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ON r-REGULAR r-CONNECTED NON-HAMILTONIAN GRAPHS

BRAD JACKSON AND T.D. PARSONS

Some methods are given for constructing regular r-valent r-connected non-hamiltonian graphs; often the graphs are also non-r-edge-colorable. The extent of the class of such graphs constructible from these methods and previous methods is discussed.

1. Introduction

In this paper the term "graph" excludes loops and multiple edges; and "multigraph" allows multiple edges but excludes loops.

Let $r \geq 3$ be an integer, and let "$r$cnh" abbreviate "r-valent regular r-connected non-hamiltonian", and let "rc" abbreviate "r-valent regular r-connected". For example, a 3c graph is cubic and 3-connected, while a 3cnh graph is 3c and non-hamiltonian.

Meredith [14] has constructed $r$cnh graphs for every $r \geq 3$; in fact, his method gives immediately an infinite family of $r$cnh graphs for each $r \geq 3$.

Starting from Meredith's graphs, we shall give some methods of constructing more $r$cnh graphs from them, so as to enlarge considerably the family of known $r$cnh graphs. All the constructions can be described in terms of operations of "cutting and splicing" graphs, familiar to many

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graph theorists.

2. Cutting and splicing

Let H be a multigraph and $v \in V(H)$. Then $H \# v$ denotes the object obtained from H by cutting out the vertex v but retaining the "pendants" or "open-ended edges" previously incident to v (see Figure 1); it may be helpful to consider H as a "topological multigraph", so that $H \# v$ is just the topological space obtained from H by removing the point v.

FIGURE 1

If G is a multigraph and $u \in V(G)$ has the same valency r in G as the vertex v has in H, then we let $G(u, H \# v)$ denote any multigraph obtained from disjoint copies of $G \# u$ and $H \# v$ by pairing in a one-to-one correspondence the r pendants of $G \# u$ and $H \# v$ and then splicing together (that is, fusing into a single edge) each corresponding pair of pendants (see Figure 2).

We say that $G(u, H \# v)$ results from "substituting $H \# v$ for u in G". (Once the pairing of the pendants is fixed, we have that $H(v, G \# u) = G(u, H \# v)$, so that we could equally well think of the operation as substituting $G \# u$ for v in H.) In general, the multigraph $G(u, H \# v)$ will depend on the pairing chosen for the pendants of $G \# u$ and $H \# v$, so that many different multigraphs will arise. Whenever we make an assertion about $G(u, H \# v)$, we are making the assertion for *every* such multigraph, independent of the pairing.

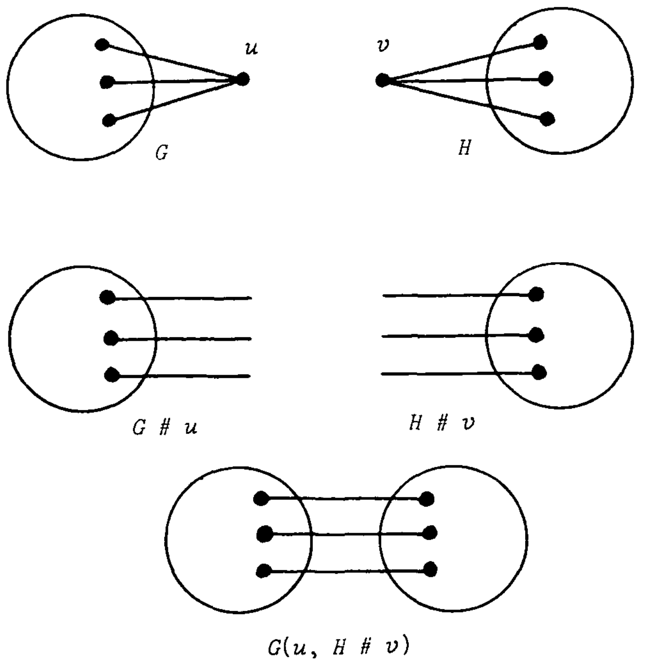

FIGURE 2

If $u_1, u_2, \ldots, u_k \in V(G)$ are distinct vertices and $v_i \in V(H_i)$ has the same valency in H_i as u_i has in G, then we use the notation $G(u_1, H_1 \# v_1; \ldots; u_k, H_k \# v_k)$ for any multigraph resulting from the substitution of disjoint copies of $H_i \# v_i$ for u_i in G, for $i = 1, 2, \ldots, k$. Also we let $G(H \# v)$ denote any substitution of disjoint copies of $H \# v$ for every vertex of G, if G is regular of the same valency as that of v in H.

