exceptional pair (see [6]). In particular, an exceptional object in such a category is determined by its class in K_0 up to a shift, which concludes the proof in this case.

It would be nice to get rid of the assumption on the vanishing of Ext^1 in Theorem 1.2. So far, we were able to do this only in the case of rank 2 bundles assuming that the index of X is ≥ 2 .

Theorem 1.3 *Let X be a Fano threefold with very ample anticanonical class, $b_2 = 1$, and index ≥ 2 . Then every exceptional bundle E of rank 2 on X satisfies $\text{Ext}^1(E_1, E_1(-K)) = 0$. Hence, such a bundle is uniquely determined by its class in $K_0(X)$.*

2 Proofs via Zube's Trick

In this section when talking about the stability of a vector bundle we always mean Mumford's stability with respect to the slope function corresponding to an ample generator of the Neron–Severi group (we will only need this for varieties with Picard number 1 and for curves).

Let S be an algebraic K3 surface. Recall that a *spherical object* $F \in D^b(S)$ is an object satisfying $\text{Hom}^i(F, F) = 0$ for $i \neq 0, 2$, and $\text{Hom}^0(F, F) = \text{Hom}^2(F, F) = \mathbb{C}$ (see [22]). If F is a coherent sheaf, then F is spherical if and only if it is simple and rigid. Let us recall some well-known properties of spherical and rigid sheaves.

Lemma 2.1 *Let S be a K3 surface, and let F be a spherical sheaf on S .*

- (i) *Let $TF \subset F$ be the torsion subsheaf. Then TF is rigid.*
- (ii) *If F is torsion free, then it is locally free.*
- (iii) *If F_1 and F_2 are stable spherical bundles with the same class in $K_0(S)$, then $F_1 \simeq F_2$.*
- (iv) *A nonzero rigid sheaf on S cannot have zero-dimensional support.*
- (v) *Assume S has Picard number 1. Then every spherical sheaf on S is locally free and stable.*

Proof Parts (i), (ii), and (iii) follow from Corollary 2.8, Proposition 3.3 and Corollary 3.5 of [17], respectively.

(iv) This follows immediately from the observation that for a sheaf F with zero-dimensional support one has $\chi(F, F) = 0$. Indeed, if F is also rigid, then we have $0 = \chi(F, F) \geq \dim \text{Hom}(F, F)$, so $F = 0$.

(v) Let $TF \subset F$ be the torsion subsheaf of a spherical sheaf F . By part (i) we know that TF is rigid. But then $-c_1(TF)^2 = \chi(TF, TF) > 0$, which is impossible, since the

Picard number of S is 1. Hence, F is torsion free, which implies that it is locally free by part (ii). Finally, the stability of F follows from [17, Proposition 3.14]. ■

The proof of the next result is based on the idea of Zube in [23]. To check the stability of a bundle on a Fano threefold X we restrict it to a smooth anticanonical divisor in X . The same trick allows us to check that an exceptional coherent sheaf on X is locally free.

Theorem 2.2 *Let X be a Fano threefold with very ample anticanonical class and $b_2 = 1$. Then every exceptional coherent sheaf on X is locally free and stable.*

Remark In the particular case when a Fano threefold is a complete intersection in the projective space, the stability of exceptional vector bundles on X was proved in [15] (also using Zube's trick).

The proof of Theorem 2.2 is based on the following well-known observation (see [12] and [22, Ex. 3.14]).

Lemma 2.3 *Let E be an exceptional object in $D^b(X)$, where X is a Fano threefold and $i: S \hookrightarrow X$ is a smooth anticanonical surface. Then Li^*E is a spherical object in $D^b(S)$.*

Proof This is derived immediately by applying the functor $\text{Hom}(E, ?)$ to the exact triangle

$$E(-S) \rightarrow E \rightarrow i_*Li^*E \rightarrow \dots$$

and using the Serre duality on X . ■

Proof of Theorem 2.2 Let E be an exceptional sheaf on X , and let $i: S \hookrightarrow X$ be a generic K3 surface in the anticanonical linear system. Since we can assume that S does not contain the associated points of E , we have $E|_S \simeq Li^*E$. Therefore, by Lemma 2.3, the sheaf $E|_S$ is spherical. Note also that by Moishezon's theorem (see [16]), the Picard number of S equals 1. Hence, by Lemma 2.1(v), $E|_S$ is locally free. Since $-K$ is ample, this immediately implies that E has constant rank (and hence is locally free) over $X \setminus \{p_1, \dots, p_n\}$, where $\{p_1, \dots, p_n\} \subset X$ is some finite set of points.

