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AVERAGING INTERPOLATION OF HERMITE-FEJÉR TYPE

BY

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1. **Introduction.** Averaging interpolation generalizes polynomials of Lagrange interpolation and also the next-to-interpolatory polynomials. This notion has recently been introduced by Motzkin, Sharma and Straus [3]. Later in [5], Saxena and Sharma have considered the convergence problem of the averaging interpolators on Tchebycheff abscissas. It appears that averaging interpolators have convergence properties similar to those of Lagrange interpolation. It is therefore reasonable to look for an extension of these operators along the lines of Hermite-Fejér interpolation. This can be done in three ways: (i) by taking assigned averages of function-values and by taking the derivatives to be zero, (ii) by taking assigned function-values and by taking the averages of derivatives to be zero, or (iii) by taking averages of function-values and by taking the averages of derivatives to be zero. The object of this note is to take the first approach. The second approach has been the subject of study by M. Botto and A. Sharma [2].

2. **The operator** $A_n(f; x)$. Let $E_n=\{x_1^{(n)}, \ldots, x_n^{(n)}\}$ denote the nth row of a triangular matrix. For simplicity of writing, we shall write x_k for $x_k^{(n)}$. Let

$$(2.1)\qquad -1 \le x_n < x_{n-1} < \cdots < x_2 < x_1 \le +1.$$

We seek to determine the polynomial $A_n(f; x)$ of degree $\le 2n-2$ such that

$$(2.2)\qquad \begin{cases} A_n(f; x_k)+A_n(f; x_{k+1})=f_k+f_{k+1}, & k=1, \ldots, n-1 \\ A_n'(f; x_k)=0, \quad k=1, \ldots, n \end{cases}$$

where we write f_k for $f(x_k)$. Following the usual notation for the fundamental polynomials of Lagrange and Hermite interpolation, we set

$$(2.3)\qquad \begin{cases} h_\nu(x)\left\{1-\dfrac{\omega''(x_\nu)}{\omega'(x_\nu)}(x-x_\nu)\right\} l_\nu^2(x), \\ l_\nu(x)=\dfrac{\omega(x)}{(x-x_\nu)\omega'(x_\nu)}, \qquad \omega(x)=\displaystyle\prod_1^n (x-x_\nu). \end{cases}$$

We shall prove the following

THEOREM 1. *Given a set of points* (2.1) *such that*

$$(2.4)\qquad H_n^* \equiv \sum_1^n (-1)^{k-1} \frac{\omega''(x_K)}{(\omega'(x_K))^3} \ne 0,$$

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there exists a unique polynomial $A_n(f; x)$ *of degree* $\le 2n-2$ *satisfying* (2.2). *Also we have*

$$A_n(f;x)=\sum_1^n f_\nu h_\nu(x)-\frac{H_n(x)}{H_n^*}\sum_1^n \frac{\omega''(x_\nu)}{(\omega'(x_\nu))^3}f_\nu \tag{2.5}$$

where

$$H_n(x)=\sum_1^n (-1)^{k-1}h_k(x). \tag{2.6}$$

If $\prod_{2n-2}^*$ *denotes the class of all polynomials* $P(x)\in\prod_{2n-2}$ *such that* $P'(x_k)=0$, $k=1,\ldots,n$, *then* $A_n(f;x)$ *is the unique polynomial which minimizes*

$$\max_k |f(x_k)-P(x_k)|$$

when $P(x)$ *runs over* $\prod_{2n-2}^*$.

