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Abstract

We study the Iitaka–Kodaira dimension of nef relative anti-canonical divisors. As a consequence, we prove that given a complex projective variety with klt singularities, if the anti-canonical divisor is nef, then the dimension of a general fibre of the maximal rationally connected fibration is at least the Iitaka–Kodaira dimension of the anti-canonical divisor.

1. Introduction

The positivity of the anti-canonical class of a projective variety is an important notion that enables us to find certain geometric features of the variety. In characteristic zero, Boucksom, Demailly, Păun and Peternell [BDPP13] proved that a projective manifold is uniruled if its canonical class is not pseudo-effective. Kollár, Miyaoka and Mori [KMM92] showed that a Fano manifold is rationally connected (see also [Cam92]). This result was generalised by Hacon and McKernan [HM07] and by Zhang [Zha06] to the case of klt log Fano pairs, that is, projective klt pairs with ample anti-log canonical class. In view of these results, it is natural to ask whether there are other positivity conditions of anti-canonical classes from which we can derive geometric consequences similar to those above.

In this article, we study the Iitaka–Kodaira dimension of nef (relative) anti-canonical divisors and prove the following theorem.

Theorem 1.1. *Let (X, Δ) be a projective klt pair over a field of characteristic zero and $r : X \dashrightarrow W$ the maximal rationally chain-connected fibration. Suppose that $-(K_X + \Delta)$ is nef. Then $\kappa(X, -(K_X + \Delta)) \leqslant \kappa(F, -(K_F + \Delta_F))$. Here, F is a general fibre of r and $K_F + \Delta_F = (K_X + \Delta)|_F$. In particular,*

$$\kappa(X, -(K_X + \Delta)) \leqslant \dim X - \dim W. \tag{1}$$

This theorem can be thought of as a generalisation of the above result of Hacon–McKernan and Zhang. Furthermore, Theorem 1.1 answers a question posed in [HM05] that asks whether inequality (1) holds or not.

To prove Theorem 1.1, we establish an injectivity theorem (Theorem 4.2). A simplified version of the injectivity theorem states the following.

Theorem 1.2. *Let k be an algebraically closed field of characteristic zero (respectively, $p > 0$). Let $f : X \to Y$ be a separable surjective morphism between smooth projective varieties over k*

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such that $f_*\mathcal{O}_X \cong \mathcal{O}_Y$, and let Δ be an effective $\mathbb{Q}$-divisor (respectively, $\mathbb{Z}_{(p)}$-divisor) on X. Take a general fibre F of f and suppose that $(F, \Delta|_F)$ is klt (respectively, strongly F-regular). Set $L = -(K_{X/Y} + \Delta)$. If L is nef, then for each $m \in \mathbb{Z}_{>0}$ the morphism

$$H^0(X, \mathcal{O}_X(\lfloor mL \rfloor)) \to H^0(F, \mathcal{O}_F(\lfloor mL|_F \rfloor))$$

induced by restriction is injective. In particular, $\kappa(X, L) \leqslant \kappa(F, L|_F)$ holds.

This theorem answers another question posed in [Eji16], which asks whether the inequality in the statement of the theorem holds.

Theorem 1.2 is actually proved in the case where f is an almost holomorphic map, that is, a rational map with proper general fibres (Theorem 4.2). Note that the maximal rationally chain-connected fibration of a projective variety is almost holomorphic. Under the assumption of Theorem 1.1, as explained in the proof of Corollary 5.2, we may assume that K_W is $\mathbb{Q}$-linearly trivial. Therefore, the relative canonical class can be identified with the absolute one, and we can apply the general version of Theorem 1.2.

Let us explain briefly the strategy of the proof of Theorem 1.2. The main ingredient of the proof is the so-called weak positivity theorem developed by several people including Fujita, Kawamata and Viehweg [Fuj78, Kaw81, Vie83]. We employ the log version of this theorem established in [Cam04, Eji17, Fuj17, Pat14], which concerns the positivity of the direct image sheaves $f_*\mathcal{O}_X(m(K_{X/Y}+\Delta))$ of log pluricanonical bundles $\mathcal{O}_X(m(K_{X/Y}+\Delta))$. Using this, we first show that given a $\mathbb{Q}$-divisor D on Y, if $-(K_X+\Delta+g^*D)$ is nef, then $-(K_Y+D)$ is pseudo-effective (Theorem 3.1). This can be viewed as a generalisation of Chen and Zhang's theorem [CZ13] that answers the Demailly–Peternell–Schneider question. Next, we prove a variant of the above result, assuming the existence of an effective $\mathbb{Q}$-Cartier divisor $\Gamma \equiv -(K_X + \Delta + g^*D)$ on X (Proposition 4.4). In this step, the pair $(F, \Delta|_F)$ is required to be klt or strongly F-regular, because we replace Δ by $\Delta + \varepsilon\Gamma$ for some small $\varepsilon > 0$. Finally, applying the above variant, we find that the support of Γ cannot contain F, from which the assertion follows.

Notation 1.3. Let k be an algebraically closed field. By a *variety* we mean an integral separated scheme of finite type over k. For a prime p, let $\mathbb{Z}_{(p)}$ denote the localisation of $\mathbb{Z}$ at $(p) = p\mathbb{Z}$. A $\mathbb{Q}$-Weil divisor D on a normal variety is said to be $\mathbb{Z}_{(p)}$-*Weil* (respectively, $\mathbb{Z}_{(p)}$-*Cartier*) if there exists an integer $m \in \mathbb{Z} \setminus p\mathbb{Z}$ such that mD is integral (respectively, Cartier).

2. Preliminaries

In this section, we review the basic terminology of the singularities of pairs and the positivity of coherent sheaves.

2.1 Singularities of pairs

We first recall several singularities that appear in the minimal model programme.