Now let us consider an arbitrary smooth anticanonical divisor $i: S \hookrightarrow X$ and the corresponding spherical object $Li^*E \in D^b(S)$. It is easy to see that the cohomology sheaves of a spherical object in $D^b(S)$ are rigid (see [9, Prop. 3.5]). Hence, the sheaf L^1i^*E is rigid. But it also has zero-dimensional support, which implies that $L^1i^*E = 0$ by Lemma 2.1(iv). Therefore, $E|_S \simeq Li^*E$ is a spherical sheaf on S , locally free outside a finite number of points. It follows that the torsion subsheaf of $E|_S$ is at the same time rigid (by Lemma 2.1(i)) and has zero-dimensional support, hence, it is zero. By Lemma 2.1(ii), this implies that $E|_S$ is locally free. Since there exists a smooth anticanonical divisor passing through every point of X , we derive that the rank of E is constant on X , therefore, E is locally free.

Assume E is not stable. Then there exists an exact sequence

$$0 \rightarrow F \rightarrow E \rightarrow Q \rightarrow 0$$

with torsion-free sheaf Q , $0 < \text{rk } F < \text{rk } E$, $\mu(F) \geq \mu(E)$. Let $i: S \hookrightarrow X$ be a generic anticanonical surface S . Since we can choose S not containing the associated points of F , E , and Q , $Li^*F \simeq F|_S$ will be a subsheaf of $Li^*E \simeq E|_S$. Since by Moishezon's theorem, S has the Picard number 1, applying Lemma 2.1(v), we derive that the bundle $E|_S$ is stable. But this contradicts the inequality

$$\mu(F|_S) = \mu(F) \geq \mu(E) = \mu(E|_S),$$

where we use $-K$ and $-K|_S$ to define the slope functions on X and on S , respectively. ■

Proof of Theorem 1.2 By Theorem 2.2, E_1 and E_2 are stable bundles of the same slope, so it is enough to construct a nonzero map between them. Consider a generic anticanonical K3 surface $S \subset X$. By Lemma 2.3, the restrictions $E_1|_S$ and $E_2|_S$ are spherical. Also, by Moishezon's theorem, S has Picard number 1. Hence, by Lemma 2.1(iii), we get an isomorphism $E_1|_S \simeq E_2|_S$. The long exact sequence

$$\cdots \rightarrow \text{Hom}(E_1, E_2) \rightarrow \text{Hom}(E_1, E_2|_S) \rightarrow \text{Ext}^1(E_1, E_2(K)) \rightarrow \cdots$$

shows that it is enough to prove the vanishing of $\text{Ext}^1(E_1, E_2(K))$ (then one can lift the nonzero element of $\text{Hom}(E_1|_S, E_2|_S)$ to a nonzero map $E_1 \rightarrow E_2$). The long exact sequences for $n \geq 1$

$$\cdots \rightarrow \text{Ext}^1(E_1, E_2((n+1)K)) \rightarrow \text{Ext}^1(E_1, E_2(nK)) \rightarrow \text{Ext}^1(E_1, E_2(nK)|_S) \rightarrow \cdots$$

together with the vanishing of $\text{Ext}^1(E_1, E_2((n+1)K)) \simeq \text{Ext}^2(E_2, E_1(-nK))^*$ for $n \gg 0$, reduce the problem to showing the vanishing of

$$\text{Ext}^1(E_1, E_2(nK)|_S) \simeq \text{Ext}^1(E_1, E_1(nK)|_S)$$

for $n \geq 1$. Now we can use the exact sequences

$$\cdots \rightarrow \text{Ext}^1(E_1, E_1(nK)) \rightarrow \text{Ext}^1(E_1, E_1(nK)|_S) \rightarrow \text{Ext}^2(E_1, E_1((n+1)K)) \rightarrow \cdots$$

