VECTOR FIELDS AND INFINITESIMAL TRANSFORMATIONS ON ALMOST-HERMITIAN MANIFOLDS WITH BOUNDARY

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Introduction. Many authors have made interesting and important contributions to the study of vector fields or infinitesimal transformations on compact orientable Riemannian manifolds and Hermitian manifolds without boundary. Recently, Hsiung (**6**, **7**, **8**) has extended some of these results to compact orientable Riemannian manifolds with boundary. The purpose of this paper is to continue Hsiung's work by studying vector fields and infinitesimal transformations on almost-Hermitian manifolds with boundary.

Section 1 contains fundamental notations and local operators and formulas for a Riemannian manifold.

In §2 fundamental formulas for Lie derivatives are given, and the infinitesimal transformations and their generating vector fields are defined in terms of Lie derivatives.

Section 3 is devoted, for compact orientable Riemannian manifolds with boundary, to a discussion of local boundary geodesic co-ordinates and the derivation of some integral formulas and theorems, which were obtained by Hsiung and will be needed in later sections of this paper.

Section 4 contains necessary and sufficient boundary conditions for a Killing vector field on a compact orientable Riemannian manifold M^n with boundary B^{n-1} to be a geodesic vector field, and also a curvature condition of the manifold M^n for the non-existence on M^n of a geodesic vector field subject to the same boundary conditions.

In §5 we define almost-Hermitian, almost-semi-Kählerian and almost-Kählerian structures with their relations. Then for almost-Hermitian structures we derive some formulas which will be useful in the next two sections.

Section 6 is devoted to contravariant analytic vector fields on an almost-Hermitian manifold M^n with boundary B^{n-1}, together with their relations to Killing, projective Killing, and conformal Killing vector fields. First on the manifold M^n we obtain some integral formulas for projective and conformal Killing vector fields and necessary and sufficient boundary conditions for a vector field to be contravariant analytic. From the integral formulas, conditions are then derived for a projective, as well as conformal, Killing vector field on M^n to define an automorphism of the manifold M^n leaving the boundary B^{n-1}

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invariant, where by an automorphism we mean an infinitesimal motion preserving the almost-Hermitian structure.

Section 7 is concerned with covariant analytic vector fields on an almost-Hermitian manifold with boundary, and contains necessary and sufficient conditions for a vector field on the manifold to be covariant analytic, obtained by calculations similar to those in §6.

Finally, §8 is devoted to a study of vector fields on an almost-Kählerian manifold with boundary. In particular, an integral formula for a contravariant analytic vector field in terms of the canonical connection on the manifold is obtained. The main results of §§6, 7, 8 constitute an extension of some recent work of Yano **(14)** and Tachinaba **(12)** on almost-Hermitian and almost-Kählerian manifolds without boundary respectively.

Throughout this paper, the dimensions of M^n and B^{n-1} are understood to be n ($\geqslant 2$) and $n-1$ respectively, all Riemannian manifolds are of class C^3, and all differential forms and vector fields are of class C^2.

1. Notations and operators.

Let M^n be a Riemannian manifold of dimension n, $\|g_{ij}\|$ with $g_{ij} = g_{ji}$ the matrix of the positive definite metric of the manifold M^n, and $\|g^{ij}\|$ the inverse matrix of $\|g_{ij}\|$. Throughout this paper all italic indices take the values $1, \ldots, n$ unless stated otherwise. We shall follow the usual tensor convention that indices can be raised and lowered by using g^{ij} and g_{ij} respectively; and that when an italic letter appears in any term as a subscript and superscript, it is understood that this letter is summed for all the values $1, \ldots, n$. We shall also use v^i and v_i to denote the contravariant and covariant components of a vector field v respectively. Moreover, if we multiply, for example, the components a_{ij} of a covariant tensor by the components b^{jk} of a contravariant tensor, it will always be understood that j is to be summed.

Let $\mathfrak{N}$ be the set $\{1, \ldots, n\}$ of positive integers less than or equal to n, and $I(p)$ denote an ordered subset $\{i_1, \ldots, i_p\}$ of the set $\mathfrak{N}$ for $p \leqslant n$. If the elements $i_1, \ldots, i_p$ are in the natural order, that is, if $i_1 < \ldots < i_p$, then the ordered set $I(p)$ is denoted by $I_0(p)$. Furthermore, let $I(p; \tilde{s}|j)$ be the ordered set $I(p)$ with the sth element i_s replaced by another element j of $\mathfrak{N}$, which may or may not belong to $I(p)$. We shall use these notations for indices throughout this paper. When more than one set of indices is needed at one time, we may use other capital letters such as $J, K, \ldots$ in addition to I.

From the metric tensor g with components g_{ij} we have

$$g_{I(n),K(n)} = g_{i_1 j_1} \cdots g_{i_n j_n} \delta^{J(n)}_{K(n)},$$

where $\delta^{J(n)}_{K(n)}$ is zero when two or more j's or k's are the same, and is $+1$ or -1 according as the j's and k's differ from one another by an even or odd number of permutations. Thus the element of area of the manifold M^n at a point P with local co-ordinates $x^1, \ldots, x^n$ is

$$dA_n = e_{1\ldots n} dx^1 \wedge \ldots \wedge dx^n, \tag{1.1}$$

where d and the wedge $\wedge$ denote the exterior differentiation and multiplication respectively, and

$$(1.2)\qquad e_{1\ldots n} = +\sqrt{g_{1\ldots n,1\ldots n}}.$$

By using orthonormal local co-ordinates $x^1, \ldots, x^n$ and the relations

$$(1.3)\qquad e_{I(n)} = \delta^{1\ldots n}_{I(n)}\, e_{1\ldots n},$$

$$(1.4)\qquad \delta^{I(p)\,J(n-p)}_{1\ldots n}\, \delta^{1\ldots n}_{I(p)K(n-p)} = p!\, \delta^{J(n-p)}_{K(n-p)},$$

we can easily obtain

$$(1.5)\qquad e_{I(p)K(n-p)}\, e^{I(p)\,J(n-p)} = p!\, \delta^{J(n-p)}_{K(n-p)}.$$

From equations (1.3), (1.4), (1.5) it follows that

$$(1.6)\qquad e_{1\ldots n}\, e^{1\ldots n} = 1.$$

On the manifold M^n let $v_{(p)}$ be a differential form of degree p given by

$$(1.7)\qquad v_{(p)} = v_{I_0(p)}\, dx^{I_0(p)},$$

where we have placed

$$(1.8)\qquad dx^{I(p)} = dx^{i_1} \wedge \ldots \wedge dx^{i_p}.$$

Then we have

$$(1.9)\qquad dv_{(p)} = (-1)^p \sum_{i_{p+1}>i_p} \left[\nabla_{i_{p+1}} v_{I_0(p)} - \sum_{s=1}^{p} \nabla_{i_s} v_{I_0(p;\, \tilde{s} \mid i_{p+1})} \right] dx^{I_0(p+1)},$$

where ∇ denotes the covariant derivation with respect to the affine connection of the Riemannian metric g_{ij}, whose components in the local co-ordinates $x^1, \ldots, x^n$ are given by

$$(1.10)\qquad \Gamma_j{}^i{}_k = \tfrac{1}{2} g^{ih}(\partial g_{hj}/\partial x^k + \partial g_{hk}/\partial x^j - \partial g_{jk}/\partial x^h).$$

Moreover, the dual and codifferential operators $*$ and δ are defined by

$$(1.11)\qquad *v_{(p)} = e_{I_0(p)J_0(n-p)}\, v^{I_0(p)}\, dx^{J_0(n-p)},$$

$$(1.12)\qquad \delta v_{(p)} = (-1)^{np+n+1} * d * v_{(p)},$$

which imply immediately

$$(1.13)\qquad \delta v_{(p)} = -p g^{ij}\, \nabla_j v_{i I_0(p-1)}\, dx^{I_0(p-1)}.$$

In particular, for a vector field v on the manifold M^n we obtain, from equations (1.9), (1.13),

$$(1.14)\qquad (dv)_{ij} = \nabla_i v_j - \nabla_j v_i,$$

$$(1.15)\qquad \delta v = -\nabla_i v^i,$$

$$(1.16)\qquad (d\delta v)_i = -\nabla_i \nabla_j v^j,$$

$$(1.17)\qquad (\delta d v)_i = \nabla^j \nabla_i v_j - \nabla^j \nabla_j v_i,$$

where $\nabla^j = g^{jk}\nabla_k$. Use of equations (1.16), (1.17), and the Ricci identity for the contravariant components v^i,

$$(1.18) \qquad [\nabla_k, \nabla_j]\, v^i = v^l R^i{}_{ljk},$$

thus gives

$$(1.19) \qquad (\Delta v)_i = -\nabla^j \nabla_j v_i + R_{ij} v^j,$$

where $[\nabla_k, \nabla_j] = \nabla_k \nabla_j - \nabla_j \nabla_k$, and Δ, $R^i{}_{jkl}$, R_{ij} are respectively the Laplace–Beltrami operator, the Riemann curvature tensor, and the Ricci tensor defined by

$$(1.20) \qquad \Delta = d\delta + \delta d,$$

$$(1.21) \qquad R^i{}_{jkl} = \partial \Gamma_j{}^i{}_k / \partial x^l - \partial \Gamma_j{}^i{}_l / \partial x^k + \Gamma_j{}^s{}_k \Gamma_s{}^i{}_l - \Gamma_j{}^s{}_l \Gamma_s{}^i{}_k,$$

$$(1.22) \qquad R_{ij} = R^k{}_{ijk}.$$

By contraction with respect to i and k, from equation (1.18) we have

$$(1.23) \qquad [\nabla_i, \nabla_j] v^i = R_{ij} v^i v^j.$$

Multiplication of equation (1.18) by g_{ih} gives the Ricci identity for the covariant components v_i,

$$(1.24) \qquad [\nabla_k, \nabla_j] v_i = -v_l R^l{}_{ijk},$$

which can also be written as

$$(1.25) \qquad -\nabla_k \nabla_i v_j + \nabla_k(\nabla_i v_j + \nabla_j v_i) - \nabla_j \nabla_k v_i = -v_l R^l{}_{ijk}.$$

Multiplying equation (1.25) by $g^{ik} g^{jh}$ and using equation (1.19), we thus obtain

$$(1.26) \qquad (Qv)^h - (\Delta v)^h = \nabla_i(\nabla^i v^h + \nabla^h v^i) - \nabla^h \nabla_i v^i,$$

where

$$(1.27) \qquad (Qv)^h = 2R_i{}^h v^i.$$

Let $u_{I(p)}$ and $v_{I(p)}$ be two tensor fields of the same order p on a compact orientable manifold M^n. Then the local and global scalar products $\langle u, v\rangle$ and (u, v) of the two tensor fields u and v are defined by

$$(1.28) \qquad \langle u, v\rangle = \frac{1}{p!}\, u^{I(p)} v_{I(p)},$$

$$(1.29) \qquad (u, v) = \int_{M^n} \langle u, v\rangle\, dA_n.$$

From equations (1.28), (1.29) it follows that (u, u) is non-negative, and that $(u, u) = 0$ implies that $u = 0$ on the whole manifold M^n.