Definition 2.1 (Cf. [KM98, Definition 2.34] and [SS10, Remark 4.2]). Let X be a normal variety over an algebraically closed field, and let Δ be an effective $\mathbb{Q}$-Weil divisor on X such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Let $\pi : \widetilde{X} \to X$ be a birational morphism from a normal variety $\widetilde{X}$. Then we can write

$$K_{\widetilde{X}} = \pi^*(K_X + \Delta) + \sum_E a(E, X, \Delta)E,$$

where E runs through all the distinct prime divisors on $\widetilde{X}$ and the $a(E, X, \Delta)$ are rational numbers. We say that the pair (X, Δ) is *log canonical* (respectively, *klt*) if $a(E, X, \Delta) \geqslant -1$

(respectively, $a(E, X, \Delta) > -1$) for every prime divisor E over X. If $\Delta = 0$, we simply say that X has only log canonical singularities (respectively, log terminal singularities). Assume that $\Delta = 0$. We say that X has only *canonical singularities* (respectively, *terminal singularities*) if $a(E, X, 0) \geqslant 0$ (respectively, $a(E, X, 0) > 0$) for every exceptional prime divisor E over X.

Next, let us discuss singularities of pairs in positive characteristic. We recall two notions of singularities defined in terms of splittings of the Frobenius morphisms.

Definition 2.2. Let X be a normal *affine* variety defined over an algebraically closed field k of characteristic $p > 0$ and let Δ be an effective $\mathbb{Q}$-divisor on X.

(i) ([HR76, p. 121] and [Sch08, Definition 3.1]) We say that (X, Δ) is *sharply F-pure* if there exists an integer $e \in \mathbb{Z}_{>0}$ for which the composition map

$$\mathcal{O}_X \to F_*^e\mathcal{O}_X \hookrightarrow F_*^e\mathcal{O}_X(\lceil (p^e - 1)\Delta \rceil)$$

of the e-times iterated Frobenius map $\mathcal{O}_X \to F_*^e\mathcal{O}_X$ with a natural inclusion $F_*^e\mathcal{O}_X \hookrightarrow F_*^e\mathcal{O}_X(\lceil (p^e - 1)\Delta \rceil)$ splits as an $\mathcal{O}_X$-module homomorphism.

(ii) ([HW02, Definition 2.1] and [HH89, p. 128, Definition]) We say that (X, Δ) is *strongly F-regular* if for every effective divisor D on X, there exists an integer $e \in \mathbb{Z}_{>0}$ such that the composition map

$$\mathcal{O}_X \to F_*^e\mathcal{O}_X \hookrightarrow F_*^e\mathcal{O}_X(\lceil (p^e - 1)\Delta \rceil + D)$$

of the e-times iterated Frobenius map $\mathcal{O}_X \to F_*^e\mathcal{O}_X$ with a natural inclusion $F_*^e\mathcal{O}_X \hookrightarrow F_*^e\mathcal{O}_X(\lceil (p^e - 1)\Delta \rceil + D)$ splits as an $\mathcal{O}_X$-module homomorphism.

For a variety X and an effective $\mathbb{Q}$-divisor Δ on X, we say that (X, Δ) is *sharply F-pure* if there exists an affine open covering $\{U_i\}$ of X such that each $(U_i, \Delta|_{U_i})$ is *sharply F-pure*. The strong F-regularity of (X, Δ) is defined in the same way. When $\Delta = 0$, we simply say that X is F-pure or strongly F-regular.

Strongly F-regular pairs are known to satisfy some properties similar to those of klt pairs. The same is true of the relationship between sharply F-pure and lc pairs. For this reason, our theorems in characteristic zero and $p > 0$ can be proved simultaneously.

2.2 Weak positivity

In this subsection, we recall the notion of weak positivity of coherent sheaves, which was originally introduced by Viehweg [Vie83, Definition 1.2]. Our definition is slightly different from the original.

Definition 2.3. Let Y be a quasi-projective normal variety over a field k, let $\mathcal{G}$ be a coherent sheaf on Y, and let H be an ample Cartier divisor.

(i) We say that $\mathcal{G}$ is *generically globally generated* if the natural morphism $H^0(Y, \mathcal{G}) \otimes_k \mathcal{O}_Y \to \mathcal{G}$ is surjective over the generic point of Y.

(ii) We say that $\mathcal{G}$ is *weakly positive* (or *pseudo-effective*) if for every $\alpha \in \mathbb{Z}_{>0}$ there exists some $\beta \in \mathbb{Z}_{>0}$ such that $(S^{\alpha\beta}\mathcal{G})^{**} \otimes \mathcal{O}_Y(\beta H)$ is generically globally generated. Here $S^{\alpha\beta}(\_\_)$ and $(\_\_)^{**}$ denote the $\alpha\beta$th symmetric product and the double dual, respectively. Note that the weak positivity of $\mathcal{G}$ does not depend on the choice of ample divisor H.

The weak positivity of $\mathcal{G}$ in the sense of the above definition is actually equivalent to pseudo-effectivity in the sense of [BKKM+15, Definition 5.1] when $\mathcal{G}$ is a vector bundle, and it is equivalent to the usual pseudo-effectivity when $\mathcal{G}$ is a line bundle.

We need Lemmas 2.4 and 2.5 below in order to prove Theorem 3.1.