Such constructions have appeared often in the literature, for example in papers of Chetwynd and Wilson [4], Grünbaum [7, pp. 33-34], Isaacs [11], Thomassen [16, p. 215], and Zamfirescu [19, p. 233].

THEOREM 1. *Let* G *and* H *be connected multigraphs, and let* $u \in V(G)$ *and* $v \in V(H)$ *both have valency* r. *Let* $X = G(u, H \# v)$.

(A) If both G *and* H *are* r*-connected, then so is* X .

(B) If G *and* H *are both* r*-edge-connected, then so is* X .

(C) If $r = 3$ *and either* G *or* H *is non-hamiltonian, then* X *is non-hamiltonian.*

(D) If there do not exist $m \le \lfloor r/2 \rfloor$ *cycles* $C_1, \ldots, C_m$ *in* H *such that* $V(C_i) \cap V(C_j) = \{v\}$ *for all* i, j *such that* $1 \le i < j \le m$ *and* $\bigcup_{i=1}^{m} V(C_i) = V(H)$ *, then* X *is non-hamiltonian.*

Proof. Parts *(A)*, *(B)* are easy exercises whose proofs we shall omit. Part *(C)* is the special case $r = 3$ of Part *(D)*. We present the easy proof of *(D)*.

Suppose C is a hamiltonian cycle in X . Then C uses some even number $2m$ of the r edges joining $G - u$ to $H - v$ in X , and if we "pinch together" all those r edges at their midpoints (to form a new vertex there, of valency $2r$), and then discard the copy of G thereby formed, then we get a copy of H in which the remaining parts of C form a closed trail T each vertex of which has valency 2 in T except the vertex v which has valency $2m$ in T . The trail T is composed of m cycles $C_1, \ldots, C_m$ having the properties forbidden by the hypotheses of *(D)*. This proves *(D)*. □

Since multigraphs of the form $G(H \# v)$ and $G(u_1, H_1 \# v_1; \ldots; u_k, H_k \# v_k)$ can be obtained by a sequence of single substitutions for vertices of G , the appropriate restatements of Theorem 1 hold for these multigraphs also, in place of $G(u, H \# v)$.

We remark that if H is a planar graph, then in any topological embedding of G in a surface S , when we cut out vertex u from G we may cut out a small disk about u on S , and then a copy of $H \# v$ can be embedded on a small disk which is sewed back onto the boundary of the resulting hole - so that $G(u, H \# v)$ is also embeddable on S . Furthermore, if the embedding of G is cellular, then so is that of $G(u, H \# v)$. In particular, if both G and H are planar, then so is $G(u, H \# v)$.

As an example of an application of Theorem 1, if H is any 3cnh graph (such as the Petersen graph) and G is any 3c graph, then every $G(u, H \# v)$ is a 3cnh graph. (Compare with Thomassen [*16*, p. 215] where this operation is used for another purpose.

When $H = K_4$, then $G(u, H \# v)$ is just the operation of "replacing u by a triangle". This operation is well-known (see Grünbaum [*6*, p. 1146]). Starting from any planar 3cnh graph such as the Tutte graph [*17*], [*2*, p. 161] and repeatedly applying this operation to vertices of successive graphs, we would obtain infinitely many planar 3cnh graphs.

3. Meredith's construction

Let $M_r = (K_{r,r} \# v)$. Then M_r is independent of the choice of $v \in V(K_{r,r})$, and there is a unique multigraph $G(u, M_r)$, the "Meredith expansion of G at u", determined by the multigraph G and the r-valent vertex u of G.