2. Lie derivatives and infinitesimal transformations. Let v be a non-zero vector field on a Riemannian manifold M^n, and let i_v and L_v denote, respectively, the interior product and the Lie derivative with respect to the vector field v. Then for a covariant tensor a of order r, the interior product $i_v a$ is a tensor of order $r - 1$ defined by

$$(i_v a)_{I(r-1)} = v^j a_{jI(r-1)}, \tag{2.1}$$

and according to H. Cartan **(2)** we have

$$L_v = i_v d + di_v, \tag{2.2}$$

from which it follows that

$$L_v d = dL_v = di_v d. \tag{2.3}$$

For later developments we shall use the following known formulas for Lie derivatives in terms of local co-ordinates $x^1, \ldots, x^n$ of the manifold M^n (for these formulas see, for instance, **9, 13**):

$$L_v u^{ij}{}_k = v^s \nabla_s u^{ij}{}_k - u^{sj}{}_k \nabla_s v^i - u^{is}{}_k \nabla_s v^j + u^{ij}{}_s \nabla_k v^s, \tag{2.4}$$

$$\begin{aligned} L_v(\nabla_l u^i{}_{jk}) - \nabla_l(L_v u^i{}_{jk}) = (L_v \Gamma_l{}^i{}_s)u^s{}_{jk} \\ -(L_v \Gamma_l{}^s{}_j)u^i{}_{sk} - (L_v \Gamma_l{}^s{}_k)u^i{}_{js}, \end{aligned} \tag{2.5}$$

$$L_v g_{ij} = \nabla_i v_j + \nabla_j v_i, \tag{2.6}$$

$$L_v \Gamma_j{}^i{}_k = \nabla_k \nabla_j v^i + R^i{}_{jkl} v^l, \tag{2.7}$$

where $u^{ij}{}_k$ and $u^i{}_{jk}$ are tensor fields of class at least C^1 on the manifold M^n, the contravariant and covariant orders of each being the numbers of superscripts and subscripts respectively. By applying equation (2.5) to g_{ij} we can easily obtain

$$L_v \Gamma_j{}^i{}_k = \tfrac{1}{2} g^{il}[\nabla_j(L_v g_{lk}) + \nabla_k(L_v g_{jl}) - \nabla_l(L_v g_{jk})]. \tag{2.8}$$

The infinitesimal transformation on the manifold M^n generated by a non-zero vector field v is called an infinitesimal motion (or isometry), affine collineation, projective motion, or conformal motion, and the corresponding v a Killing, an affine Killing, a projective Killing, or a conformal Killing vector field according as

$$L_v g_{ij} = 0, \tag{2.9}$$

$$L_v \Gamma_j{}^i{}_k = 0, \tag{2.10}$$

$$L_v \Gamma_j{}^i{}_k = p_j \delta_k{}^i + p_k \delta_j{}^i, \tag{2.11}$$

or

$$L_v g_{ij} = 2\phi g_{ij}, \tag{2.12}$$

where

$$p_i = \partial p/\partial x^i = \nabla_i p \tag{2.13}$$

is a gradient vector field on the manifold M^n, and ϕ is a scalar. An infinitesimal conformal motion defined by equation (2.12) is called a homothetic motion if ϕ is constant.

From equations (2.6), (2.9) it follows that v is a Killing vector field if

$$\nabla_i v_j + \nabla_j v_i = 0, \tag{2.14}$$

which with equation (1.15) implies

$$\delta v = 0. \tag{2.15}$$

From equations (2.7), (2.11) for any projective Killing vector field v we have

$$L_v \Gamma_j{}^i{}_k = \nabla_k \nabla_j v^i + R^i{}_{jkl} v^l = p_j \delta_k{}^i + p_k \delta_j{}^i. \tag{2.16}$$

Contraction with respect to i and j in equation (2.16) and use of the identity $R^i{}_{ikl} = 0$ give

$$p_j = \frac{1}{n+1} \nabla_j \nabla_i v^i. \tag{2.17}$$

By means of equations (1.16), (1.19), (1.27), (2.17), and the equation obtained by multiplying equation (2.16) by g^{jk} we thus have

$$\Delta v - \frac{2}{n+1} d\delta v = Qv. \tag{2.18}$$

Similarly, for a conformal Killing vector field v, from equations (2.7), (2.15), (2.8) we have

$$\nabla_i v_j + \nabla_j v_i = 2\phi g_{ij}, \tag{2.19}$$

$$-\delta v = n\phi, \tag{2.20}$$

$$\begin{aligned} L_v \Gamma_j{}^i{}_k &= \nabla_k \nabla_j v^i + R^i{}_{jkl} v^l \\ &= \phi_j \delta_k{}^i + \phi_k \delta_j{}^i - \phi^i g_{jk}, \end{aligned} \tag{2.21}$$

where we have placed

$$\phi_j = \nabla_j \phi = \partial\phi/\partial x^j, \qquad \phi^i = g^{ij} \phi_j. \tag{2.22}$$

Multiplication of equation (2.21) by g^{jk} and substitution of equation (2.20) in the resulting equation yield immediately

$$\Delta v + \left(1 - \frac{2}{n}\right) d\delta v = Qv. \tag{2.23}$$

3. Local boundary geodesic co-ordinates and integral formulas. Throughout this paper, by an $(n-1)$-dimensional boundary B^{n-1} of a compact n-dimensional submanifold M^n of an n-dimensional manifold $\mathfrak{M}^n$ $(n \geqslant 2)$ we mean either an empty or a non-empty subdomain on the submanifold M^n satisfying the following condition: At every point P of the boundary B^{n-1} there is a full neighbourhood $U(P)$ of the point P on the manifold $\mathfrak{M}^n$ and admissible local co-ordinates $x^1, \ldots, x^n$ such that $U(P) \cap M^n$ appears in the space of the x's as a hemisphere

$$(3.1) \qquad \sum_{i=1}^{n} (x^i)^2 < \epsilon^2, \qquad x^1 < 0,$$

the base $x^1 = 0$ of the hemisphere corresponding to the boundary B^{n-1}. For non-empty boundary B^{n-1} we shall choose the local co-ordinates $x^1, \ldots, x^n$ to be boundary geodesic co-ordinates so that at each point P of the boundary the x^1-curve is a geodesic of the manifold M^n, with x^1 as its arc length measured from the boundary B^{n-1}, and is orthogonal to the x^i-curves, $i = 2, \ldots, n$. Thus on the boundary B^{n-1} we can easily obtain (cf. **3**, p. 57)

$$(3.2) \qquad g_{11} = g^{11} = 1, \qquad g_{1i} = g^{1i} = 0 \qquad (i = 2, \ldots, n).$$

Moreover, by equation (3.1) the unit tangent vector N of the x^1-curve at every point P of the boundary B^{n-1} is the unit outer normal vector of the boundary B^{n-1} in the sense that x^1 is increasing along the x^1-curve in the direction of the vector N.

By using local boundary geodesic co-ordinates, from equations (1.10), (3.2), it is easily seen that on the boundary B^{n-1}

$$(3.3) \qquad \Gamma_1{}^1{}_1 = 0, \qquad \Gamma_1{}^i{}_1 = 0, \qquad \Gamma_1{}^1{}_i = 0, \qquad 2\Gamma_1{}^i{}_j = g^{ik}\partial g_{jk}/\partial x^1,$$

$$(3.4) \qquad \begin{aligned} b_{ij} &= (\overline{\nabla}_j \overline{\nabla}_i x^h) g_{hk} N^k + g_{lr} N^r \Gamma_h{}^l{}_k \nabla_i x^h \nabla_j x^k \\ &= \Gamma_i{}^1{}_j = -\tfrac{1}{2}\partial g_{ij}/\partial x^1 \qquad (i, j = 2, \ldots, n), \end{aligned}$$

where b_{ij} are the coefficients of the second fundamental form on the boundary B^{n-1} relative to the outer normal vector N on the manifold M^n, and $\overline{\nabla}$ denotes the covariant derivation with respect to the metric tensor g_{ij} $(i, j = 2, \ldots, n)$ of the boundary B^{n-1} (cf. **3**, p. 147). Equations (3.3), (3.4) imply immediately that

$$(3.5) \qquad b_j{}^i \equiv g^{ik} b_{kj} = -\Gamma_1{}^i{}_j.$$

The boundary B^{n-1} is said to be convex or concave on the manifold M^n according as the matrix $||b_{ij}||$ for $i, j = 2, \ldots, n$ is negative or positive definite. If $b_{ij} = 0$ for $i, j = 2, \ldots, n$, then all the geodesics of the boundary B^{n-1} are geodesics of the manifold M^n, and the boundary B^{n-1} is said to be totally geodesic on the manifold. Moreover, in terms of local boundary geodesic co-ordinates the tangential and normal components of a vector v are respectively v_i, $i \neq 1$, and v_1.