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Lemma 2.4. *Let $f : Y' \to Y$ be a projective surjective morphism between geometrically normal quasi-projective varieties over a field. Let $\mathcal{G}$ be a torsion-free coherent sheaf on Y.*

(1) *[Vie95, Lemma 2.15(1)] Suppose that there are no f-exceptional divisors on Y'. If $\mathcal{G}$ is weakly positive, then so is $f^*\mathcal{G}$.*
(2) *Let E be an effective f-exceptional Weil divisor on $\tilde{Y}$. If $\mathcal{O}_{Y'}(E) \otimes f^*\mathcal{G}$ is weakly positive, then so is $\mathcal{G}$.*

Proof. We first prove (1). Note that if $\mathcal{G}$ is locally free, the assertion is proved in the same way as [Vie95, Lemma 2.15]. Let $V \subseteq Y$ be the maximal open subset such that $\mathcal{G}|_V$ is locally free. Then the sheaf $(f|_{f^{-1}(V)})^*(\mathcal{G}|_V)$ is weakly positive and is isomorphic to $(f^*\mathcal{G})|_{f^{-1}(V)}$. We then see that $f^*\mathcal{G}$ is weakly positive, since $\operatorname{codim}_{Y'}(Y' \backslash f^{-1}(V)) \geqslant 2$ by the assumption. Next, we show (2). We first consider the case where $E = 0$ and $\mathcal{G}$ is locally free. If f is generically finite, then the assertion is proved in the same way as [Vie95, Lemma 2.15, part 2]. Note that the proof in [Vie95] uses a splitting map $f_*\mathcal{O}_{Y'} \to \mathcal{O}_Y$ of f, which does not always exist, but the map can be replaced by a nonzero map $f_*\mathcal{O}_{Y'} \to \mathcal{H}^l$ in our case, where $\mathcal{H}$ is a given ample line bundle and $l \gg 0$ is an integer depending only on $f_*\mathcal{O}_{Y'}$ and $\mathcal{H}$. When f is not generically finite, we need to recall a definition in [BKKM+15]. For a vector bundle $\mathcal{E}$ on a quasi-projective variety X, we define $\mathbf{Bs}(\mathcal{E})$ to be the set of points $x \in X$ such that the stalk $\mathcal{E}_x$ is *not* generated by global sections of $\mathcal{E}$ [BKKM+15, Definition 2.4]. Put $\mathbf{B}_-(\mathcal{E}) := \bigcup_{\alpha \geqslant 1} \bigcap_{\beta \geqslant 1} \mathbf{Bs}((S^{\alpha\beta}\mathcal{E}) \otimes \mathcal{H}^{\beta})$, where $\mathcal{H}$ is an ample line bundle on X. Note that $\mathbf{B}_-(\mathcal{E})$ is independent of the choice of $\mathcal{H}$ and is a union of countably many closed subsets of X. In accordance with [BKKM+15, Definition 5.1], we use the terminology 'pseudo-effective' instead of 'weakly positive'. Since $f^*\mathcal{G}$ is pseudo-effective, $\mathbf{B}_-(f^*\mathcal{G}) \neq Y'$. Extending the base field, we may assume that it is uncountable. Let Z be an intersection of r very general hyperplanes on Y', where $r := \dim Y' - \dim Y$. Then $Z \not\subseteq \mathbf{B}_-(f^*\mathcal{G})$ and $f|_Z : Z \to Y$ is a generically finite surjective morphism. One can easily check that $\mathbf{B}_-((f^*\mathcal{G})|_Z) \neq Z$, so $(f^*\mathcal{G})|_Z \cong (f|_Z)^*\mathcal{G}$ is pseudo-effective, and hence so is $\mathcal{G}$. Finally, we treat the general case. Let V be the maximal open subset of $Y \setminus f(\operatorname{Supp} E)$ such that $\mathcal{G}|_V$ is locally free. Then $E|_{f^{-1}(V)} = 0$, so $(f|_{f^{-1}(V)})^*(\mathcal{G}|_V) \cong (\mathcal{O}_{Y'}(E) \otimes f^*\mathcal{G})|_{f^{-1}(V)}$ is pseudo-effective, and hence so is $\mathcal{G}|_V$. Since $\operatorname{codim}_Y(Y \setminus V) \geqslant 2$, we see that $\mathcal{G}$ is also pseudo-effective. □

The next lemma follows directly from the definition of weak positivity.

Lemma 2.5 [Vie95, Lemma 2.16(c)]. *Let $\tau : \mathcal{F} \to \mathcal{G}$ be a generically surjective morphism between coherent sheaves on a normal quasi-projective variety over a field. If $\mathcal{F}$ is weakly positive, then so is $\mathcal{G}$.*

3. A version of weak positivity theorem

In this section, we give a version of a weak positivity theorem for the direct image sheaves of log pluricanonical bundles. This is a generalisation of [CZ13, Main Theorem] and [Eji16, Theorem 1.3(1)]. In §§ 4 and 5, we give applications of this weak positivity theorem.

Theorem 3.1. *Let $f : X \to Y$ be a separable surjective morphism between normal projective varieties over an algebraically closed field k of characteristic zero (respectively, $p > 0$) such that $f_*\mathcal{O}_X = \mathcal{O}_Y$ and K_Y is $\mathbb{Q}$-Cartier. Let $\Delta = \Delta^+ - \Delta^-$ be a $\mathbb{Q}$-Weil (respectively, $\mathbb{Z}_{(p)}$-Weil) divisor with the decomposition by the effective $\mathbb{Q}$-divisors, and let D be a $\mathbb{Q}$-Cartier divisor on Y. Suppose that (Z, Δ_Z^+) is lc (respectively, sharply F-pure) for a general fibre Z of f, where*

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$K_Z + \Delta_Z^+ = (K_X + \Delta^+)|_Z$, and suppose that $\operatorname{Supp}\Delta^-$ does not dominate Y. Moreover, assume that $-(K_X + \Delta + f^*D)$ is a nef $\mathbb{Q}$-Cartier (respectively, $\mathbb{Z}_{(p)}$-Cartier) divisor. Fix an integer $l > 0$ such that $l(K_X + \Delta)$ and $l(K_Y + D)$ are Cartier and $l\Delta^-$ is integral. Then there exists an effective f-exceptional divisor B on X such that

$$\mathcal{O}_X\big(l(-f^*(K_Y + D) + \Delta^- + B)\big)$$

is a weakly positive sheaf. Furthermore, this B can be replaced by 0 if Y has only canonical singularities.