Meredith [*14*] proved that $G(u, M_r)$ is hamiltonian if and only if G is hamiltonian, and further that for r-regular multigraphs G, the graph (with no multiple edges!) $G(M_r)$ is r-connected if and only if the multigraph G is r-edge-connected. To construct $r$cnh graphs $G(M_r)$, it therefore suffices to construct r-regular r-edge-connected non-hamiltonian multigraphs G. Meredith constructed such multigraphs by replicating edges carefully in the Petersen graph. Letting $H_1 = G(M_r)$, $H_2 = H_1(M_r)$, $H_3 = H_2(M_r)$, and so on, one gets infinitely many $r$cnh graphs for each $r \geq 3$. In fact, one has more flexibility than this, for if G_1 is an $r$cnh graph, then so is $G_1(u, M_r)$ by Meredith's results and Theorem 1 *(A)*. Therefore substitutions by M_r may occur one vertex at a time, giving G_1, $G_2 = G_1(u_1, M_r)$, $G_3 = G_2(u_2, M_r)$, and so on, as $r$cnh graphs. (This is used for $r = 3$ by Holton, McKay and Plummer [*10*].)

To extract the fullest generality from Meredith's construction, let us define the notion of an "r-good" graph. An *r-assignment* for a graph G is a function $\mu : E(G) \to \{1, 2, 3, \ldots\}$ such that the multigraph $G * \mu$, obtained by replacing each edge $e = uv$ of G by $\mu(e)$ multiple edges from u to v, is r-regular. An r-assignment μ for G is an *r-good assignment* for G, and the resulting multigraph $G * \mu$ is an r-good extension of G, if $G * \mu$ is r-edge-connected. We say G is r-good if it has an r-good assignment.

In terms of our definition, Meredith showed that the Petersen graph is r-good for every $r \geq 3$, and that for every r-good extension $G * \mu$ of an r-good non-hamiltonian graph G, the graph $[G * \mu](M_r)$ is an $r$cnh graph.

The question now arises as to which graphs are r-good, and in particular which 3c graphs are r-good.

THEOREM 2. *Every* 3c *graph is* r*-good for all* $r \geq 3$.

Proof. We shall prove the theorem for all $r \neq 5$; the case $r = 5$ is more involved, and we discuss it in Remark 1 below.

Let G be a cubic 3-connected graph. If μ is an r-good assignment for G, then for any integer $m \geq 0$ the function $\mu + m : E(G) \to \{1, 2, 3, \ldots\}$ defined by $(\mu+m)(e) = \mu(e) + m$, is clearly an $(r+3m)$-good assignment. Therefore to show that G is r-good for all $r \geq 3$ such that $r \neq 5$, it suffices to prove that G is r-good for $r = 3, 4, 8$. Trivially, G is 3-good. By a well-known theorem of Petersen [15], [2, p. 79], [9, p. 89], G has a 1-factor F. Define functions μ_4 and μ_8 on $E(G)$ by

$$\mu_4(e) = \begin{cases} 2, & e \in F, \\ 1, & e \notin F, \end{cases}$$

and $\mu_8(e) = 2\mu_4(e)$ for all $e \in E(G)$. Clearly, the multigraphs $G * \mu_4$ and $G * \mu_8$ are regular of valencies 4 and 8, respectively. Let K be any (minimal) edge cutset of G, and let $s_j = \sum_{e \in K} \mu_j(e)$ for $j = 4, 8$. If $|K| \geq 4$, obviously $s_4 \geq 4$ and $s_8 \geq 8$. If $|K| = 3$, then the two components of $G - K$ each have an odd number of vertices (since G is cubic), so some edge of F is in K, giving $s_4 \geq 1 + 1 + 2 = 4$ and $s_8 \geq 2 + 2 + 4 = 8$. It follows that $G * \mu_4$ and

$G * \mu_8$ are 4-edge-connected and 8-edge-connected, respectively, so G is 4-good and 8-good. □

REMARK 1. We first proved Theorem 2 only for all $r \geq 3$ such that $r \neq 5$, *via* a longer argument involving the "Max-flow Min-cut Theorem" of Ford and Fulkerson [5], and we conjectured that every 3c graph was 5-good. Carsten Thomassen (in a Private Communication) then pointed out to us the simpler proof given above for $r \neq 5$, and informed us that he had proved the conjecture for the case $r = 5$. His latter proof derives a contradiction from the necessary properties of a minimal non-5-good 3c graph.