Now consider a compact orientable Riemannian manifold M^n with boundary B^{n-1}, and let u be a vector field of class C^2 on M^n. Then on the manifold M^n we can construct the differential form

$$(3.6) \qquad \omega = {}^* u_i \, dx^i.$$

By means of equations (1.11), (1.3) we can easily obtain

$$(3.7) \qquad \omega = \delta^{1\ldots n}_{i J_0(n-1)} e_{1\ldots n} u^i \, dx^{J_0(n-1)},$$

which becomes, on the boundary B^{n-1} in terms of local boundary geodesic co-ordinates,

$$\omega = u_1 \, dA_{n-1}, \tag{3.8}$$

where

$$dA_{n-1} = e_{1\dots n} \, dx^2 \wedge \dots \wedge dx^n \tag{3.9}$$

is the element of area of the boundary B^{n-1}. Use of equations (3.7), (1.1) gives immediately

$$\begin{aligned} d\omega &= \delta^{1\dots n}_{j J_0(n-1)} e_{1\dots n} \nabla_k u^j \, dx^k \wedge dx^{J_0(n-1)} \\ &= \nabla_j u^j \, dA_n. \end{aligned} \tag{3.10}$$

By applying Stokes' theorem we thus obtain the integral formula

$$\int_{M^n} \nabla_j u^j \, dA_n = \int_{B^{n-1}} u_1 \, dA_{n-1}. \tag{3.11}$$

For a vector field v on a compact orientable Riemannian manifold M^n with boundary B^{n-1}, replacement of the vector field u^j in equation (3.11) by the vectors $v^i \nabla^j v_i$, $v^i \nabla_i v^j$, $v^j \nabla_i v^i$, and use of equations (1.19), (1.27), (1.28), (1.29), (1.15) yield the integral formulas, respectively,

$$(\tfrac{1}{2} Qv - \Delta v, v) + 2(\nabla v, \nabla v) = \int_{B^{n-1}} v^i \nabla_1 v_i \, dA_{n-1}, \tag{3.12}$$

$$2(T\nabla v, \nabla v) + \int_{M^n} v^j \nabla_i \nabla_j v^i \, dA_n = \int_{B^{n-1}} v^i \nabla_i v_1 \, dA_{n-1}, \tag{3.13}$$

$$(\delta v, \delta v) + \int_{M^n} v^j \nabla_j \nabla_i v^i \, dA_n = \int_{B^{n-1}} v_1 \nabla_i v^i \, dA_{n-1}, \tag{3.14}$$

where for a covariant tensor u_{ij}

$$(Tu)_{ij} = u_{ji}. \tag{3.15}$$

Subtraction of equation (3.14) from equation (3.13) and substitution of equations (1.23), (1.27) give immediately

$$(\tfrac{1}{2} Qv, v) + 2(\mathrm{T}\nabla v, \nabla v) - (\delta v, \delta v) = \int_{B^{n-1}} (v^i \nabla_i v_1 - v_1 \nabla_i v^i) \, dA_{n-1}. \tag{3.16}$$

By subtracting equation (3.16) from equation (3.12) we obtain, in consequence of equation (1.14),

$$\begin{aligned} -(\Delta v, v) + (dv, dv) + (\delta v, \delta v) \\ = \int_{B^{n-1}} [v^i(\nabla_1 v_i - \nabla_i v_1) + v_1 \nabla_i v^i] \, dA_{n-1}. \end{aligned} \tag{3.17}$$

Similarly, addition of equations (3.12), (3.16) and use of equation (2.6) yield

$$\begin{aligned} (Qv - \Delta v, v) + (L_v g, L_v g) - (\delta v, \delta v) \\ = \int_{B^{n-1}} [v^i(\nabla_1 v_i + \nabla_i v_1) - v_1 \nabla_i v^i] \, dA_{n-1}. \end{aligned} \tag{3.18}$$

The integral formulas (3.11), . . . , (3.14), (3.16), (3.17), (3.18) were first obtained by Hsiung **(6, 8)**.

From equations (2.14), (2.15), (2.16) it follows immediately that a Killing vector field v on any Riemannian manifold satisfies

$$\Delta v - Qv = 0. \tag{3.19}$$

For the converse, suppose that on a compact orientable Riemannian manifold M^n with boundary B^{n-1} a vector field v has zero tangential component on the boundary B^{n-1} and satisfies equations (2.15), (3.19). If $\nabla_1 v_1 = 0$ on the boundary B^{n-1} in local boundary geodesic co-ordinates, then by using equations (2.15), (3.19), from equation (3.18) it follows that the vector field v satisfies equation (2.9), and therefore is a Killing vector field on the manifold M^n. Hence we obtain.

THEOREM 3. 1T. *On a compact orientable Riemannian manifold M^n with boundary B^{n-1}, a necessary and sufficient condition for a vector field v with zero tangential component on the boundary B^{n-1} to be a Killing vector field is that it satisfy equations* (2.15), (3.19) *on the manifold M^n and*

$$\nabla_1 v_1 = 0 \qquad \text{on } B^{n-1} \tag{3.20}$$

in local boundary geodesic co-ordinates.

Similarly, from equation (3.18) we have

THEOREM 3. 1N. *On a compact orientable Riemannian manifold M^n with boundary B^{n-1}, a necessary and sufficient condition for a vector field v with zero normal component on the boundary B^{n-1} to be a Killing vector field is that it satisfy equations* (2.15), (3.19) *on the manifold M^n and*

$$\sum_{i=2}^{n} v^i(\nabla_1 v_i + \nabla_i v_1) = 0 \qquad \text{on } B^{n-1} \tag{3.21}$$

in local boundary geodesic co-ordinates.

It should be noted that the letters T and N in Theorems 3.1T and 3.1N are used to denote similar theorems on vector fields with zero tangential and normal components on the boundary B^{n-1} of the manifold M^n respectively; for convenience we shall use this notation throughout this paper.

4. Geodesic vector fields. It is well known that on a Riemannian manifold a necessary and sufficient condition that an infinitesimal transformation generated by a vector field v transform a geodesic into a geodesic and preserve the affine arc length s is that

$$(\nabla_j \nabla_i v^h + R^h{}_{ijk} v^k) \frac{dx^i}{ds}\frac{dx^j}{ds} = 0. \tag{4.1}$$

Accordingly, for a unit vector u at any point P of the manifold M^n, the vector U defined by

$$U^h = (\nabla_j \nabla_i v^h + R^h{}_{ijk} v^k) u^i u^j \tag{4.2}$$

is called the geodesic deviation vector of the vector u with respect to the vector field v. For n mutually orthogonal unit vectors $u_{|a}$ $(a = 1, \ldots, n)$ at the point P of the manifold M^n, we can then form the mean of the geodesic deviation vectors $U_{|a}$ of $u_{|a}$ with respect to v:

$$\frac{1}{n}\sum_{a=1}^{n} U^h_{|a} = \frac{1}{n}\sum_{a=1}^{n} (\nabla_j \nabla_i v^h + R^h{}_{ijk})u^i_{|a} u^j_{|a}. \tag{4.3}$$

By using the relation

$$g^{ij} = \sum_{a=1}^{n} u^i_{|a} u^j_{|a}, \tag{4.4}$$

(cf. **3**, p. 98), equation (4.3) becomes

$$\frac{1}{n}\sum U^h_{|a} = \frac{1}{n}(\nabla^i\nabla_i v^h + R_i{}^h v^i). \tag{4.5}$$

Since the right side of equation (4.5) is independent of the choice of the n mutually orthogonal unit vectors $u_{|a}$, we call the left side of equation (4.5) the mean geodesic deviation vector at the point P with respect to the vector field v. A vector field v, with respect to which the mean geodesic deviation vector vanishes, is called a geodesic vector field. Thus a vector field v satisfying equation (3.19) is geodesic.

The following theorem is an immediate consequence of equations (1.26), (2.14), (2.15).

THEOREM 4.1. *On any Riemannian manifold M^n every Killing vector field is geodesic.*

Substituting equation (3.19) in equation (3.17), for any geodesic vector field v we have

$$\begin{aligned}(Qv, v) &= (dv, dv) + (\delta v, \delta v) \\ &\quad - \int_{B^{n-1}} (v^i\nabla_1 v_i - v^j\nabla_j v_1 + v_1 \nabla_i v^i)\, dA_{n-1}.\end{aligned} \tag{4.6}$$

If the boundary B^{n-1} is convex or totally geodesic, and on the boundary B^{n-1} the geodesic vector field v has zero tangential component and satisfies equation (3.20), then the integrand of the boundary integral in equation (4.6) is non-positive. Since the integrands of the first two integrals on the right side of equation (4.6) are non-negative, we thus have

$$(Qv, v) \geqslant 0. \tag{4.7}$$

If the equality holds in equation (4.7), from equation (4.6) it follows that on the manifold M^n

$$dv = 0, \qquad \delta v = 0, \tag{4.8}$$

that is, v is a harmonic vector field on M^n. A combination of equation (3.19) with the second equation of (4.8) and an application of Theorem 3.1T show that v is also a Killing vector field. From equation (2.14) and the first equation of (4.8) we can thus conclude that $\nabla_i v_j = 0$.

Similar arguments can be applied to the case where the geodesic vector field v has zero normal component on the boundary B^{n-1}. Hence we have

THEOREM 4.2. *On a compact orientable Riemannian manifold M^n with a convex or totally geodesic boundary B^{n-1} let v be a geodesic vector field, which on the boundary B^{n-1} has zero tangential component and satisfies equation* (3.20) (*or has zero normal component and satisfies equations* (3.21) *and* $\nabla_1 v_i = 0$, $i \neq 1$). *Then we have*

$$(Qv, v) \geqslant 0, \tag{4.9}$$

where the equality implies that the geodesic vector field v is a parallel vector field, that is, v has zero covariant derivative over the whole manifold M^n.