Proof. We first prove the assertion under the assumption that f is equi-dimensional. Note that in this case, the pull-back of a Weil divisor by f is well-defined, and coincides with the usual one if the divisor is $\mathbb{Q}$-Cartier. For this reason, in this step we need not assume that K_Y is $\mathbb{Q}$-Cartier. Set $\mathcal{F} := \mathcal{O}_X(l(-f^*(K_Y + D) + \Delta^-))$. Fix $0 < \alpha \in \mathbb{Z}$ (respectively, $0 < \alpha \in \mathbb{Z} \setminus p\mathbb{Z}$) and an ample Cartier divisor A on X. It suffices to show that there is some $\beta \in \mathbb{Z}_{>0}$ such that $\mathcal{F}^{[\alpha\beta]} \otimes \mathcal{O}_X(\beta l A)$ is weakly positive, where $\mathcal{F}^{[n]} := (\mathcal{F}^{\otimes n})^{**}$. Indeed, for given $\alpha' \in \mathbb{Z}_{>0}$, if $\mathcal{F}^{[2\alpha'\beta']} \otimes \mathcal{O}_X(\beta' l A)$ is weakly positive for some $\beta' \in \mathbb{Z}_{>0}$, then there is $\gamma \gg 0$ such that

$$\big(\mathcal{F}^{[2\alpha'\beta']} \otimes \mathcal{O}_X(\beta' l A)\big)^{[\alpha'\gamma]} \otimes \mathcal{O}_X(\gamma l A) \cong \mathcal{F}^{[2(\alpha')^2\beta'\gamma]} \otimes \mathcal{O}_X\big((\alpha'\beta'\gamma + \gamma) l A\big)$$

is generically globally generated, and hence so is $\mathcal{F}^{[\alpha'\gamma']} \otimes \mathcal{O}_X(\gamma' l A)$ because of the inequality $\alpha'\beta'\gamma + \gamma \leqslant \gamma'$, where $\gamma' := 2\alpha'\beta'\gamma$.

Since $-(K_X + \Delta + f^*D) + \alpha^{-1}A$ is an ample $\mathbb{Q}$-Cartier (respectively, $\mathbb{Z}_{(p)}$-Cartier) divisor, by [SW13, Corollary 6.10] we can take a $\mathbb{Q}$-Cartier (respectively, $\mathbb{Z}_{(p)}$-Cartier) divisor $\Gamma \geqslant 0$ on X such that $\Gamma \sim_{\mathbb{Q}} -(K_X + \Delta + f^*D) + \alpha^{-1}A$ and $(Z, \Delta_Z^+ + \Gamma|_Z)$ is lc (respectively, sharply F-pure). Note that in characteristic $p > 0$, $(X_{\overline{\eta}}, (\Delta^+ + \Gamma)|_{X_{\overline{\eta}}})$ is also sharply F-pure by [PSZ18, Corollary 3.31], where $X_{\overline{\eta}}$ is the geometric generic fibre of f. Applying [Cam04, Theorem 4.13] or [Fuj17, Theorem 1.1] (respectively, [Eji17, Theorem 5.1]), we see that for any sufficiently divisible $m \in \mathbb{Z}_{>0}$, the restriction of the sheaf

$$\big(f_*\mathcal{O}_X(lm(K_X + \Delta^+ + \Gamma))\big) \otimes \mathcal{O}_Y(-lmK_Y)$$

to an open subset Y_0 with $\operatorname{codim}_Y(Y \setminus Y_0) \geqslant 2$ is weakly positive, which means that the sheaf on Y is already weakly positive. Note that in characteristic $p > 0$, since

$$(K_X + \Delta^+ + \Gamma)|_{X_{\overline{\eta}}} \sim_{\mathbb{Q}} (\Delta^- + f^*D + \alpha^{-1}A)|_{X_{\overline{\eta}}} \sim_{\mathbb{Q}} \alpha^{-1}A|_{X_{\overline{\eta}}},$$

we find that for $(K_X + \Delta^+ + \Gamma)|_{X_{\overline{\eta}}}$ the hypothesis of [Eji17, Theorem 5.1] is satisfied as shown in [Eji17, Example 3.11]. Take an integer $\beta \gg 0$ that is sufficiently divisible. Then we see that $\mathcal{F}^{[\alpha\beta]} \otimes \mathcal{O}_X(\beta l A)$ is weakly positive, by combining Lemmas 2.4 and 2.5 with the following sequence of generically surjective morphisms:

$$\begin{aligned}
&f^*((f_*\mathcal{O}_X(\alpha\beta l(K_X + \Delta^+ + \Gamma))) \otimes \mathcal{O}_Y(-\alpha\beta l K_Y)) && \\
&\quad \to \mathcal{O}_X(\alpha\beta l(K_X + \Delta^+ + \Gamma)) \otimes f^*\mathcal{O}_Y(-\alpha\beta l K_Y) && \text{by } f^*f_*(\underline{\quad}) \to (\underline{\quad}) \\
&\quad \cong \mathcal{O}_X(\alpha\beta l(\Delta^- - f^*D + \alpha^{-1}A)) \otimes f^*\mathcal{O}_Y(-\alpha\beta l K_Y) && \text{by definition of } \Gamma \\
&\quad \to \mathcal{O}_X(\alpha\beta l(\Delta^- - f^*(K_Y + D) + \alpha^{-1}A)) && (\underline{\quad}) \to (\underline{\quad})^{**} \\
&\quad \cong \mathcal{F}^{[\alpha\beta]} \otimes \mathcal{O}_X(\beta l A) && \text{by definition of } \mathcal{F}.
\end{aligned}$$

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Next, we consider the general case. By the flattening theorem [AO00, 3.3, flattening lemma], we have a birational morphism $\sigma : \tilde{Y} \to Y$ from a normal projective variety $\tilde{Y}$ such that $\tilde{f} : \tilde{X} \to \tilde{Y}$ is equi-dimensional, where $\tilde{f}$ is the natural morphism from the normalisation $\tilde{X}$ of the main component of $\tilde{Y} \times_Y X$ to $\tilde{Y}$. We have the following commutative diagram.