Proof. In order to prove (2.5), we first construct the $n-1$ fundamental polynomials $\lambda_\nu(x)$ with the properties:

$$\begin{cases}\lambda_\nu(x_k)+\lambda_\nu(x_{k+1})=\delta_{\nu k}, & k=1,\ldots,n-1\\ \lambda_\nu'(x_k)=0, & k=1,\ldots,n.\end{cases}$$

Denoting $\lambda_\nu(x_1)$ by $P_{\nu 1}$, we see that

$$\begin{aligned}\lambda_\nu(x_k)&=(-1)^{k-1}P_{\nu 1}, \qquad k=1,\ldots,\nu\\ &=(-1)^{k-\nu-1}[1-(-1)^{\nu-1}P_{\nu 1}], \qquad k=\nu+1,\ldots,n.\end{aligned}$$

Hence by Hermite interpolation we have

$$\begin{aligned}\lambda_\nu(x)&=\sum_{k=1}^{\nu}(-1)^{k-1}P_{\nu 1}h_k(x)+\sum_{k=\nu+1}^{n}(-1)^{k-\nu-1}[1-(-1)^{\nu-1}P_{\nu 1}]h_k(x)\\ &=H_n(x)P_{\nu 1}+\sum_{k=\nu+1}^{n}(-1)^{k-\nu-1}h_k(x).\end{aligned}$$

Since $\lambda_\nu(x)$ is a polynomial of degree $\le 2n-2$, on equating to zero the coefficients of x^{2n-1}, we have

$$H_n^*P_{\nu 1}=\sum_{k=\nu+1}^{n}(-1)^{k-\nu}\frac{\omega''(x_k)}{(\omega'(x_k))^3}.$$

Then

$$\lambda_\nu(x)=\frac{H_n(x)}{H_n^*}\sum_{k=\nu+1}^{n}(-1)^{k-\nu}\frac{\omega''(x_k)}{(\omega'(x_k))^3}+\sum_{k=\nu+1}^{n}(-1)^{k-\nu-1}h_k(x).$$

Hence $A_n(f;x)=\sum_1^n (f_\nu+f_{\nu+1})\lambda_\nu(x)$, which on simplification yields (2.5).

In order to prove second part of the theorem, we observe that a polynomial

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$P(x) \in \prod_{2n-2}^{*}$ will have the representation

$$P(x) = \sum_{1}^{n} g_\nu h_\nu(x) \tag{2.7}$$

where

$$\sum_{1}^{n} g_\nu \frac{\omega''(x_\nu)}{(\omega'(x_\nu))^3} = 0. \tag{2.8}$$

In order that $\max_\nu |f(x_\nu) - P(x_\nu)|$ is minimized, it is enough to minimize $\max |f_\nu - g_\nu|$, where g_ν runs over all numbers satisfying (2.8). Hence

$$f_\nu - g_\nu = (-1)(f_{\nu+1} - g_{\nu+1}), \qquad \nu = 1, \ldots, n-1. \tag{2.9}$$

Solving (2.8) and (2.9) for g_ν, we see after some simplification that $P(x) = A_n(f; x)$ which completes the proof of theorem 1.

If we denote by $H_n(f; x)$ the Hermite-Fejeŕ interpolation of degree $\leq 2n-1$ satisfying the conditions

$$H_n(f; x_\nu) = f_\nu, \qquad H_n'(f; x_\nu) = 0; \qquad \nu = 1, \ldots, n,$$

then

$$H_n(f; x) = \sum_{\nu=1}^{n} f_\nu h_\nu(x) \tag{2.10}$$

and we can write (2.5) as

$$A_n(f; x) = H_n(f; x) - \chi_n(f, x), \tag{2.11}$$

where

$$\chi_n(f, x) = \frac{H_n(x)}{H_n^*} \sum_{\nu=1}^{n} \frac{\omega''(x_\nu)}{(\omega'(x_\nu))^3} f_\nu. \tag{2.12}$$

We have the following general result on the convergence of the sequence of polynomials $A_n(f; x)$ to $f(x)$:

COROLLARY. *Let $f(x)$ be given continuous function on $[-1, 1]$ and if*

$$\|\chi_n(f, x)\| \to 0 \quad \text{as} \quad n \to \infty$$

then

$$\|A_n(f; x) - f(x)\| \to 0$$

whenever

$$\|H_n(f; x) - f(x)\| \to 0 \quad \text{as} \quad n \to \infty$$

Here $\|\cdot\| = \max_{-1 \leq x \leq 1} |\cdot|$.