An examination of the integrand of (4.9) gives immediately

COROLLARY 4.2.1. *On a compact orientable Riemannian manifold M^n with a convex or totally geodesic boundary B^{n-1}, if the Ricci curvature $R_{ij}v^iv^j$ is negative definite everywhere, then there exists no non-zero geodesic vector field v satisfying the same boundary conditions on B^{n-1} as those in Theorem* 4.2; *if the Ricci curvature is negative semi-definite, then such a geodesic vector field v is a parallel vector field.*

For the case of empty boundary B^{n-1}, Theorems 4.1, 4.2, and Corollary 4.2.1 were obtained by Yano and Nagano **(15)**.

Now suppose that a Riemannian manifold is an Einstein manifold so that

$$R_{ij} = Rg_{ij}/n, \tag{4.10}$$

which implies that

$$R = g^{ij}R_{ij}. \tag{4.11}$$

From Theorem 4.2 we thus obtain

COROLLARY 4.2.2. *On a compact orientable Einstein manifold M^n with a convex or totally geodesic boundary B^{n-1}, if the scalar curvature R is negative definite everywhere, then there exists no non-zero geodesic vector field v satisfying the same boundary conditions on B^{n-1} as those in Theorem* 4.2.

5. Almost-Hermitian structures. On a Riemannian manifold M^n with the metric tensor g_{ij}, if there exists a mixed tensor field $F_i{}^j$ of the second order satisfying

$$F_i{}^j F_j{}^k = -\delta_i^k, \tag{5.1.}$$

then $F_i{}^j$ is said to define an almost-complex structure on the manifold M^n, and M^n is called an almost-complex manifold. If an almost-complex structure tensor $F_i{}^j$ further satisfies

$$g_{ij} F_k{}^i F_h{}^j = g_{kh}, \tag{5.2}$$

then $F_i^{\ j}$ is said to define an almost-Hermitian structure on the manifold M^n, and M^n is called an almost-Hermitian manifold. From equations (5.1), (5.2) it follows that the tensor F_{ij} defined by

$$F_{ij} = g_{jr} F_i^{\ r} \tag{5.3}$$

is skew-symmetric, since $F_{ij} = g_{hk} F_j^{\ h} F_r^{\ k} F_i^{\ r} = -F_{ji}$. By using multiplication of determinants, from equation (5.1) we readily see that a necessary condition for the existence of an almost-complex structure on a Riemannian manifold M^n is that the dimension n of the manifold M^n should be even. It should also be remarked that an almost-complex manifold is always orientable, and the orientation depends only on the tensor $F_i^{\ j}$.

On a Riemannian manifold M^n an almost-Hermitian structure $F_i^{\ j}$ with vanishing Nijenhuis tensor $N_{hi}^{\ \ j}$ defined by

$$N_{hi}^{\ \ j} = F_h^{\ r}(\nabla_r F_i^{\ j} - \nabla_i F_r^{\ j}) - F_i^{\ r}(\nabla_r F_h^{\ j} - \nabla_h F_r^{\ j}) \tag{5.4}$$

is called a pseudo-Hermitian structure, and the manifold a pseudo-Hermitian manifold. An almost (pseudo)-Hermitian structure $F_i^{\ j}$ defined on a manifold M^n is called an almost (pseudo)-Kählerian structure if the associated differential form

$$\omega = F_{ij}\, dx^i \wedge dx^j \tag{5.5}$$

is closed, that is,

$$d\omega = 0; \tag{5.6}$$

and the manifold M^n is accordingly called an almost (pseudo)-Kählerian manifold. From equations (5.5), (5.6) it is easily seen that an almost (pseudo)-Kählerian structure $F_i^{\ j}$ satisfies

$$F_{hij} \equiv \nabla_h F_{ij} + \nabla_i F_{jh} + \nabla_j F_{hi} = 0. \tag{5.7}$$

The tensor F_{hij} is obviously skew-symmetric in all indices.

An almost-Hermitian structure $F_i^{\ j}$ (an almost-Hermitian manifold) with vanishing vector F_i defined by

$$F_i = -\nabla_j F_i^{\ j} \tag{5.8}$$

is called an almost-semi-Kählerian structure (an almost-semi-Kählerian manifold). In particular the structure $F_i^{\ j}$ is Kählerian if $\nabla_i F_j^{\ k} = 0$.

Multiplying equation (5.3) by F^{ki} we obtain

$$F_{ij} F^{ki} = -\delta_j^{\ k}. \tag{5.9}$$

Making use of equation (5.8) and covariant differentiation of both sides of equation (5.9), from equation (5.7) it is easily seen that

$$F_{hij} F^{ij} = 2F_h^{\ i} F_i. \tag{5.10}$$

Thus an almost-semi-Kählerian structure satisfies

$$F_{hij}\, F^{ij} = 0. \tag{5.11}$$

Multiplying equation (5.10) by $F_k^{\ h}$ and using equation (5.9) we obtain

$$F_k = -\tfrac{1}{2} F_{hij}\, F^{ij} F_k^{\ h}, \tag{5.12}$$

which shows that an almost-Kählerian structure (an almost-Kählerian manifold) is also an almost-semi-Kählerian structure (an almost-semi-Kählerian manifold).

The remainder of this section is devoted to the establishment of some formulas, which will be needed later, for an almost-Hermitian structure $F_i^{\ j}$.

Covariant differentiation of equation (5.1) gives immediately

$$F_i^{\ j} \nabla_h F_j^{\ k} = -F_j^{\ k} \nabla_h F_i^{\ j}. \tag{5.13}$$

From equation (5.8) it follows that

$$F^j = g^{jk} F_k = \nabla_i F^{ij}. \tag{5.14}$$

Since F^{jr} is skew-symmetric and $\Gamma_j{}^i{}_r$ is symmetric with respect to j and r, we have

$$F^{jr} L_v\, \Gamma_j{}^i{}_r = 0. \tag{5.15}$$

By means of equations (5.14), (5.15), (2.7), (2.4), from equation (2.5) we can easily obtain

$$F_i^{\ h} L_v\, F^i = F_i^{\ h} v^r \nabla_j \nabla_r F^{ji} - F_i^{\ h} \nabla_j F^{jr} \nabla_r v^i - F_i^{\ h} F^{jr} \nabla_j \nabla_r v^i + R_i^{\ h} v^i. \tag{5.16}$$

Similarly, by putting

$$F_{rs}{}^j = g^{jk} F_{rsk} \tag{5.17}$$

and using equations (5.7), (2.4) we find that

$$\begin{aligned} -\tfrac{1}{2} F_{rs}{}^h L_v\, F^{rs} = &-(\nabla_r F_s^{\ h}) v^i \nabla_i F^{rs} - \tfrac{1}{2} (\nabla^h F_{rs}) v^i \nabla_i F^{rs} \\ &+ (\nabla_r F_s^{\ h}) F^{ri} \nabla_i v^s - (\nabla_s F_r^{\ h}) F^{ri} \nabla_i v^s + (\nabla^h F_{rs}) F^{ri} \nabla_i v^s. \end{aligned} \tag{5.18}$$

Suppose that

$$S^{ij} = g^{ir} L_v\, F_r^{\ j}, \tag{5.19}$$

from which it follows that

$$S_{ij} = g_{ih} L_v\, F_j^{\ h}. \tag{5.20}$$

A simple calculation with use of equations (5.19), (5.20), (2.4), (5.1), (5.13) suffices to demonstrate that

$$\begin{aligned} \tfrac{1}{2} S^{ij} S_{ij} = \tfrac{1}{2} (v^r \nabla_r F^{ji}) v^s \nabla_s F_{ji} &- 2 (F_{js} \nabla^s v_i) v^r \nabla_r F^{ji} \\ &- (F^{jr} \nabla_r v^i) F_{si} \nabla_j v^s + \nabla^j v^i \nabla_j v_i. \end{aligned} \tag{5.21}$$

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By putting

$$a_{jk}(v) = F_{rk} L_v F_j{}^r, \tag{5.22}$$

applying the ordinary product rule for differentiation to $\nabla^j(a_{jk} v^k)$, and making use of equations (2.4), (5.1), (5.16), (5.18), (5.21) we can finally arrive at

$$-\nabla^j(a_{jk} v^k) = (\nabla^i\nabla_i v^h + R_i{}^h v^i - F_i{}^h L_v F^i - \tfrac{1}{2}F_{rs}{}^h L_v F^{rs})v_h + \tfrac{1}{2}S^{ij}S_{ij}. \tag{5.23}$$

6. Contravariant analytic vector fields. On an almost-Hermitian manifold M^n a vector field v is called a contravariant almost-analytic vector, or simply a contravariant analytic vector, if it satisfies

$$L_v F_i{}^j = 0. \tag{6.1}$$

By applying the ordinary product rule for differentiation to $\nabla^j(F_r{}^h L_v F_j{}^r)$, making use of equations (2.4), (5.13), (5.16), (5.18), and noticing that

$$g^{jr}(\nabla^h F_{rs})L_v F_j{}^s = \nabla^h F_{rs}(v^i\nabla_i F^{rs} - 2F^{ri}\nabla_i v^s),$$

we can easily see that condition (6.1) implies

$$\nabla^r\nabla_r v^j + R_r{}^j v^r - F_i{}^j L_v F^i - \tfrac{1}{2}F_{rs}{}^j L_v F^{rs} = 0. \tag{6.2}$$

On a pseudo-Kählerian manifold a contravariant almost-analytic vector is a contravariant pseudo-analytic vector (cf. **11**).