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\rho} & X \\ {\scriptstyle \tilde{f}}\downarrow & & \downarrow{\scriptstyle f} \\ \tilde{Y} & \xrightarrow{\sigma} & Y \end{array}$$

Let $E \geqslant 0$ be a σ-exceptional divisor on $\tilde{Y}$ such that $-K_{\tilde{Y}} \leqslant -\sigma^* K_Y + E$. Let $\tilde{\Delta}$ be a $\mathbb{Q}$-Weil divisor on $\tilde{X}$ such that $K_{\tilde{X}} + \tilde{\Delta} = \rho^*(K_X + \Delta)$. Then $-(K_{\tilde{X}} + \tilde{\Delta} + \tilde{f}^*(\sigma^* D))$ is equal to the nef divisor $\rho^*(-(K_X + \Delta + f^* D))$, so it follows from the previous step that $\mathcal{O}_{\tilde{X}}(l(-\tilde{f}^*(K_{\tilde{Y}}+\sigma^* D)+\tilde{\Delta}^-))$ is weakly positive. Here, $\tilde{\Delta}^-$ is defined by the natural decomposition $\tilde{\Delta} = \tilde{\Delta}^+ - \tilde{\Delta}^-$ by the effective $\mathbb{Q}$-divisors. Since $-\tilde{f}^*(K_{\tilde{Y}} + \sigma^* D) \leqslant -\rho^* f^*(K_Y + D) + \tilde{f}^* E$, the sheaf $\mathcal{O}_{\tilde{X}}(l(-\rho^* f^*(K_Y+D)+\tilde{\Delta}^-+\tilde{f}^* E))$ is also weakly positive, and hence so is $\mathcal{O}_X(l(-f^*(K_Y + D) + \Delta^- + \rho_*(\tilde{f}^* E)))$, where $\rho_* \tilde{\Delta}^- = \Delta^-$. Set $B := \rho_*(\tilde{f}^* E)$. Then $f_* B = \sigma_* E = 0$. If Y is canonical, we can put $E = 0$, and then $B = 0$. This completes the proof. □

When $\Delta^- = 0$, Lemma 2.4(2) tells us that $-(K_Y + D)$ is pseudo-effective. In particular, one can recover [CZ13, Main Theorem] and [Eji16, Theorem 1.3(1)].

Remark 3.2. (1) Our proof differs from that in [CZ13], even when X and Y are smooth and both Δ and D are zero. We explain briefly the difference. In the proof of Theorem 3.1, we use the natural morphism

$$f^*((f_* \mathcal{O}_X(\alpha\beta(K_X + \Gamma))) \otimes \mathcal{O}_Y(-\alpha\beta K_Y)) \to \mathcal{O}_X(\beta(-\alpha f^* K_Y + A))$$

to derive the weak positivity of the target from that of the source. The proof in [CZ13] employs another morphism, which is induced by a trace map,

$$\begin{aligned} &S^n((f_* \mathcal{O}_X(\alpha\beta(K_X + \Gamma))) \otimes \omega_Y^{-\alpha\beta}) \otimes S^n((f_* \mathcal{O}_X(\alpha\beta(K_X + \Gamma)))^* \otimes \mathcal{O}_Y(\beta L)) \\ &\quad \to \mathcal{O}_Y(\beta n(-\alpha K_Y + L)). \end{aligned}$$

Here, L is an ample divisor on Y, which is needed to get the weak positivity of the source.

(2) Next, we give a supplement to the proof of [CZ13, Main Theorem]. Suppose for simplicity that X and Y are smooth and D is zero in the notation of [CZ13, Main Theorem]. Then f' and $\pi : X' \to X$ in [CZ13] coincide with f and $\mathrm{id} : X \to X$, respectively. Take $\delta, \epsilon \in \mathbb{Q}_{>0}$ with $\epsilon/\delta \ll 1$. In [CZ13, p. 1855, line 12 in the proof of the Main Theorem], we could not get the global generation of

$$\mathcal{O}_Y(m\delta L) \otimes (f'_* \mathcal{O}_{X'}(\pi^* m\epsilon A))^*$$

for any sufficiently divisible m. However, if we fix some k, we can get the global generation for any m with $m\epsilon < k$. Thanks to this fact, we only need to make a small modification. For example, this can be done as follows. Let A be sufficiently ample. Considering Castelnuovo–Mumford regularity together with Fujita's vanishing theorem, we see that $A + N$ is free for any nef divisor N on X. Fix $n \in \mathbb{Z}_{>0}$. Since $A - nK_X$ is free, there is an $E \in |A - nK_X|$ such that $(X, (1/n)E)$ is lc. Then

$$(f_* \mathcal{O}_X(A)) \otimes \omega_Y^{-n} \cong f_* \mathcal{O}_X(A - nf^* K_Y) \cong f_* \mathcal{O}_X(nK_{X/Y} + E),$$

so we find that this sheaf is weakly positive by [Cam04, Theorem 4.13] or [Fuj17, Theorem 1.1]. Take $m_0 \in \mathbb{Z}_{>0}$ such that $\mathcal{O}_Y(m_0L) \otimes (f_*\mathcal{O}_X(A))^*$ is globally generated. Then a non-trivial trace map

$$(\mathcal{O}_Y(m_0L) \otimes (f_*\mathcal{O}_X(A))^*) \otimes ((f_*\mathcal{O}_X(A)) \otimes \omega_Y^{-n}) \to \mathcal{O}_Y(m_0L) \otimes \omega_Y^{-n}$$

implies that $-K_Y + (m_0/n)L$ is pseudo-effective, and hence so is $-K_Y = \lim_{n\to\infty}(-K_Y + (m_0/n)L)$. Thus we complete the proof.