that show that it would be enough to know that

$$\text{Ext}^1(E_1, E_1(nK)) = \text{Ext}^2(E_1, E_1(nK)) = 0$$

for $n \geq 1$. Set $F = \underline{\text{End}}_0(E_1)$, the bundle of traceless endomorphisms of E_1 . Since E_1 is exceptional and $\text{Ext}^1(E_1, E_1(-K)) = 0$ by our assumption, we get

$$H^3(F(K)) \simeq H^0(F)^* = 0, \quad H^2(F) = 0, \quad H^1(F(-K)) = 0.$$

Hence, the sheaf F is 2-regular in the sense of Castelnuovo–Mumford with respect to $-K$ (see [18]). It follows that $H^2(F(-nK)) = 0$ for $n \geq 0$, and $H^1(F(-nK)) = 0$ for $n \geq 1$. In addition we have $H^1(F) = 0$ because E_1 is exceptional. Using Serre duality we derive that $H^1(F(nK)) = H^2(F(nK)) = 0$ for all $n \in \mathbb{Z}$. On the other hand, Kodaira vanishing together with Serre duality imply that $H^1(\mathcal{O}_X(nK)) = H^2(\mathcal{O}_X(nK)) = 0$ for $n \in \mathbb{Z}$. Hence, $\text{Ext}^1(E_1, E_1(nK)) = \text{Ext}^2(E_1, E_1(nK)) = 0$ for $n \in \mathbb{Z}$. ■

Proof of Theorem 1.3 Let $S \subset X$ be a generic anticanonical K3 surface. Set $F = E|_S$. The exact sequence

$$0 = \text{Ext}^1(E, E) \rightarrow \text{Ext}^1(E, E(-K)) \rightarrow \text{Ext}^1(E, E(-K)|_S) \rightarrow \dots$$

shows that it suffices to check the vanishing of $\text{Ext}_S^1(F, F(L))$, where $L = -K|_S$. By Serre duality on S , this is equivalent to the vanishing of $\text{Ext}_S^1(F, F(-L))$. Let $C \subset S$ be a smooth curve in the linear system $|L|$. Then the exact sequence

$$\dots \rightarrow \text{Hom}(F, F) \rightarrow \text{Hom}(F|_C, F|_C) \rightarrow \text{Ext}_S^1(F, F(-L)) \rightarrow \text{Ext}_S^1(F, F) = 0$$

shows that $\text{Ext}_S^1(F, F(-L)) = 0$ if and only if the restriction $F|_C$ is a simple vector bundle on C . Therefore, it is enough to check that $F|_C$ is stable. To this end we will use the effective version of Bogomolov's theorem on the restriction of stable bundles from surfaces to curves (see [2]). Recall that since F is a spherical bundle and S has Picard number 1, F is stable (by Lemma 2.1(v)). Using the condition $\chi(F, F) = 2$ one easily computes that $\Delta(F) = 4c_2(F) - c_1^2(F) = 2$. Let H be the fundamental ample divisor class on X , so that $-K = kH$, where k is the index of X . Then $L = kH|_S$, where $k \geq 2 = \Delta(F)/2 + 1$. Hence, the restriction $F|_C$ is stable (cf. [10, Thm. 7.3.5] or [14, Thm. 1.4.1]). ■