The proof follows at once if we note that, from (2.11),

$$\|A_n(f; x) - f(x)\| \leq \|H_n(f; x) - f\| + \|\chi_n(f, x)\|.$$

3. When $\omega(x)=2^{-(n-2)}(1-x^2)U_{n-2}(x)$. If $U_{n-2}(x)=\sin(n-1)\theta/\sin\theta$, $x=\cos\theta$, then the choice of $\omega(x)=2^{-(n-2)}(1-x^2)U_{n-2}(x)$ gives simple formulae for $A_n(f; x)$. In fact

$$x_\nu=\cos\frac{\nu-1}{x-1}\pi, \qquad \nu=1,\ldots,n$$

and since

$$(1-x^2)U''_{n-2}(x)-3xU'_{n-2}(x)+n(n-2)U_{n-2}(x)=0,$$

it follows that

$$\left.\begin{aligned}\omega'(x_\nu)&=(-1)^{\nu-1}(n-1)2^{-(n-2)}\\ \frac{\omega''(x_\nu)}{\omega'(x_\nu)}&=-\frac{x_\nu}{1-x_\nu^2}\end{aligned}\right\}\nu=2,\ldots,n-1$$

and

$$\omega'(1)=-(n-1)2^{-(n-3)}=(-1)^{n-1}\omega'(-1)$$

$$\frac{\omega''(1)}{\omega'(1)}=\frac{2n^2-4n+3}{3}=-\frac{\omega''(-1)}{\omega'(-1)}.$$

Hence

$$l_i(x)=\frac{1+x}{2}\cdot\frac{U_{x-2}(x)}{n-1}$$

$$l_n(x)=(-1)^n\frac{1-x}{2}\cdot\frac{U_{n-2}(x)}{n-1}$$

$$l_\nu(x)=\frac{(-1)^{\nu-1}}{n-1}\cdot\frac{(1-x^2)U_{n-2}(x)}{x-x_\nu}, \qquad \nu=2;\ldots,n-1.$$

From (2.6), it follows that $H_n(x)$ is the polynomial which interpolates the data $\{(-1)^{\nu-1}\}$ at x_ν and whose derivatives at these nodes is zero. Hence it is easy to see that

$$H_n(x)=\begin{cases}T_{n-1}(x)+\frac{1}{2}x(1-x^2)U_{n-2}^2(x), & n \text{ even}\\ T_{n-1}(x)+\frac{1}{2}(1-x^2)U_{n-2}^2(x), & n \text{ odd},\end{cases} \tag{3.1}$$

where $T_{n-1}(x)=\cos(n-1)\text{arc}\cos x$ denotes the Tchebycheff polynomial of first kind. Here we use the formulae

$$\left.\begin{aligned}T_{n-1}(x_\nu)&=(-1)^{\nu-1}\\ T'_{n-1}(x_\nu)&=0\end{aligned}\right\}\quad\nu=2,\ldots,n-1$$

and

$$T_{n-1^{(1)}}=1=(-1)^{n-1}T_{n-1}(-1)$$

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$$T'_{n-1}(1)=(n-1)^2=(-1)^n T'_{n-1}(-1)$$
$$U_{n-2}(1)=(n-1)=(-1)^n U_{n-2}(-1).$$

We see from (2.4) that H_n^* is the coefficient of $-x^{2n-1}$ in $H_n(x)$, so that

$$H_n^* = 2^{2n-5} \quad n \text{ even} \qquad (3.2)$$
$$= 0 \quad n \text{ odd}.$$

Thus on account of (2.4) and (3.2), the polynomials $A_n(f; x)$ exist uniquely only for even values of n. Hence from now onward we shall take n to be even.