4. Injectivity for restrictions on fibres

In this section, we discuss the restriction map of global sections of some line bundles (which are basically relative anti-log canonical bundles) under some semi-positivity assumptions. Throughout this section, we use the notation below.

Notation 4.1. Let $f : X \dashrightarrow Y$ be a separable almost holomorphic dominant rational map between normal projective varieties over an algebraically closed field k of characteristic zero (respectively, $p > 0$); that is, f is a separable surjective regular morphism over some non-empty open set of Y. Let Δ be an effective $\mathbb{Q}$-Weil (respectively, $\mathbb{Z}_{(p)}$-Weil) divisor on X. Let Y_0 be the maximal open set that f is regular on, and let $X_0 = f^{-1}(Y_0)$. Suppose that the following conditions hold:

(1) $K_X + \Delta$ is $\mathbb{Q}$-Cartier;
(2) a general fibre F of f is normal and (F, Δ_F) is klt (respectively, strongly F-regular), where $K_F + \Delta_F = (K_X + \Delta)|_F$;
(3) Y has only canonical singularities;
(4) $(f|_{X_0})_*\mathcal{O}_{X_0} = \mathcal{O}_{Y_0}$.

Moreover, let $\pi : W \to X$ be a resolution of the indeterminacy locus of f with a normal variety W and let $g : W \to Y$ be the induced morphism.

We now have the following diagram.

$$\begin{tikzcd} & W \arrow[dl, "\pi"'] \arrow[dr, "g"] & \\ X \arrow[rr, dashed, "f"] & & Y \end{tikzcd}$$

The aim of this section is to prove the following theorem.

Theorem 4.2. *With the notation above, let $Z := f^{-1}(y)$ be a closed fibre of f over a regular point $y \in Y_0$. Suppose further that the following conditions hold:*

(i) *$L := -(\pi^*(K_X + \Delta) - g^*K_Y)$ is nef;*
(ii) *f is flat at every point in Z;*
(iii) *$\operatorname{Supp}\Delta$ does not contain Z;*
(iv) *Z is normal.*

Then the support of any effective $\mathbb{Q}$-Cartier divisor Γ on W with $\Gamma \equiv L$ does not contain $\pi^{-1}(Z)$.

Before proving the theorem, let us consider the case of ruled surfaces.

Example 4.3. Let Y be a smooth projective curve and D a Cartier divisor on Y with $\deg D \leqslant 0$. Let $f : X := \mathbb{P}(\mathcal{O}_Y \oplus \mathcal{O}_Y(D)) \to Y$ be the projective bundle. Set $C_0 := \mathbb{P}(\mathcal{O}_Y) \subset X$. As shown in [Har77, § 5.2], we have $C_0^2 = \deg D$ and $-K_{X/Y} \sim 2C_0 - f^*D$. Suppose now that $-K_{X/Y}$ is nef. Then

$$0 \leqslant -K_{X/Y} \cdot C_0 = 2C_0^2 - f^*D \cdot C_0 = \deg D \leqslant 0,$$

and so $-K_{X/Y} \cdot C_0 = C_0^2 = 0$. Take an effective $\mathbb{Q}$-divisor Γ on X with $\Gamma \equiv -K_{X/Y}$. Since $\Gamma \cdot C_0 = 0$, we conclude that $\operatorname{Supp} \Gamma$ does not contain a fibre of f.

By applying our weak positivity theorem (Theorem 3.1), we obtain the following proposition.

Proposition 4.4. *With Notation 4.1, let D and E be $\mathbb{Q}$-Cartier divisors on Y. Suppose that the following conditions hold:*

(a) $L := -(\pi^*(K_X + \Delta) + g^*D)$ *is nef;*
(b) *there exists a $\mathbb{Q}$-Cartier divisor $\Gamma \geqslant 0$ on W with $\Gamma \equiv L - g^*E$.*

Then there exists some $\varepsilon \in \mathbb{Q}_{>0}$ such that $\mathcal{O}_X(-m\pi_*g^*(K_Y + D + \varepsilon E))$ is weakly positive for any sufficiently divisible $m \in \mathbb{Z}_{>0}$.

Proof. We take effective $\mathbb{Q}$-divisors Δ_W and G on W such that they have no common components and

$$\pi^*(K_X + \Delta) + G = K_W + \Delta_W.$$

Then G is π-exceptional. For any $\varepsilon \in [0, 1) \cap \mathbb{Q}$, we fix the notation as follows:

$$\Delta_W^{(\varepsilon)} := \Delta_W + \varepsilon\Gamma; \quad D^{(\varepsilon)} := D + \varepsilon E; \quad L^{(\varepsilon)} := -(K_W + \Delta_W^{(\varepsilon)} - G + g^*D^{(\varepsilon)}).$$