References

- [1] A. A. Beilinson, *Coherent sheaves on \mathbb{P}^n and problems in linear algebra*. (Russian) Funktsional Anal. i Prilozhen. **12**(1978), no. 3, 68–69.
- [2] F. A. Bogomolov, *Stable vector bundles on projective surfaces*. (Russian) Mat. Sb. **185**(1994), no. 4, 3–26; translation in Russian Acad. Sci. Sb. Math. **81**(1995), no. 2, 397–419.
- [3] A. I. Bondal, *Representations of associative algebras and coherent sheaves*. Math. USSR-Izv. **34**(1990), no. 1, 23–42.
- [4] A. I. Bondal and A. E. Polishchuk, *Homological properties of associative algebras: the method of helices*. Russian Acad. Sci. Izv. Math. **42**(1994), no. 2, 219–260.
- [5] T. Bridgeland, *t -structures on some local Calabi-Yau varieties*. J. Algebra **289**(2005), no. 2, 453–483. doi:10.1016/j.jalgebra.2005.03.016
- [6] W. Crawley-Boevey, *Exceptional sequences of representations of quivers*. In: Representations of algebras (Ottawa, ON, 1992), CMS Conf. Proc., American Mathematical Society, Providence, RI, 1993, pp. 117–124.
- [7] A. L. Gorodentsev and S. A. Kuleshov, *Helix theory*. Mosc. Math. J. **4**(2004), no. 2, 377–440, 535.
- [8] V. A. Iskovskikh and Yu. G. Prokhorov, *Fano varieties*. Algebraic geometry, V, Encyclopaedia Math. Sci., 47, Springer, Berlin, 1999, pp. 1–247.
- [9] A. Ishii and H. Uehara, *Autoequivalences of derived categories on the minimal resolutions of A_n -singularities on surfaces*. J. Differential Geom. **71**(2005), no. 3, 385–435.
- [10] D. Huybrechts and M. Lehn, *The geometry of moduli spaces of sheaves*. Friedr. Vieweg & Sohn, Braunschweig, 1997.
- [11] M. M. Kapranov, *Derived category of coherent bundles on a quadric*. (Russian) Funktsional Anal. i Prilozhen. **20**(1986), no. 2, 67.
- [12] S. A. Kuleshov, *Exceptional bundles on K3 surfaces*. In: Helices and vector bundles, London Math. Soc. Lecture Note Ser., Cambridge University Press, Cambridge, 1990, pp. 105–114.
- [13] A. G. Kuznetsov, *An exceptional set of vector bundles on the varieties V_{22}* . Moscow Univ. Math. Bull. **51**(1996), no. 3, 35–37.
- [14] A. Langer, *Lectures on torsion-free sheaves and their moduli*. In: Algebraic cycles, sheaves, shtukas, and moduli, Trends Math., Birkhäuser, Basel, 2008, pp. 69–103.
- [15] R. M. Miró-Roig and H. Soares, *The stability of exceptional bundles on complete intersection 3-folds*. Proc. Amer. Math. Soc. **136**(2008), no. 11, 3751–3757. doi:10.1090/S0002-9939-08-09258-7

- [16] B. Moišezon, *Algebraic homology classes on algebraic varieties*. (Russian) *Izv. Akad. Nauk SSSR Ser. Mat.* **31**(1967), 225–268.
- [17] S. Mukai, *On the moduli space of bundles on K3 surfaces. I*. In: *Vector bundles on algebraic varieties* (Bombay, 1984), Tata Inst. Fund. Res., Bombay, 1987, pp. 341–413.
- [18] D. Mumford, *Lectures on curves on an algebraic surface*. *Annals of Mathematics Studies*, 59, Princeton University Press, Princeton, NJ, 1966.
- [19] D. Yu. Nogin, *Helices on some Fano threefolds: constructivity of semiorthogonal bases of K_0* . *Ann. Sci. École Norm. Sup. (4)* **27**(1994), no. 2, 129–172.
- [20] D. O. Orlov, *Exceptional set of vector bundles on the variety V_5* . *Moscow Univ. Math. Bull.* **46**(1991), no. 5, 48–50.
- [21] L. Positselski, *All strictly exceptional collections in $\mathcal{D}_{coh}^b(\mathbb{P}^n)$ consist of vector bundles*. arXiv:alg-geom/9507014v1.
- [22] P. Seidel and R. Thomas, *Braid group actions on derived categories of coherent sheaves*. *Duke Math. J.* **108**(2001), no. 1, 37–108. doi:10.1215/S0012-7094-01-10812-0
- [23] D. Yu. Zube, *The stability of exceptional bundles on three-dimensional projective space*. In: *Helices and vector bundles*, London Math. Soc. Lecture Note Ser., 148, Cambridge University Press, Cambridge, 1990, pp. 115–117.

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