It follows that for every 3cnh graph G and for every $r \geq 3$ we may construct an $r$cnh graph $[G * \mu](M_r)$, based on G, for every r-good assignment μ for G. Thus what Meredith did for the Petersen graph may be done just as well for any 3cnh graph G serving as the "base" - and there is a great variety of such graphs G.

4. Construction 1

Let k be an integer such that $1 \leq k \leq r$, let $H_1, H_2, \ldots, H_k$ be disjoint copies of rc graphs (possibly some pairs of these graphs are isomorphic), and suppose further that H_1 is non-hamiltonian. Let B be the complete bipartite graph with bipartition $X \cup Y$, where $X = \{x_1, \ldots, x_r\}$ and $Y = \{y_1, \ldots, y_r\}$. For $i = 1, 2, \ldots, k$ choose a vertex v_i in graph H_i, and let $w_1^i, \ldots, w_r^i$ be the vertices adjacent to v_i in H_i. Consider any graph G of the form $B(y_1, H_1 \# v_1; \ldots; y_k, H_k \# v_k)$. In Figure 3 (p. 212) we picture the cases $k = r$ and $k < r$.

THEOREM 3. *The graph* G *is* $r$cnh.

Proof. Clearly G is r-regular, and G is r-connected by Theorem 1 *(A)*. It remains to show that G is non-hamiltonian. Suppose that G has a hamiltonian cycle C. Then $C - \{x_1, \ldots, x_r\}$ is the disjoint union of r paths $P_1, \ldots, P_r$. Any of these paths which consists of a single

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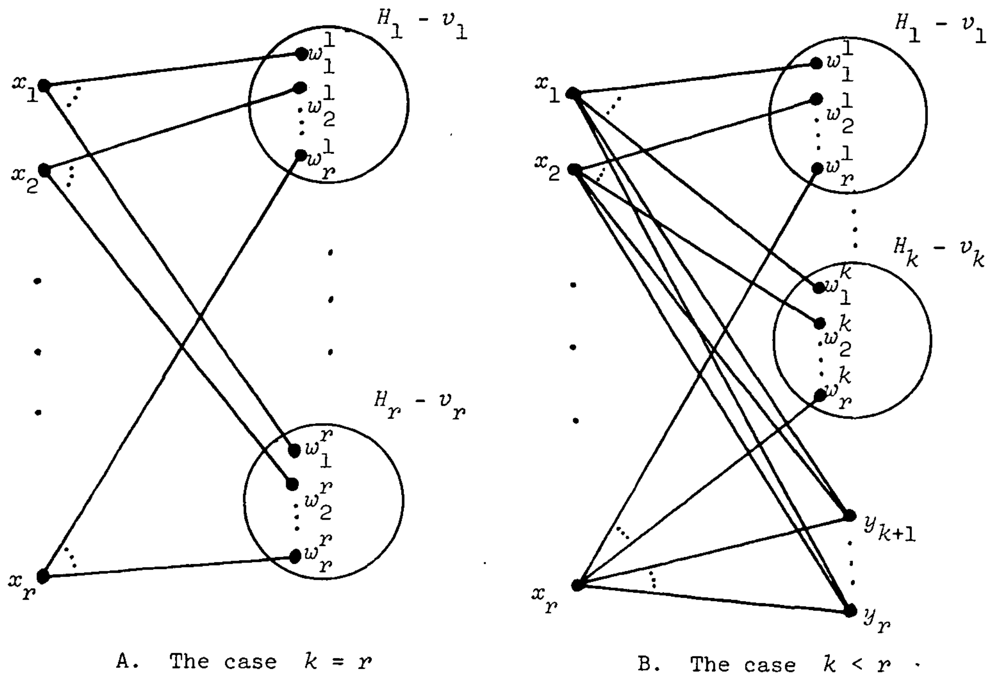