Using equation (5.22) to replace u^r in the integral formula (3.11) by $a^r{}_k v^k$ we thus obtain

THEOREM 6.1. *For a vector field v on a compact almost-Hermitian manifold M^n with boundary B^{n-1}, the following integral formula is valid:*

$$\begin{aligned}(S_{ij}, S_{ij}) + \int_{M^n} (\nabla^r\nabla_r v^j + R_r{}^j v^r - F_i{}^j L_v F^i - \tfrac{1}{2}F_{rs}{}^j L_v F^{rs})v_j\, dA_n \\ = -\int_{B^{n-1}} [v^r(\nabla_r F_1{}^j)F_{jk} - F_1{}^i(\nabla_i v^j)F_{jk} - \nabla_1 v_k]v^k dA_{n-1}.\end{aligned} \tag{6.3}$$

Now let us consider a compact almost-Hermitian manifold M^n with boundary B^{n-1} and use local boundary geodesic co-ordinates. The boundary B^{n-1} is called a semi-pseudo boundary if on B^{n-1} the normal component of the structure tensor $F_i{}^j$ is covariant constant, that is, if

$$\nabla_k F_1{}^i = 0 \qquad (k = 2, \ldots, n;\ i = 1, \ldots, n) \text{ on } B^{n-1}. \tag{6.4}$$

Since F_{ij} is skew-symmetric in i and j, we have $F_{11} = 0$, which implies that $F_1{}^1 = 0$ on B^{n-1}. On the other hand, from equation (5.1) it follows that $(\nabla_h F_i{}^j)F_{jk}$ is skew-symmetric in i and k. Thus we have

$$(\nabla_h F_1{}^j)F_{j1} = 0 \qquad \text{on } B^{n-1}. \tag{6.5}$$

If B^{n-1} is concave or totally geodesic and on B^{n-1} a vector field v has zero tangential component and satisfies equation (3.20), then the integrand of the boundary integral in equation (6.3) is non-negative in consequence of equation

(6.5). By using equations (6.1), (6.2) and noticing from equation (5.19) that $S^{ij} = 0$ for all i, j implies that $L_v F_i{}^j = 0$, we therefore obtain

THEOREM 6.2T. *On a compact almost-Hermitian manifold M^n with a concave or totally geodesic boundary B^{n-1} let v be a vector field which on the boundary B^{n-1} has zero tangential component and satisfies condition* (3.20). *Then a necessary and sufficient condition that the vector field be contravariant analytic is that it satisfy*

$$\nabla^r\nabla_r v^j + R_i{}^j v^i - F_i{}^j L_v F^i - \tfrac{1}{2}F_{rs}{}^j L_v F^{rs} = 0. \tag{6.6}$$

Now on a compact almost-Hermitian manifold M^n with boundary B^{n-1} let v be a vector field having zero normal component, instead of zero tangential component, on B^{n-1}. It is easily seen that if the boundary B^{n-1} is semi-pseudo and the vector field v satisfies different boundary conditions on B^{n-1}, by using the same arguments as those in the proof of Theorem 6.2T we can obtain

THEOREM 6.2N. *On a compact almost-Hermitian manifold M^n with a semi-pseudo boundary B^{n-1} let v be a vector field which on the boundary B^{n-1} has zero normal component and satisfies*

$$\begin{aligned} &\nabla_i v_j = 0 \qquad (i = 2, \ldots, n; j = 1, \ldots, n), \\ &\nabla_1 v_k = 0 \qquad (k = 2, \ldots, n). \end{aligned} \tag{6.7}$$

Then a necessary and sufficient condition that the vector field v be contravariant analytic is that it satisfy equation (6.6).

From equations (3.19), (6.6) and Theorems 6.2T, 6.2N follows immediately

THEOREM 6.3. *On an almost-Hermitian manifold M^n let v be a vector field satisfying*

$$F_i{}^j L_v F^i + \tfrac{1}{2}F_{rs}{}^j L_v F^{rs} = 0. \tag{6.8}$$

If the vector field v is contravariant analytic, then it is geodesic. Conversely, on an almost-Hermitian manifold M^n with boundary B^{n-1} a geodesic vector field v satisfying the condition (6.8) *is contravariant analytic, if* (i) *the boundary B^{n-1} is concave or totally geodesic, and on B^{n-1} the vector field v has zero tangential component and satisfies equation* (3.20), *or if* (ii) *the boundary B^{n-1} is semi-pseudo, and on B^{n-1} the vector field v has zero normal component and satisfies the conditions* (6.7).

From the first part of Theorem 6.3 follows immediately

COROLLARY 6.3.1. *Theorem* 4.2 *and Corollaries* 4.2.1, 4.2.2 *are also true for a contravariant analytic vector field v satisfying equation* (6.8) *on an almost-Hermitian manifold M^n with a convex or totally geodesic boundary B^{n-1}.*

By means of equations (5.12), (5.14) it is readily seen that every vector field v on an almost-Kählerian manifold M^n satisfies equation (6.8). Thus from the first part of Theorem 6.3 we also have

COROLLARY 6.3.2. *Theorem* 4.2 *and Corollaries* 4.2.1, 4.2.2 *are also true for a contravariant analytic vector field* v *on an almost-Kählerian manifold* M^n *with a convex or totally geodesic boundary* B^{n-1}.

For the case of empty boundary B^{n-1}, Corollaries 6.3.1, 6.3.2 were obtained by Tachibana **(12)**.

An application of Theorems 3.1T, 3.1N gives the following two theorems.

THEOREM 6.4T. *On a compact almost-Hermitian manifold* M^n *with boundary* B^{n-1} *let* v *be a contravariant analytic vector field which satisfies equation* (6.8) *on the manifold* M^n *and which on the boundary* B^{n-1} *has zero tangential component and satisfies equation* (3.20). *Then a necessary and sufficient condition that* v *be a Killing vector field is that it satisfy equation* (2.15) *on* M^n.

THEOREM 6.4N. *On a compact almost-Hermitian manifold* M^n *with boundary* B^{n-1} *let* v *be a contravariant analytic vector field which satisfies equation* (6.8) *on the manifold* M^n *and which on the boundary* B^{n-1} *has zero normal component and satisfies equations* (6.7). *Then a necessary and sufficient condition that* v *be a Killing vector field is that* v *satisfy equations* (2.15) *and* (3.21) *on* M^n *and* B^{n-1} *respectively*.

Theorems 6.2T, 6.2N, 6.4T, 6.4N were obtained by Tachibana **(12)** for almost-Kählerian manifolds with empty boundary.

From equations (6.3), (2.18) we obtain

THEOREM 6.5. *For any projective Killing vector field* v *satisfying equation* (6.8) *on a compact almost-Hermitian manifold* M^n *with boundary* B^{n-1}, *the following integral formula is valid:*

$$(6.9)\quad (S_{ij}, S_{ij}) - \frac{2}{n+1}(v, d\delta v) = -\int_{B^{n-1}} [v^r(\nabla_r F_1{}^j)F_{jk} - F_1{}^i(\nabla_i v^j)F_{jk} - \nabla_1 v_k]v^k\, dA_{n-1}.$$

If on a compact almost-Hermitian manifold M^n with boundary B^{n-1} a vector field v generates an infinitesimal transformation T_v leaving the boundary B^{n-1} invariant, then from the definition of an infinitesimal transformation, the vector field v must have zero normal component on B^{n-1}. The infinitesimal transformation T_v is called an automorphism of the manifold M^n if v is an infinitesimal motion and preserves the almost-Hermitian structure $F_i{}^j$, that is, $L_v F_i^j = 0$.

Now let v be a projective Killing vector field satisfying equation (6.8) on a compact almost-Hermitian manifold M^n with boundary B^{n-1}. Substituting

$$(6.10)\qquad \langle v, d\delta v\rangle = \langle \delta v, \delta v\rangle + \nabla^j(v_j\, \delta v)$$

in the integral formula (6.9) and applying equation (3.11) to the integral $\int_{M^n} \nabla_j(v^j\nabla_i v^i)\, dA_n$, we obtain

$$(6.11)\qquad (S_{ij}, S_{ij}) - \frac{2}{n+1}(\delta v, \delta v) = -\int_{B^{n-1}}\Big\{[v^i(\nabla_i F_1{}^j)F_{jk} - F_1{}^i(\nabla_i v^j)F_{jk} - \nabla_1 v_k]v^k - \frac{2}{n+1}v_1\,\delta v\Big\}\,dA_{n-1}.$$

Making use of equations (6.11), (6.1), (6.2), (6.5), (5.19) and Theorems 6.4T, 6.4N we thus arrive at the following two theorems.

THEOREM 6.6T *On a compact almost-Hermitian manifold M^n with a totally geodesic boundary B^{n-1} let v be a projective Killing vector field which satisfies equation* (6.8) *on the manifold M^n and which on the boundary B^{n-1} has zero tangential component and satisfies equation* (3.20). *Then the following integral formula holds:*

$$(6.12)\qquad (S_{ij}, S_{ij}) = \frac{2}{n+1}(\delta v, \delta v).$$

In particular, if the vector field v is further contravariant analytic, then it is a Killing vector field.

THEOREM 6.6N. *On a compact almost-Hermitian manifold M^n with a semi-pseudo boundary B^{n-1} let v be a projective Killing vector field which satisfies equation* (6.8) *on the manifold M^n and which on the boundary B^{n-1} has zero normal component and satisfies equations* (6.7). *Then the integral formula* (6.12) *is still valid. In particular, if the vector field v is further contravariant analytic, then it defines an automorphism of the manifold M^n leaving the boundary B^{n-1} invariant.*

For the case of almost-Kählerian manifolds with empty boundary, Theorems 6.6T, 6.6N are due to Tachibana **(12)**; in this case a vector field v always satisfies equation (6.8).