We write (2.5) on account of (2.3) as

$$A_n(f; x) = \sum_{\nu=1}^{n} f_\nu \mu_\nu(x), \qquad (3.3)$$

where

$$\mu_\nu(x) = l_\nu^2(x) - \frac{\omega''(x_\nu)}{(\omega'(x_\nu))^3}\left[\frac{\omega^2(x)}{x-x_\nu} + \frac{H_n(x)}{H_n^*}\right].$$

Hence on using (3.1) and (3.2) etc., we get after simple calculations:

$$\mu_1(x) = \left[\frac{1+x}{2}\cdot\frac{U_{n-2}(x)}{n-1}\right]^2 + \frac{2n^2-4n+3}{12(n-1)^2}\left[(1-x^2)U_{n-2}^2(x) - 2T_{n-1}(x)\right], \qquad (3.4)$$

$$\mu_n(x) = \left[\frac{1-x}{2}\cdot\frac{U_{n-2}(x)}{n-1}\right]^2 + \frac{2n^2-4n+3}{12(n-1)^2}[(1-x^2)U_{n-2}^2(x) + 2T_{n-1}(x)] \qquad (3.5)$$

and for $\nu = 2, \ldots, n-1$

$$\mu_\nu(x) = \left[\frac{(1-x^2)U_{n-2}(x)}{(n-1)(x-x_\nu)}\right]^2 + \frac{x_\nu}{(n-1)^2(1-x_\nu^2)} \times \left[\frac{1-xx_\nu}{x-x_\nu}(1-x^2)U_{n-2}^2(x) + 2T_{n-1}(x)\right] \qquad (3.6)$$

which are, in fact polynomials of degree $2n-2$. Thus $A_n(f; x)$ is given by (3.3), (3.4), (3.5) and (3.6).

4. **Convergence of the sequence** $A_n(f; x)$ when $(x) = 2^{-(n-2)}(1-x^2)U_{n-2}(x)$. We have the following

THEOREM 2. *Let $f(x)$ be a given continuous function in $[-1, 1]$ and n even then the sequence $A_n(f; x)$ constructed on the zeros of*

$$\omega(x) = 2^{-(n-2)}(1-x^2)U_{n-2}(x) \qquad (4.1)$$

converges uniformly to $f(x)$ for $-1 \le x \le 1$.

Proof. Owing to the known convergence of the Hermite-Fejér process, a simple proof of this theorem can be given by using the Corollary to theorem 1.

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In this way we shall have no occasion to use the explicit formulae obtained in §3. In the sequel we shall also need the following

LEMMA. *For any* $f(x)\in C[-1,1]$ *and* n *even we have*

$$\lim_{n\to\infty}\frac{1}{(n-1)^2}\sum_{\nu=2}^{n-1}\frac{x_\nu}{1-x_\nu}\cdot f_\nu=\frac{f(1)-f(-1)}{6}$$

where

$$x_\nu=\cos\frac{\nu-1}{n-1}\pi$$

The proof of this lemma is similar to that of theorem 3 in [1].

From (2.12) we can easily see that

$$\chi_n(f,x)=2H_n(x)\left[\frac{f(1)-f(-1)}{6}-\frac{1}{(n-1)^2}\sum_{\nu=2}^{n-1}\frac{x_\nu}{1-x_\nu^2}f_\nu\right]+\frac{H_n(x)}{(n-1)^2}\cdot\frac{f(1)-f(-1)}{6}.$$

Hence on account of the lemma and the inequality

$$|H_n(x)|\le 3,\qquad -1\le x\le 1,$$

which easily follows from (3.1), we have

$$\|\chi_n(f,x)\|\to 0\quad\text{as}\quad n\to\infty\quad\text{for}\quad -1\le x\le 1. \tag{4.2}$$

Further it is known from [4] that for $f\in C[-1,1]$ and

$$x_\nu=\cos\frac{\nu-1}{n-1}\pi$$

$\nu=1,\ldots;n,$

$$\|H_n(f;x)-f(x)\|\to 0\quad\text{as}\quad n\to\infty,\qquad -1\le x\le 1. \tag{4.3}$$

Thus (4.2) and (4.3) satisfy the conditions of the corollary and this completes the proof of theorem 2.

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