We then have

$$\begin{aligned} L^{(\varepsilon)} &= -(K_W + \Delta_W + \varepsilon\Gamma - G + g^*(D + \varepsilon E)) \\ &= -(K_W + \Delta_W - G + g^*D) - \varepsilon(\Gamma + g^*E) \\ &\equiv L - \varepsilon L, \end{aligned}$$

so (a) indicates that $L^{(\varepsilon)}$ is nef. Take a general fibre F of f. Note that now F is also a general fibre of g. Since (F, Δ_F) is klt (respectively, strongly F-regular), so is $(F, \Delta_F^{(\varepsilon)})$ for $\varepsilon \ll 1$, where $K_F + \Delta_F^{(\varepsilon)} = (K_W + \Delta_W^{(\varepsilon)})|_F$. Fix such ε. Applying Theorem 3.1 to $g, W, Y, \Delta_W^{(\varepsilon)} - G$ and $D^{(\varepsilon)}$, we get that $\mathcal{O}_W(m(-g^*(K_Y + D^{(\varepsilon)}) + G))$ is weakly positive for any sufficiently divisible $m \in \mathbb{Z}_{>0}$. Thus, the sheaf

$$(\pi_*\mathcal{O}_W(m(-g^*(K_Y + D^{(\varepsilon)}) + G)))^{**} \cong \mathcal{O}_X(-m\pi_*g^*(K_Y + D^{(\varepsilon)}))$$

is also weakly positive, which is our assertion. □

The next corollary follows directly from the proposition.

Corollary 4.5. *With Notation 4.1, suppose that $L := -(\pi^*(K_X + \Delta) - g^*K_Y)$ is nef. Let E be a nonzero effective $\mathbb{Q}$-Cartier divisor on Y. If $L - g^*E$ is numerically equivalent to an effective $\mathbb{Q}$-Cartier divisor on W, then g^*E is π-exceptional.*

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Proof. Fix an ample Cartier divisor A on X. Applying Proposition 4.4 with $D = -K_Y$, we see that $\mathcal{O}_X(-m\pi_* g^* E)$ is weakly positive for some sufficiently divisible $m \in \mathbb{Z}_{>0}$. Set $P := mg^*E \geqslant 0$. We show that $\pi_* P = 0$. Fix $\alpha \in \mathbb{Z}_{>0}$. Put $M := -\alpha P + \pi^* A$. The weak positivity of $\mathcal{O}_X(-\pi_* P)$ implies that $\pi_* M = -\alpha \pi_* P + A \sim_{\mathbb{Q}} \Gamma \geqslant 0$ for a $\mathbb{Q}$-Weil divisor Γ on X. Note that Γ is not necessarily $\mathbb{Q}$-Cartier. Thanks to [dJon96, 4.1, Theorem], we have a regular alteration $\rho : \tilde{W} \to W$ of W, i.e. a generically finite morphism from a smooth projective variety $\tilde{W}$ to W. Set $\tilde{\pi} : \tilde{W} \xrightarrow{\rho} W \xrightarrow{\pi} X$. Let $V \subseteq X$ be the maximal open subset such that $U := \tilde{\pi}^{-1}(V)$ is flat over V. Since $(\rho^* M)|_U \sim_{\mathbb{Q}} (\tilde{\pi}|_U)^*(\Gamma|_V) \geqslant 0$, there is a divisor $B \geqslant 0$ supported on $\tilde{W} \setminus U$ such that $\rho^* M + B$ is $\mathbb{Q}$-linearly equivalent to an effective $\mathbb{Q}$-divisor on $\tilde{W}$. Note that $\tilde{\pi}_* B = 0$. Then

$$M \cdot (\pi^* A)^{\dim X - 1} = \frac{(\rho^* M) \cdot (\tilde{\pi}^* A)^{\dim X - 1}}{\deg \rho} = \frac{(\rho^* M + B) \cdot (\tilde{\pi}^* A)^{\dim X - 1}}{\deg \rho} \geqslant 0,$$

and so $-\alpha P \cdot (\pi^* A)^{\dim X - 1} + (\pi^* A)^{\dim X} \geqslant 0$. This means that $P \cdot (\pi^* A)^{\dim X - 1} = 0$, since α can be any positive integer. Hence $\pi_* P = 0$ as desired. $\square$

Using this corollary, we now prove the main theorem. We start with a brief explanation of the strategy. Suppose for simplicity that $\dim Y = 1$. Then we can regard $y \in Y_0$ as a divisor. Let Γ be a $\mathbb{Q}$-Cartier divisor on X with $0 \leqslant \Gamma \equiv L$. Let $b \geqslant 0$ be the coefficient of $B := g^* y$ in Γ. We then have $L - g^*(ay) \equiv \Gamma - bB \geqslant 0$. The assertion is equivalent to saying that $b = 0$. If $b > 0$, then Corollary 4.5 implies that B is π-exceptional, which is a contradiction.

In the case where $\dim Y \geqslant 2$, we need to consider the blowing-up of Y at y.

Proof of Theorem 4.2. Let $\tilde{W}$ (respectively, $\tilde{X}$ and $\tilde{Y}$) be the blowing-up of W (respectively, of X and Y) along $g^{-1}(y)$ (respectively, along Z and y), let φ_W (respectively, φ_X and φ_Y) be the natural projection, and let B (respectively, $\tilde{Z}$ and E) denote its exceptional locus. We now have the following commutative diagram.

$$\begin{tikzcd}
\tilde{Z} \arrow[d, hook] & B \arrow[l, "\cong"'] \arrow[r] \arrow[d, hook] & E \arrow[d, hook] \\
\tilde{X} \arrow[d, "\varphi_X"'] & \tilde{W} \arrow[l, "\tilde{\pi}"'] \arrow[r, "\tilde{g}"] \arrow[d, "\varphi_W"] & \tilde{Y} \arrow[d, "\varphi_Y"] \\
X & W \arrow[l, "\pi"'] \arrow[r, "g"] & Y
\end{tikzcd}$$