FIGURE 3

point is either a w_j^i or (if $k < r$), a y_{k+j} . And any such path which is nontrivial must have both its endpoints in some set $\{w_1^i, \ldots, w_r^i\}$, and must be contained entirely in some induced subgraph $H_i - v_i$. Since every $H_i - v_i$ $(1 \le i \le k)$ must have its vertices among those vertices of the paths $P_1, \ldots, P_r$, and since if $k < r$ each y_{k+j} (for $1 \le j \le r-k$) must be some P_i , we see that we may assume without loss of generality that P_i is a hamiltonian path in $H_i - v_i$ for $1 \le i \le k$ and (if $k < r$) that $P_{k+j} = y_{k+j}$ for $1 \le j \le r-k$. But then P_1 is a hamiltonian path in $H_1 - v_1$, and the endpoints of P_1 are in $\{w_1^1, \ldots, w_r^1\}$, so that $v_1 P_1 v_1$ is a hamiltonian cycle in H_1 . This

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contradicts the hypothesis that H_1 is $r$cnh. Therefore G is non-hamiltonian, and the theorem is proved. □

We note that this construction requires an initial $r$cnh graph H_1, which can be obtained from Meredith's construction, based on any 3cnh graph. Since there are many choices for H_1, for $v_1 \in V(H_1)$, and since $H_2, \ldots, H_k$ are arbitrary rc graphs, the construction is quite flexible in creating new $r$cnh graphs. In particular, it can first be used to create many new 3cnh graphs, which will be r-good and so in turn can be used to create more $r$cnh graphs.

We remark that when $k = 1$, the graph $G = B(y_1, H_1 \# v_1)$ is just the same as the graph $H_1(v_1, M_r)$ obtained by performing a Meredith expansion of v_1 in H_1, as in Section 3.

5. Construction 2

Let $k > \lfloor (r-2)/2 \rfloor$, and first construct a graph $G = B(y_1, H_1 \# v_1; \ldots, y_k, H_k \# v_k)$ as in Construction 1, but where we require all of $H_1, \ldots, H_j$ to be $r$cnh, where j is some (fixed) integer such that $j \le k$ and $j > \lfloor (r-2)/2 \rfloor$. Next let H be an arbitrary rc graph and v an arbitrary vertex of H, and let $J = H(v, G \# x_1)$. The graph J is the result of Construction 2.

THEOREM 4. *The graph J is $r$cnh.*

Theorem 4 follows at once from Theorem 1 and the following lemma.

LEMMA 5. *Let m be an integer such that $1 \le m \le \lfloor r/2 \rfloor$, and let G be the $r$cnh graph described just prior to Theorem 4. Then there do not exist cycles $C_1, \ldots, C_m$ of G such that $\bigcup_{i=1}^{m} V(C_i) = V(G)$ and $1 \le p < q < m$ implies $V(C_p) \cap V(C_q) = \{x_1\}$.*

Proof. Suppose that such cycles $C_1, \ldots, C_m$ exist. Then they would give a closed trail T in G in which vertex x_1 would have valency $2m$

and every other vertex of T would have valence 2 in T. Now $T - \{x_1, \ldots, x_r\}$ consists of a family Π of $m + r - 1$ disjoint paths, each of which is either a single y_{k+t} (if $k < r$) or else is contained entirely in some induced subgraph $H_i - v_i$ of G for $i \in \{1, \ldots, k\}$. Furthermore, since every vertex outside $\{x_1, \ldots, x_r\}$ must lie in exactly one of these paths, every singleton y_{k+t} is one of the paths, and each vertex set $V(H_i - v_i)$ must be covered by those paths $P \in \Pi$ such that $P \subseteq V(H_i - v_i)$, for $1 \le i \le k$. Since H_i is non-hamiltonian for $1 \le i \le j$, the graph $H_i - v_i$ contains no hamiltonian path for $1 \le i \le j$. Therefore it takes at least two members of Π to cover each $V(H_i - v_i)$, for $1 \le i \le j$. Since it takes at least one member of Π to cover each $V(H_i - v_i)$ for $j \le i \le k$, and to cover each y_{k+t} for $1 \le t \le r-k$, we have that

$$|\Pi| \ge 2j + (k-j) + (r-k) = r + j > r + \lfloor (r-2)/2 \rfloor .$$

But $|\Pi| = m + r - 1$ and $m \le \lfloor r/2 \rfloor$, so $|\Pi| \le \lfloor r/2 \rfloor + r - 1$, so that we get $r + \lfloor (r-2)/2 \rfloor < \lfloor r/2 \rfloor + r - 1$, a contradiction.