Now let v be a conformal Killing vector field which satisfies equation (6.8). Substituting equations (6.8), (2.23) in the integral formula (6.3), noticing that

$$(6.13)\qquad -v_j\,\nabla^j\nabla_i v^i = \langle\delta v, \delta v\rangle + \nabla^j(v_j\,\delta v),$$

and applying equation (3.11) to the integral $\int_{M^n}\nabla_j(v^j\,\nabla_i v^i)\,dA_n$, we can obtain

THEOREM 6.7. *For any conformal Killing vector field v satisfying equation* (6.8) *on a compact almost-Hermitian manifold M^n with boundary B^{n-1}, the following integral formula is valid:*

$$(6.14)\qquad (S_{ij}, S_{ij}) + \frac{n-2}{n}(\delta v, \delta v) = -\int_{B^{n-1}}\Big\{[v^r(\nabla_r F_1{}^j)F_{jk} - F_1{}^i(\nabla_i v^j)F_{jk} - \nabla_1 v_k]v^k + \frac{n-2}{n}v_1\,\delta v\Big\}\,dA_{n-1}.$$

By means of equations (6.14), (5.19), (6.1), (6.2), (6.5) and Theorems 6.4T, 6.4N, we are led to the following two theorems.

THEOREM 6.8T. *On a compact almost-Hermitian manifold M^n with a concave or totally geodesic boundary B^{n-1} let v be a conformal Killing vector field which satisfies equation* (6.8) *on the manifold M^n and which on the boundary B^{n-1} has zero tangential component and satisfies equation* (3.20). *Then v is a contravariant analytic vector field for $n = 2$, and is a Killing vector field for $n > 2$.*

THEOREM 6.8N. *On a compact almost-Hermitian manifold M^n with a semi-pseudo boundary B^{n-1} let v be a conformal Killing vector field which satisfies equation* (6.8) *on the manifold M^n and which on the boundary B^{n-1} has zero normal component and satisfies equations* (6.7). *Then the vector field v is contravariant analytic for $n = 2$, and defines an automorphism of the manifold M^n leaving the boundary B^{n-1} invariant for $n > 2$.*

Applying Lie differentiation to equation (5.11) we have

$$F_{hij} L_v F^{ij} + F^{ij} L_v F_{hij} = 0, \tag{6.15}$$

from which it follows immediately that on an almost-semi-Kählerian manifold M^n if a vector field v satisfies

$$F_{hij} L_v F^{ij} = 0, \quad \text{or} \quad F^{ij} L_v F_{hij} = 0, \tag{6.16}$$

it also satisfies equation (6.8). On the other hand, a vector field v on an almost-Kählerian manifold always satisfies (6.16). Thus the following two corollaries are an immediate consequence of Theorems 6.8T, 6.8N.

COROLLARY 6.8.1. *Theorems* 6.8T, 6.8N *are still true if the almost-Hermitian manifold M^n is replaced by an almost-semi-Kählerian manifold M^n, and the condition* (6.8) *by* (6.16).

COROLLARY 6.8.2. *Theorems* 6.8T, 6.8N *are still true for a conformal Killing vector field v on an almost-Kählerian manifold M^n with equation* (6.8) *automatically satisfied.*

For the case of empty boundary B^{n-1}, Theorems 6.8T, 6.8N and Corollary 6.8.1 were obtained by Yano **(14)**, and Corollary 6.8.2 by Lichnerowicz **(10)** and Goldberg **(4)** for a Kählerian manifold M^n and by Goldberg **(5)** for an almost-Kählerian manifold M^n.

7. Covariant analytic vector fields. On an almost-Hermitian manifold M^n a vector u is called a covariant almost-analytic vector, or simply a covariant analytic vector, if

$$\nabla_i (F_j{}^r u_r) = u_r \nabla_j F_i{}^r + F_i{}^r \nabla_r u_j. \tag{7.1}$$

In particular, for an almost-Kählerian manifold, condition (7.1) becomes, in consequence of equations (5.7) and $F_{ij} = -F_{ji}$,

$$u^r \nabla_r F_{ij} = -F_i{}^r \nabla_r u_j + F_j{}^r \nabla_i u_r. \tag{7.2}$$

On a pseudo-Kählerian manifold a covariant almost-analytic vector is a covariant pseudo-analytic vector (cf. **11**).

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LEMMA 7.1. *On an almost-semi-Kählerian manifold M^n with boundary B^{n-1} let λ, μ be two scalar functions satisfying*

$$\nabla_i \lambda = F_i^{\,r} \nabla_r \mu. \tag{7.3}$$

If the normal derivative of one of the two functions λ, μ on the boundary B^{n-1} vanishes, that is, if either $\nabla_1 \lambda = 0$ or $\nabla_1 \mu = 0$ on B^{n-1} in terms of local boundary geodesic co-ordinates, then the two functions λ, μ are constant over the whole manifold M^n.

Proof. Multiplying equation (7.3) by $F_j^{\,i}$ and using equation (5.1), we obtain $\nabla_j(-\mu) = F_j^{\,i} \nabla_i \lambda$. So without loss of generality we may assume that $\nabla_1 \lambda = 0$ on the boundary B^{n-1}. From the definition of an almost-semi-Kählerian manifold and equation (7.3) it follows that

$$\nabla^i \nabla_i \lambda = F^{ir}(\nabla_i \nabla_r \mu),$$

the right side of which is zero, since $\nabla_i \nabla_r \mu$ and F^{ir} are, respectively, symmetric and skew-symmetric with respect to i and r. Thus we have

$$\nabla^i \nabla_i(\lambda^2) = 2\nabla^i \lambda \nabla_i \lambda.$$

Integrating the above equation over the manifold M^n and applying the integral formula (3.11) to the left side of the equation, we obtain

$$2(\nabla_i \lambda, \nabla_i \lambda) = \int_{B^{n-1}} \nabla_1(\lambda^2)\, dA_{n-1} = 0,$$

which implies that λ is constant over the whole manifold M^n, and therefore μ is also, because of equations (7.3), (5.1). Hence the lemma is proved.

For an almost-Kählerian manifold M^n with empty boundary B^{n-1}, Lemma 7.1 was obtained by Tachibana **(12)** and is a generalization of Liouville's theorem in the theory of functions of a complex variable.

On an almost-semi-Kählerian manifold M^n let u be a covariant analytic vector, v a contravariant analytic vector, and λ, μ two scalar functions defined by

$$\lambda = F_r^{\,s} u_s v^r, \tag{7.4}$$

$$\mu = u_r v^r. \tag{7.5}$$

Since v is contravariant analytic, by definition we have

$$L_v F_i^{\,j} = v^r \nabla_r F_i^{\,j} - F_i^{\,r} \nabla_r v^j + F_r^{\,j} \nabla_i v^r = 0. \tag{7.6}$$

From equations (7.1), (7.4), (7.5), (7.6) it is easily seen that the two functions λ, μ satisfy equation (7.3). By Lemma 7.1 we thus obtain

THEOREM 7.1. *On a compact almost-semi-Kählerian manifold M^n with boundary B^{n-1} let v be a contravariant analytic vector and u a covariant analytic vector. If the normal derivative of the inner product μ of u and v vanishes on the boundary B^{n-1}, then the inner product μ is constant over the whole manifold M^n.*

Now consider a vector field u on an almost-Hermitian manifold M^n, and define the vector field $\tilde{u}$ by

$$\tilde{u}_i = F_i{}^t u_t, \tag{7.7}$$

from which it follows that

$$\tilde{v}_i = F_i{}^t v_t = -u_i, \tag{7.8}$$

where

$$v_i = \tilde{u}_i. \tag{7.9}$$

If u is covariant analytic, then by equations (7.1), (7.7), (7.8), (7.9) we obtain

$$\nabla_i v_j = \nabla_j v_i + F_i{}^r[\nabla_j \tilde{v}_r - \nabla_r(F_j{}^s v_s)]. \tag{7.10}$$

Multiplication of equation (7.10) by $F_t{}^i$ and use of equations (5.1), (7.8), (7.9) give immediately equation (7.1) for $\tilde{u}$. Thus we arrive at

THEOREM 7.2. *On an almost-Hermitian manifold M^n if a vector field u is covariant analytic, then the vector field $\tilde{u}$ is also.*

For almost-Kählerian manifolds with empty boundary, Theorems 7.1, 7.2 were obtained by Tachibana **(12)**.

Now let us consider an almost-Kählerian manifold M^n with boundary B^{n-1}, and use the structure tensor $F_i{}^j$ to define the tensor

$$R^*_{jk} \equiv \tfrac{1}{2}F^{st}R_{jrts}\,F_k{}^r. \tag{7.11}$$

It should be noted that on a pseudo-Kählerian manifold $R^*_{jk} = R_{jk}$. By using the relation $F^{ir} = -F^{ri}$ and the Bianchi identity

$$R_{hrji} + R_{hjir} + R_{hirj} = 0,$$

we have

$$R_{hrji}\,F^{ir} = \tfrac{1}{2}F^{ir}(R_{hrji} - R_{hijr}) = \tfrac{1}{2}F^{ri}R_{hjir}. \tag{7.12}$$

Multiplying by g^{ik} the Ricci identity for the tensor F_{ih},

$$[\nabla_k, \nabla_j]F_{ih} = -F_{rh}\,R^r{}_{ijk} - F_{ir}\,R^r{}_{hjk},$$

and making use of equations (7.12) and $\nabla_i F^{ij} = 0$, we are led to

$$\nabla^r\nabla_j\,F_{rh} = \tfrac{1}{2}F^{st}R_{hjts} + R_j{}^rF_{rh}. \tag{7.13}$$

Application of the operator $\nabla^j = g^{jr}\nabla_r$ to the equation $F_{jir} = 0$ and use of equation (7.13) yield

$$\nabla^r\nabla_r\,F_{ji} = F^{st}R_{ijts} + R_j{}^rF_{ri} - R_i{}^rF_{rj}. \tag{7.14}$$