Since π and g are flat at every point in $g^{-1}(y)$, each square in the above diagram is cartesian. Then $B = \tilde{g}^* E$ as divisors and $B \cong g^{-1}(y) \times E \cong Z \times E$. Now we see that $\tilde{W}$ is normal, because $\tilde{g}$ is a flat morphism with normal fibres in a neighbourhood of B. Hence $\tilde{X}$ is also normal. Set $\tilde{\Delta} := {\varphi_X}_*^{-1} \Delta$ and $\tilde{L} := -\tilde{\pi}^*(K_{\tilde{X}} + \tilde{\Delta}) + \tilde{g}^* K_{\tilde{Y}}$. Then $\tilde{L}$ is nef. Indeed, one can easily check that

$$\tilde{\pi}^*(K_{\tilde{X}} + \tilde{\Delta}) - \tilde{g}^* K_{\tilde{Y}} = {\varphi_W}^*(\pi^*(K_X + \Delta) - g^* K_Y),$$

which means that $\tilde{L} = {\varphi_W}^* L$. Let Γ be a $\mathbb{Q}$-Cartier divisor on W with $0 \leqslant \Gamma \equiv L$, and let $b \geqslant 0$ be the coefficient of B in ${\varphi_W}^* \Gamma$. Then $\tilde{L} - \tilde{g}^*(bE) \equiv {\varphi_W}^* \Gamma - bB \geqslant 0$. Our claim is equivalent to saying that $b = 0$. If $b > 0$, then Corollary 4.5 implies that B is $\tilde{\pi}$-exceptional, which is a contradiction. $\square$

We can use Theorem 4.2 to study the section rings of nef relative anti-canonical divisors.

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DEFINITION 4.6. Let D be a $\mathbb{Q}$-Weil divisor on a normal variety. The *section ring of D* is given by

$$R(X, D) := \bigoplus_{m \geqslant 0} H^0(X, \lfloor mD \rfloor).$$

Recall that $H^0(X, \lfloor mD \rfloor)$ is defined as $\{f \in K(X) \mid \mathrm{div}(f) + mD \geqslant 0\}$, so $R(X, D)$ naturally forms a graded k-algebra.

COROLLARY 4.7. *In the same situation as in Theorem 4.2, the ring homomorphism*

$$R(X, L) \to R(Z, L|_Z)$$

induced by restriction is injective.

Proof. Take an $m \in \mathbb{Z}_{>0}$ and an $s \in H^0(X, \lfloor mL \rfloor)$ with $s|_Z = 0$. Then $s^d|_Z = (s|_Z)^d = 0$ for some $d \in \mathbb{Z}_{>0}$ that is sufficiently divisible. Since $s^d \in H^0(X, d\lfloor mL \rfloor)$, Theorem 4.2 implies $s^d = 0$ by passing through the natural injection $H^0(X, d\lfloor mL \rfloor) \to H^0(X, dmL)$, and so $s = 0$. □

5. Main theorems

In this section, we discuss applications of Theorem 4.2. Throughout this section, we use Notation 4.1 and suppose that $L := -\pi^*(K_X + \Delta) + g^*K_Y$ is nef. Take a general fibre F of f. Corollary 4.7 implies that

$$\kappa(X, L) \leqslant \kappa(F, L|_F).$$

Combining this with the inequality

$$\kappa(F, L|_F) \leqslant \dim F = \dim X - \dim Y,$$

we obtain the following result.

COROLLARY 5.1. *In this situation, the inequality*

$$\kappa(X, L) \leqslant \dim X - \dim Y$$

holds. In particular, if L is nef and big, then $\dim Y = 0$.

The next corollary gives an affirmative answer to a question of Hacon and McKernan.

COROLLARY 5.2. *Let (X, Δ) be a projective klt pair over a field in characteristic zero and let $r : X \dashrightarrow W$ be the maximal rationally chain-connected fibration. Suppose that $-(K_X + \Delta)$ is nef. Then $\kappa(X, -(K_X + \Delta)) \leqslant \kappa(F, -(K_F + \Delta_F))$. Here, F is a general fibre of r and $K_F + \Delta_F = (K_X + \Delta)|_F$. In particular,*

$$\kappa(X, -(K_X + \Delta)) \leqslant \dim X - \dim W.$$

Proof. By [HM07, Corollary 1.5], r is the maximal rational connected fibration. First we may assume that W is smooth since r is a rational map. [GHS03, Corollary 1.4] implies that W is not uniruled. By [Zha05, Remark 1], we see that the numerical Kodaira dimension $\kappa_\sigma(K_W)$ is zero. Thus we have a good minimal model of W by [Dru11] and [Nak04, V, 4.9, Corollary]. Hence we may assume that W has only $\mathbb{Q}$-factorial terminal singularities and $K_W \sim_{\mathbb{Q}} 0$. Thus the desired inequality follows from Corollary 5.1. □

6. Questions

Surprisingly, it is proved in [CH19, 1.4, Theorem] that the maximal rationally connected fibration of a projective manifold with nef anti-canonical bundle can be taken as a regular morphism. In view of this result, it seems natural to ask the following question.

Question 6.1. Let (X, Δ) be a projective (even compact Kähler) klt pair over the complex number field such that $-(K_X + \Delta)$ is nef, and let $r : X \dashrightarrow W$ be the maximal rationally chain-connected fibration. Can r be represented by a regular morphism?

We also should ask the following question for the Kähler case of Theorem 1.1.

Question 6.2. Let (X, Δ) be a compact Kähler klt pair and $r : X \dashrightarrow W$ the maximal rationally chain connected fibration. Suppose that $-(K_X + \Delta)$ is nef. Does the inequality

$$\kappa(X, -(K_X + \Delta)) \leqslant \dim X - \dim W$$

hold?

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