Therefore no such cycles $C_1, \ldots, C_m$ exist. This proves the lemma, and Theorem 4 follows. $\square$

We see from Construction 2 that, if in Construction 1 enough of the building blocks H_i are $r$cnh, then the resulting $r$cnh graph G may be "substituted" into an arbitrary rc graph H to create a new $r$cnh graph J. This method has been refined to produce $r$cnh graphs whose longest cycles contain an arbitrarily small fraction of the vertex set; we discuss this briefly in the next section.

There are many special constructions for $r$cnh graphs, and we indicate a few more, but omit the proofs.

Let L be the join of a cycle $y_1 y_2 \cdots y_r y_1$ with $r - 2$ independent points $x_1, \ldots, x_{r-2}$, and let H_i be $r$cnh and $v_i \in V(H_i)$ for $i = 1, \ldots, r$. Then $L(y_1, H_1 \# v_1; \ldots; y_r, H_r \# v_r)$ is $r$cnh.

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Let H_1, H_2, H_3, H_4 be 4cnh, and let K_5 be the complete graph with vertices y_i $(1 \le i \le 5)$. Then $K_5(y_1, H_1 \# v_1; \ldots; y_4, H_4 \# v_4)$ is 4cnh, for any choices of $v_i \in V(H_i)$, $1 \le i \le 4$.

6. Longest cycles in $r$cnh graphs

Our work in this paper was motivated by a question of Babai. Let $l_r(n)$ be the minimum length of a longest cycle in any rc graph on exactly n vertices. (The domain of the function l_r is the set of all integers $n \ge r+1$ if r is even, or all *even* $n \ge r+1$ if r is odd, since rc graphs exist precisely for these n, by a theorem of Harary [8].) Babai asked [1] whether $l_r(n) < n^{1-\varepsilon}$ for some positive constant ε and all sufficiently large n for which l_r is defined. For the case $r = 3$, it was well-known that $\liminf(\log l_3(n))/\log n < 1$; and recently, Bondy and Simonovits [3] showed that $e^{a\sqrt{\log n}} < l_3(n) < n^{1-\varepsilon}$ for all large n and positive constants a, ε.

Applying constructions first obtained while working on this paper, we have been able now to answer Babai's question affirmatively for all $r \ge 3$ (see [12], [13]).

We remark that for those $r$cnh graphs G on n vertices whose longest cycles have fewer than $\lfloor n/r \rfloor$ vertices, the graphs $H(v, G \# x)$ will be $r$cnh for any rc graph H (similar to the phenomenon of our Construction 2).

7. Non-r-edge-colorable $r$cnh graphs

Meredith [14] noted that a regular r-valent multigraph H is r-edge-colorable if and only if the multigraph $H(v, M_r)$ is r-edge-colorable (where we defined M_r in Section 3), and he obtained appropriate multigraphs H by replicating edges in Petersen's graph so that $H(M_r)$ was an r-regular non-r-edge-colorable graph. Meredith constructed such a graph (of order $20r - 10$) for each $r \ge 3$; these were all non-hamiltonian, of course, and all were r-connected except for the cases

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$r = 5, 6, 7$. For any such graph G then, all of G , $G(u_1, M_r) = G_2$, $G_2(u_2, M_r) = G_3$, and so on, are r-regular non-r-edge-colorable, so Meredith constructed an infinite family of such graphs for each $r \geq 3$, and these were also $r$cnh except for $r = 5, 6, 7$.