On the other hand, by means of the Ricci identity (1.24) for a vector field v and the relation $F_{ij} = -F_{ji}$, we obtain

$$F^{st}\nabla_s\,\nabla_t\,v_i = -\tfrac{1}{2}F^{st}R_{rits}\,v^r. \tag{7.15}$$

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From equations (7.11), (7.13), (7.15) it follows that

$$F_k{}^i v^r \nabla^j \nabla_r F_{ji} = -v^r \overset{*}{R}_{rk} + v^r R_{rk}, \tag{7.16}$$

$$F_k{}^i F^{jr} \nabla_j \nabla_r v_i = -v^r \overset{*}{R}_{rk}. \tag{7.17}$$

Let the vector $\tilde{v}$ be defined by equation (7.8) from the vector v. Then

$$\tilde{v}^i = g^{it} \tilde{v}_t = -F_t{}^i v^t. \tag{7.18}$$

By computing $\nabla^r \nabla_r \tilde{v}_i$ directly from equation (7.8) and making use of equations (7.14), (5.2) we are readily led to

$$(\nabla^r \nabla_r \tilde{v}_i - R_{ri} \tilde{v}^r) \tilde{v}^i = (\nabla^r \nabla_r v_i + R_{ri} v^r) v^i - 2\overset{*}{R}_{ri} v^r v^i + 2(\nabla^r v^t)(\nabla_r F_{ti}) F_j{}^i v^j. \tag{7.19}$$

Now for a covariant analytic vector u, by applying the operator $F_k{}^i \nabla^j$ to equation (7.2) and using $\nabla_j F^{jr} = 0$, we obtain, in consequence of equations (7.16), (7.17),

$$T_i(u) = 0, \tag{7.20}$$

where

$$T_i(u) \equiv \nabla^r \nabla_r u_i + R_{ri} u^r - 2\overset{*}{R}_{ri} u^r + \nabla^j u^r (\nabla_j F_{rs} + \nabla_r F_{js}) F_i{}^s. \tag{7.21}$$

Furthermore, for any vector field v we can define the tensor

$$\begin{aligned} b_{jk}(v) &\equiv (v^r \nabla_r F_j{}^s + F_r{}^s \nabla_j v^r + F_j{}^r \nabla_r v^s) F_{sk} \\ &= v^r (\nabla_r F_j{}^s) F_{sk} + F_j{}^r (\nabla_r v^s) F_{sk} - \nabla_j v_k. \end{aligned} \tag{7.22}$$

Making use of equation (5.7), from equations (5.22), (7.22) it is easily seen that $a_{jk}(v) = 0$ (that $b_{jk}(v) = 0$) is equivalent to the fact that the vector field v is contravariant analytic (is covariant analytic). Noticing the similarity between b_{jk} and a_{jk} defined by equation (5.22), by calculations analogous to those in §5 we can show that

$$\tfrac{1}{2} b^2(v) = \tfrac{1}{2} b_{jk} b^{jk} = -v^s v^r \nabla_s F^{jt} \nabla_j F_{tr} + F^{rj} F^{sk} \nabla_r v_s \nabla_j v_k + \nabla_j v_k \nabla^j v^k, \tag{7.23}$$

$$\begin{aligned} \nabla^j (b_{jk} v^k) = &- (\nabla^r \nabla_r v_k + v^r R_{rk}) v^k + 2\overset{*}{R}_{rk} v^r v^k \\ &+ v^r v^k \nabla_r F_j{}^s \nabla^j F_{sk} - 2(\nabla^j v^r) F_j{}^s (\nabla_r F_{sk}) v^k \\ &- F^{rj} F^{sk} \nabla_r v_s \nabla_j v_k - \nabla_j v_k \nabla^j v^k. \end{aligned} \tag{7.24}$$

Addition of equations (7.23), (7.24) and the use of equation (5.13) yield

$$\nabla^j (b_{jk} v^k) + \tfrac{1}{2} b^2(v) = -(\nabla^r \nabla_r v^i + v^r R_{ri}) v^i + 2\overset{*}{R}_{ri} v^r v^i - 2\nabla^j v^r (\nabla_r F_{js}) F_k{}^s v^k. \tag{7.25}$$

Equations (7.25) and (7.19) imply, by subtraction,

$$\tfrac{1}{2}[(\nabla^r \nabla_r \tilde{v}_i - \tilde{v}^r R_{ri}) \tilde{v}^i - \nabla^j (b_{jk} v^k)] = v^i T_i(v) + \tfrac{1}{4} b^2(v). \tag{7.26}$$

Integrating equation (7.26) over the manifold M^n, applying the integral formulas (3.11), (3.17) for the vector field $\tilde{v}$, and using equation (7.21), we obtain

THEOREM 7.3. *For any vector field v on a compact almost-Kählerian manifold M^n with boundary B^{n-1}, the following integral formula is valid:*

$$\int_{M^n} [v^i T_i(v) + \tfrac{1}{4}b^2(v) + \tfrac{1}{2}S(\tilde{v})]\, dA_n = \tfrac{1}{2}\int_{B^{n-1}} (\tilde{v}_i \nabla_1 \tilde{v}^i - \tilde{v}^j \nabla_j \tilde{v}_1 + \tilde{v}_1 \nabla_i \tilde{v}^i - b_{1k} v^k)\, dA_{n-1}, \tag{7.27}$$

where $\tilde{v}$, b_{jk}, $b^2(v)$, $T_i(v)$ are defined by equations (7.8), (7.22), (7.23), (7.21) *respectively, and*

$$S(\tilde{v}) = \langle d\tilde{v}, d\tilde{v}\rangle + \langle \delta\tilde{v}, \delta\tilde{v}\rangle. \tag{7.28}$$

By means of equations (7.8), (7.18), (7.22), (5.1), (6.5) and $F_i{}^r v_r v^t \nabla_1 F_t{}^i = 0$, which can be derived from equation (5.13), it is easily seen that if the vector field v has zero tangential component on the boundary B^{n-1} and satisfies equations (3.20) and (7.20) on B^{n-1} and M^n respectively, then equation (7.27) is reduced to

$$\int_{M^n} [b^2(v) + 2S(\tilde{v})]\, dA_n + 4\int_{B^{n-1}} v^1 v_1\, b_{rs}\, F_1{}^r F_1{}^s\, dA_{n-1} = 0. \tag{7.29}$$

If the boundary B^{n-1} is convex or totally geodesic, then from equation (7.29) it follows that $b^2(v) = 0$, showing that v is covariant analytic. Since we have already shown that a covariant analytic vector v satisfies equation (7.20), we therefore arrive at

THEOREM 7.4T. *On a compact almost-Kählerian manifold M^n with a convex or totally geodesic boundary B^{n-1} let v be a vector field which on the boundary B^{n-1} has zero tangential component and satisfies equation* (3.20). *Then a necessary and sufficient condition that the vector v be covariant analytic is that it satisfy equation* (7.20) *on the manifold M^n.*

Similarly, we have

THEOREM 7.4N. *On a compact almost-Kählerian manifold M^n with boundary B^{n-1} on which $\nabla_i F_1{}^j = 0$ for $i, j = 1, \ldots, n$, let v be a vector field which on the boundary B^{n-1} has zero normal component and satisfies $\nabla_i v_j = 0$ for all i and j not equal to* 1 *at the same time. Then a necessary and sufficient condition that the vector v be covariant analytic is that it satisfy equation* (7.20) *on the manifold M^n.*

If the boundary B^{n-1} is convex or totally geodesic, from equation (7.29) it follows also that $S(\tilde{v}) = 0$, which with equations (7.28), (4.8) for the vector $\tilde{v}$ implies that the vector $\tilde{v}$ is harmonic. Hence we have

THEOREM 7.5T. *Under the same assumptions as those in Theorem* 7.4T, *if the vector field v is covariant analytic, then the vector field $\tilde{v}$ is harmonic.*

Similarly, we obtain

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THEOREM 7.5N. *Under the same assumptions as those in Theorem* 7.4N, *if the vector field v is covariant analytic, then the vector field $\tilde{v}$ is harmonic.*

For the case of an empty boundary, Theorems 7.3, 7.4T, 7.4N, 7.5T, 7.5N were obtained by Tachibana **(12)**.

8. Almost-Kählerian manifolds. Let M^n be an almost-Kählerian manifold. Then from equation (5.7) we have

$$2\nabla_r F_{ti} = \nabla_r F_{ti} + \nabla_t F_{ri} - \nabla_i F_{rt}. \tag{8.1}$$

Substituting equation (8.1) in equation (7.19) and making use of the integral formula (3.17) for $\tilde{v}$ we obtain

LEMMA 8.1. *On a compact almost-Kählerian manifold M^n with boundary B^{n-1} for any vector v satisfying*

$$\nabla^r v^t (\nabla_r F_{ti} + \nabla_t F_{ri}) F_j{}^i v^j = 0, \tag{8.2}$$

the following integral formula is valid:

$$\int_{M^n} [(\nabla^r \nabla_r v_i + R_{ri} v^r) v^i - (2R^*_{rj} v^r + F_j{}^i \nabla_i F_{rt} \nabla^r v^t) v^j + S(\tilde{v})] \, dA_n = \int_{B^{n-1}} (\tilde{v}^i \nabla_1 \tilde{v}_i - \tilde{v}^j \nabla_j \tilde{v}_1 + \tilde{v}_1 \nabla_i \tilde{v}^i) \, dA_{n-1}, \tag{8.3}$$

where $\tilde{v}$, $S(\tilde{v})$ are defined by equations (7.8), (7.28) *respectively.*

Let v be a contravariant analytic vector field on an almost-Kählerian manifold M^n. Then from equation (7.6) we have

$$v^r \nabla_r F_{ij} = F_i{}^r \nabla_r v_j + F_j{}^r \nabla_i v_r. \tag{8.4}$$

By means of equations (8.4), (7.2), (5.1) we obtain immediately

LEMMA 8.2. *On an almost-Kählerian manifold M^n if a contravariant analytic vector field v is also covariant analytic, then it is a parallel vector field.*