To extract the fullest generality from Meredith's methods, let us define a graph G to be *r-nice* if there is an r-assignment μ for which $G * \mu$ (defined in Section 3) is non-r-edge-colorable; in this case we say that μ is an *r-nice assignment* for G , and that $G * \mu$ is an *r-nice extension* of G , and then by Meredith's results, $[G * \mu](M_r)$ will be an r-regular non-r-edge-colorable graph, which will be non-hamiltonian if G is non-hamiltonian, and r-connected if μ is also r-good. In terms of our definition, for G the Petersen graph, Meredith constructed one r-nice extension $G * \mu$ for each $r \geq 3$, such that these extensions were also r-good for $r \geq 8$.

Of course if G has odd order and $G * \mu$ is r-regular (so r is even), then $G * \mu$ will trivially be non-r-edge-colorable; thus the question of r-nice extensions is interesting only for graphs G of even order.

Many of the $r$cnh graphs arising from our constructions will be non-r-edge-colorable, as follows from Lemma 6 and its Corollary 7, below. Lemma 6 is trivial; when $r = 3$, it is the same as the well-known (and equally trivial) "Parity Lemma" (see [4]), but for $r > 3$ it has stronger hypotheses and a stronger conclusion than the "Generalized Parity Lemma".

LEMMA 6. *Let $r \geq 3$ and let G be a connected r-regular multigraph with a proper r-edge coloring using the colors $1, 2, \ldots, r$. Suppose that K is an edge-cutset of cardinality r , such that at least one of the components of $G - K$ has odd order. Then K contains one edge of each color i , for $i = 1, 2, \ldots, r$.*

Proof. The edges of color i form a 1-factor F_i in G , but at least one component of $G - K$ cannot contain any 1-factor, so K contains an edge of F_i . □

COROLLARY 7. *Let G and H be connected r-regular multigraphs at least one of which has even order. Let $u \in V(G)$ and $v \in V(H)$. Then*

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$G(u, H \# v)$ is r-edge-colorable if and only if both G and H are r-edge-colorable. □

It follows that if, say, in our Construction 1 one of the graphs H_i is of even order and non-r-edge-colorable, then the resulting graph G will be $r$cnh and non-r-edge-colorable. Similar remarks apply to the other constructions.

Because of its connection with the famous "four color problem", the construction of cubic non-3-edge-colorable graphs has received much attention. Certain such graphs have been called "snarks", and similarly certain r-regular non-r-edge-colorable graphs have been called "r-snarks" or "supersnarks" [*4*]. The r-regular non-r-edge colorable graphs constructed by Meredith are examples of r-snarks.

The definitions of "snark" and "r-snark" are burdened with various *ad hoc* conditions designed to exclude so called "trivial" r-regular non-r-edge-colorable graphs. In Isaacs' paper [*11*], much ado is made about what should or should not be considered "trivial", and infinite families of "non-trivial" snarks are constructed. Such usage of the word "trivial" is fraught with difficulties, in our opinion; for, what mathematicians regard as trivial changes with the passage of time (a fact noted also by Isaacs). Consequently, the definitions of "elusive" objects such as "snarks" will become increasingly encumbered by *ad hoc* exclusions. For example, in light of our Lemma 6, it seems that the definition of "r-snark" for $r > 3$ in [*4*] should further exclude the existence of cutsets K of the type in our lemma.

It seems more productive to regard certain objects as "primitive" or "irreducible" (rather than "trivial") with respect to some specified set of constructions or operations; if this is done, then truly few primitive $r$nch graphs, or r-regular non-r-edge-colorable graphs, are known.

8. Concluding remarks

We conclude with some questions for further research.

It is natural to want to be able to characterize the objects of some family - such as $r$cnh graphs, or r-regular non-r-edge-colorable graphs - as being constructible from some more tractable family of "primitive"

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objects by means of some finite list of computable constructions or operations. Sometimes this can be done completely, such as in the fundamental theorem of arithmetic where the primitives are primes, and in the fundamental theorem of algebra where the primitives are monic linear polynomials and complex constants. We know of no such characterization of $r$nch graphs, even relative to a large family of primitives such as the rc graphs.

In our constructions, the primitives are the rc graphs and the "smallest" $r$cnh graphs relative to decompositions of graphs $G(u, H \# v)$ into graphs G, H having each more than one vertex. In turn, these smallest $r$cnh graphs arose from Meredith's construction, for which the primitives could be regarded as arising from smallest 3cnh graphs G from the operation $[G * \mu](M_r)$ for r-good assignments μ. Do all 3cnh graphs arise somehow from the Petersen graph? Tutte [18] has conjectured that every bridgeless cubic graph which is not 3-edge-colorable contains a subgraph contractible to the Petersen graph.

Does our Theorem 2 generalize to "Every k-regular k-connected graph is r-good for all $r \geq k$ "? Which graphs G of even order have r-nice assignments for all appropriate values of r ? In particular, given any 3cnh graph, for which $r \geq 3$ must it have an r-nice assignment?

References

[1] L. Babai, Problem 18, Unsolved problems, *Summer Research Workshop in Algebraic Combinatorics* (Mathematics Department, Simon Fraser University, July 1979).

[2] J.A. Bondy and U.S.R. Murty, *Graph theory with applications* (American Elsevier, New York, 1976).

[3] J.A. Bondy and M. Simonovits, "Longest cycles in 3-connected 3-regular graphs", *Canad. J. Math.* 32 (1980), 987-992.

[4] Amanda G. Chetwynd and Robin J. Wilson, "Snarks and supersnarks", *Proceedings of the Fourth International Conference on Graph Theory*, Kalamazoo, Michigan (John Wiley & Sons, New York, to appear).

[5] L.R. Ford, Jr. and D.R. Fulkerson, "Maximal flow through a network", *Canad. J. Math.* 8 (1956), 399-404.

[6] Branko Grünbaum, "Polytopes, graphs, and complexes", *Bull. Amer. Math. Soc.* 76 (1970), 1131-1201.

[7] Branko Grünbaum, "Vertices missed by longest paths or circuits", *J. Combin. Theory Ser. A* 17 (1974), 31-38.

[8] Frank Harary, "The maximum connectivity of a graph", *Proc. Nat. Acad. Sci. U.S.A.* 48 (1962), 1142-1146.

[9] Frank Harary, *Graph theory* (Addison-Wesley, Reading, Massachusetts; Menlo Park, California; London; 1969).

[10] D.A. Holton, B.D. McKay & M.D. Plummer, "Cycle through specified vertices in 3-connected cubic graphs" (Research Report 38. University of Melbourne, Parkville, Victoria 1979).

[11] Rufus Isaacs, "Infinite families of nontrivial trivalent graphs which are not Tait colorable", *Amer. Math. Monthly* 82 (1975), 221-239.

[12] Brad Jackson and T.D. Parsons, "Longest cycles in r-regular r-connected graphs", submitted.

[13] Brad Jackson and T.D. Parsons, "A shortness exponent for r-regular r-connected graphs", submitted.

[14] G.H.J. Meredith, "Regular n-valent n-connected nonhamiltonian non-n-edge-colorable graphs", *J. Combin. Theory Ser. B* 14 (1973), 55-60.

[15] Julius Petersen, "Die Theorie der regulären Graphs", *Acta Math.* 15 (1891), 193-220.

[16] Carsten Thomassen, "A minimal condition implying a special K_4-subdivision in a graph", *Arch. Math.* 25 (1974), 210-215.

[17] W.T. Tutte, "On Hamiltonian circuits", *J. London Math. Soc.* 21 (1946), 98-101.

[18] W.T. Tutte, "A geometrical version of the four color problem", *Combinatorial mathematics and its applications*, 553-560 (Proc. Conf. Univ. North Carolina, Chapel Hill, North Carolina, 1967. University of North Carolina Press, Chapel Hill, New York, 1969).

[19] Tudor Zamfirescu, "On longest paths and circuits in graphs", *Math. Scand.* **38** (1976), 211-239.

Department of Mathematics,
University of California, Santa Cruz,
Santa Cruz,
California 95064,
USA;

Department of Mathematics,
College of Science,
Pennsylvania State University,
University Park,
Pennsylvania 16802,
USA.