Substitution of equation (5.7) in equation (8.4) and use of equation (7.8) give readily

$$\nabla_j \tilde{v}_i - \nabla_i \tilde{v}_j = -F_j{}^r (\nabla_r v_i + \nabla_i v_r), \tag{8.5}$$

which with equations (5.1), (5.13) implies that

$$(\nabla_j \tilde{v}_i - \nabla_i \tilde{v}_j)(\nabla_k F_r{}^j) F^{ir} = -\nabla_k F^{it} (\nabla_t v_i + \nabla_i v_t) = 0, \tag{8.6}$$

since $\nabla_k F^{it}$ and $\nabla_t v_i + \nabla_i v_t$ are, respectively, skew-symmetric and symmetric in i and t. Applying ∇^i to equation (8.4) and using equations (5.7), (5.8), (5.12), we obtain

$$\nabla^i v^r \nabla_r F_{ij} + v^r \nabla^i \nabla_r F_{ij} - F_i{}^r \nabla^i \nabla_r v_j - \nabla^i F_j{}^r \nabla_i v_r - F_j{}^r \nabla^i \nabla_i v_r = 0. \tag{8.7}$$

Multiplication of equations (8.7), (8.4) by $F_k{}^j$ and use of equations (7.16), (7.17), respectively, yield

(8.8) $$\nabla^r\nabla_r v_k + R_{rk} v^r + \nabla^i v^r(\nabla_i F_{rj} + \nabla_r F_{ij})F_k{}^j = 0,$$

(8.9) $$\nabla_i v_k = F_i{}^r F_k{}^j \nabla_r v_j - F_k{}^j v^r \nabla_r F_{ij}.$$

On the other hand, from equations (5.1), (5.7), (5.13) and $F_{ij} = -F_{ji}$ it follows that

(8.10) $$F_h{}^j F_s{}^r(\nabla^h F^{st})F_t{}^i = -F_h{}^j(\nabla^i F^{rh} + \nabla^r F^{hi}).$$

Interchanging r and j in equation (8.10) and adding the resulting equation to equation (8.10) we obtain, in consequence of equation (5.13),

(8.11) $$\nabla_j v_r(\nabla^j F^{rt} + \nabla^r F^{jt})F_t{}^i = -F_h{}^j F_s{}^r \nabla_j v_r(\nabla^h F^{st} + \nabla^s F^{ht})F_t{}^i.$$

Substituting equation (8.9) for $\nabla_j v_r$ on the right side of equation (8.11) and noticing that $F_h{}^j \nabla_l F_{js}$ and $(\nabla^h F^{st} + \nabla^s F^{ht})F_t{}^i$ are, respectively, skew-symmetric and symmetric in h and s so that their product is zero, for any contravariant analytic vector field v on the manifold M^n we thus have

(8.12) $$\nabla_j v_r(\nabla^j F^{rt} + \nabla^r F^{jt})F_t{}^i = 0.$$

Making use of equations (5.7), (6.2), (6.8), (6.16), (7.21), (8.12), Theorem 7.3, and Lemma 8.2, and applying to equation (7.27) the arguments on the boundary conditions in the proofs of Theorems 7.4T, 7.4N, we can easily arrive at

THEOREM 8.1. *On a compact almost-Kählerian manifold M^n with a convex or totally geodesic boundary B^{n-1} let v be a contravariant analytic vector field which on the boundary B^{n-1} has zero tangential component and satisfies equation* (3.20) (*or on a compact almost-Kählerian manifold M^n with boundary B^{n-1} on which $\nabla_i F_k{}^j = 0$ for $i, j, k = 1, \ldots, n$, let v be a contravariant analytic vector field which on the boundary B^{n-1} has zero normal component and satisfies $\nabla_i v_j = 0$ for all i and j not equal to* 1 *at the same time*). *Then*

$$\int_{M^n} \overset{*}{R}_{ir} v^i v^r \, dA_n \geqslant 0,$$

where the equality implies that v is a parallel vector field.

For the case of empty boundary B^{n-1}, Theorem 8.1 is due to Tachibana **(12)**.

On an almost-Kählerian manifold M^n let us now consider the canonical connection defined by

(8.13) $$\bar{\Gamma}_i{}^h{}_j = \Gamma_i{}^h{}_j + t^h{}_{ij},$$

where

(8.14) $$t^h{}_{ij} = -\tfrac{1}{2}F_r{}^h \nabla_j F_i{}^r.$$

It is easily seen that the tensors g_{ij} and $F_i{}^j$ both are covariantly constant with respect to the connection $\bar{\Gamma}^h_{ij}$, and it should be noted that on the manifold M^n there are many other connections having this property. Let $\bar{R}^i{}_{jkr}$ be the curvature tensor of the manifold M^n with respect to the connection $\bar{\Gamma}_i{}^h{}_j$. Then from equations (1.21), (8.13) we have

$$(8.15)\qquad \bar{R}^h{}_{ijk} = R^h{}_{ijk} + \nabla_k t^h{}_{ij} - \nabla_j t^h{}_{ik} + t^s{}_{ij} t^h{}_{sk} - t^s{}_{ik} t^h{}_{sj}.$$

By means of equations (8.14), (8.15), (5.13), an elementary calculation suffices to demonstrate that

$$(8.16)\qquad \bar{R}^h{}_{ijk} = \tfrac{1}{2}R^h{}_{ijk} - \tfrac{1}{2}R^r{}_{sjk} F_r{}^h F_i{}^s - \tfrac{1}{4}\nabla_k F_r{}^h \nabla_j F_i{}^r + \tfrac{1}{4}\nabla_j F_r{}^h \nabla_k F_i{}^r.$$

Multiplying equation (8.16) by $F_h{}^i$ and using equations (5.1), (5.13) we obtain

$$(8.17)\qquad \bar{R}^h{}_{ijk} F_h{}^i = R^h{}_{ijk} F_h{}^i + \tfrac{1}{2}T_{kj},$$

where

$$(8.18)\qquad T_{kj} = -F_h{}^i \nabla_k F_r{}^h \nabla_j F_i{}^r.$$

On the other hand, use of equations (5.1), (5.7), (5.13) yields immediately

$$\begin{aligned}(8.19)\qquad (\nabla_k F_{rh})F^{hi}F_l{}^j \nabla_j F_i{}^r &= (\nabla_k F_h{}^r)F^{hi} F_l{}^j(\nabla_i F_{rj} + \nabla_r F_{ji})\\ &= (\nabla_k F_h{}^r)F^{hi}(-F_{rj} \nabla_i F_l{}^j - F_{ji} \nabla_r F_l{}^j)\\ &= \nabla_k F^{ji}(\nabla_l F_{ji} + \nabla_j F_{il}) - \nabla_k F_h{}^r \nabla_r F_l{}^h = \nabla_k F^{ji} \nabla_l F_{ji}.\end{aligned}$$

From equations (8.17), (8.18), (8.19), (7.11) it follows that

$$\begin{aligned}(8.20)\qquad \bar{R}^*_{kl} &= \tfrac{1}{2}\bar{R}^h{}_{ijk} F_h{}^i F_l{}^j = R^*_{kl} + \tfrac{1}{4}T_{kj} F_l{}^j\\ &= R^*_{kl} - \tfrac{1}{4}\nabla_k F_{ij} \nabla_l F^{ij}.\end{aligned}$$

For any vector field v on the manifold M^n use of equations (5.7), (7.8) shows immediately that

$$\begin{aligned}(8.21)\qquad -(\nabla_k F_{rh})v^k &= (\nabla_r F_{hk} - \nabla_h F_{rk})v^k\\ &= \nabla_r \tilde{v}_h - \nabla_h \tilde{v}_r + F_{rk} \nabla_h v^k - F_{hk} \nabla_r v^k,\end{aligned}$$

from which with equations (8.18), (5.13) we have

$$(8.22)\qquad T_{kj} v^k = (\nabla_r \tilde{v}_h - \nabla_h \tilde{v}_r)(\nabla_j F_i{}^r)F^{hi} + 2\nabla_j F_{hk} \nabla^h v^k.$$

If v is a contravariant analytic vector field on the manifold M^n, then by means of equations (8.6), (8.20), (8.22) we obtain

$$(8.23)\qquad v^j \bar{R}^*_{jk} = v^j R^*_{jk} + \tfrac{1}{2}F_k{}^l \nabla_l F_{hj} \nabla^h v^j.$$

Thus for a contravariant analytic vector field v on a compact amost-Kählerian manifold M^n with boundary B^{n-1}, from Lemma 8.1 and equations (5.7), (6.2), (6.8), (6.16), (8.12), (8.23), we have

$$\begin{aligned}(8.24)\qquad &\int_{M^n} [S(\tilde{v}) - 2\bar{R}^*_{jk} v^j v^k]\, dA_n\\ &= \int_{B^{n-1}} (\tilde{v}^i \nabla_1 \tilde{v}_i - \tilde{v}^j \nabla_j \tilde{v}_1 + \tilde{v}_1 \nabla_i \tilde{v}^i)\, dA_{n-1}.\end{aligned}$$

If $S(\tilde{v}) = 0$, then from equations (7.28), (4.8) for the vector $\tilde{v}$ it follows that $\tilde{v}$ is harmonic, and therefore v is a Killing vector field due to equations (8.5), (2.14). Applying to equation (8.24) the arguments on the boundary conditions in the proofs of Theorems 7.4T, 7.4N, we thus arrive at

THEOREM 8.2. *Under the same assumptions as those in Theorem* 8.1,

$$\int_{M^n} \bar{R}^*_{jk} v^j v^k \geqslant 0,$$

where the equality implies that v *is a Killing vector field and* $\tilde{v}$ *is harmonic.*

For the case of empty boundary B^{n-1}, Theorem 8.1 is due to Apte **(1)** for a Killing vector field v, and due to Tachibana **(12)** for a contravariant analytic vector field v.

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