A GENERALIZED RING OF QUOTIENTS I

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Introduction. According to a well-known theorem of algebra [3, p.47], an integral domain can be embedded in a field, called its field of quotients. Every freshman is familiar with the simplest form of this theorem concerning the integers and the rational numbers. Many generalisations have been given in which a "ring of quotients" is constructed for a given ring [e.g. 6, 7, 11, 12, 13]. Of course, we cannot expect a ring of quotients to be a field, or even a skew field. As Malcev [9] has shown, there even exist rings without proper divisors of zero which cannot be embedded isomorphically in a skew field. The most recent construction, by Utumi [12], gives a ring of quotients for any ring with zero left annihilator. We show in this paper that this construction can be extended to arbitrary rings, in fact, to arbitrary modules. The method used is more abstract: a fundamental relation between rings, defined by Utumi in terms of their elements, is here replaced by a corresponding relation between modules, defined by means of homomorphisms.

As an illustration, let us see how the rational numbers can be defined by the method of [12], [13] and the present paper.

A rational number may be regarded as a linear operator, or partial endomorphism, of the additive group of the integers. For example, 2/3 is the mapping of the ideal (3), composed of the multiples of 3, onto the ideal (2), which sends 3k onto 2k. It is easily seen that 2/3, as a partial endomorphism, cannot be extended. It is called irreducible [1]. On the other hand, 4/6, which sends 6k onto 4k, can be extended to the irreducible partial endomorphism 2/3. This extension is unique; in fact, it can easily be shown that any partial endomorphism of the additive group of the integers whose domain is a non-zero ideal can be extended in one and only one way to an irreducible partial endomorphism.

How are arithmetic operations performed when we use this definition of the rational numbers?

Consider for example the addition of 1/6 and 3/10. Their domains, (6) and (10), have the intersection (30). Restricting

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the given summands to this domain, we obtain $5/30 + 9/30 = 14/30$, by the usual method of adding homomorphisms. This result may be extended to the irreducible partial homomorphism $7/15$, which is therefore the required sum.

Again, let us consider the multiplication of $35/6$ by $10/7$. We must restrict the domain of $10/7$ so that its image will be contained in the domain (6) of $35/6$, that is, to $\{7k \mid 10k \in (6)\} = (21)$. Then we have $(35/6)(30/21) = 175/21$, by the usual method of multiplying homomorphisms. This result may be extended to the irreducible partial homomorphism $25/3$, which is therefore the required product.

When rationals are defined as linear operators, rather than as sets of pairs of integers [3, p.45], the operations of addition and multiplication need not be defined by a tour de force, but are already determined by the corresponding definitions for partial homomorphisms. This has the advantage that the verification of the associative and distributive laws requires little or no computation.

If R is a ring, and B and A are right R-modules, and if D is any R-submodule of B, then an R-homomorphism ϕ of D into A is called a partial homomorphism from B into A. Its domain, image and kernel are denoted thus: $\mathrm{dom}\,\phi = D$, $\mathrm{im}\,\phi$, $\ker \phi = \{d \in D \mid \phi d = 0\}$. ϕ is called irreducible if it cannot be extended to a partial R-homomorphism from B into A, whose domain properly contains D.

We shall require a few well-known facts about partial homomorphisms.

PROPOSITION 0.1. Every partial homomorphism can be extended to an irreducible one.

Proof. The partial homomophisms from B into A are partially ordered by the inclusion relation between their graphs. It is easily seen that the union of an increasing sequence of such graphs is again the graph of a partial homomorphism. The existence of irreducible partial homomorphisms extending a given one then follows by Zorn's Lemma [2, p.42].

PROPOSITION 0.2. Two partial homomorphisms from B into A possess a common extension if and only if they coincide on the intersection of their domains.

Proof. Let ϕ and ψ be partial homomorphisms from B into A, such that $\phi z = \psi z$ for all $z \in \operatorname{dom}\phi \cap \operatorname{dom}\psi$. Put

$$\chi(x + y) = \phi x + \psi y$$

for $x \in \operatorname{dom}\phi$, $y \in \operatorname{dom}\psi$. To show that this definition makes sense we must verify that equal values of $x + y$ imply equal values of $\phi x + \psi y$. By linearity, this is the same as showing that $x + y = 0$ implies $\phi x + \psi y = 0$. Now if $x + y = 0$, then $y = -x \in \operatorname{dom}\phi \cap \operatorname{dom}\psi$, so that $\psi y = \phi y = -\phi x$. It is easily seen that χ is a homomorphism of $\operatorname{dom}\phi + \operatorname{dom}\psi$ into A, extending both ϕ and ψ.

The converse is obvious.

COROLLARY 0.3. If two partial homomorphisms from B into A coincide on the intersection of their domains, then if one of them is irreducible it extends the other.

PROPOSITION 0.4. If ϕ is an irreducible partial homomorphism from B into A, C a submodule of B such that $\operatorname{dom}\phi \cap C = 0$, then $C = 0$.

Proof. Let ψ be the zero mapping of C into A, then ψ coincides with ϕ on the intersection of their domains (which is 0), hence by 0.3 $C \subseteq \operatorname{dom}\phi$, so that $C = \operatorname{dom}\phi \cap C = 0$.

1. A relation among three modules.

In sections 1 to 4, R denotes an associative ring, and A, B, C ..... right R-modules. Homomorphisms and submodules are understood to be R-homomorphisms and R-submodules.

Let C be a submodule of B. We write $C \leq B\ (A)$ if every homomorphism of C into A can be extended uniquely to an irreducible partial homomorphism from B into A. That is, $C \leq B\ (A)$ if and only if, for any partial homomorphism ϕ from B into A, $C \subseteq \ker\phi$ implies $\operatorname{im}\phi = 0$.

For it is obvious that the first statement implies the second. Assume the second statement and let ψ, χ be two partial homomorphisms from B into A coinciding on C. Then $C \subseteq \ker(\psi - \chi)$ and so $(\phi - \chi)(\operatorname{dom}\psi \cap \operatorname{dom}\chi) = 0$. By 0.1 and 0.2, ψ and χ possess a common irreducible extension. In particular, if ψ and χ are already irreducible, they must be equal.

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PROPOSITION 1.1. (i) $B \leq B\ (A)$.
(ii) if $C \leq B\ (A)$ and $D \subseteq A$, then $C \leq B\ (D)$.
(iii) if $C \leq B\ (A)$ and $C \subseteq D \subseteq B$, then $C \leq D\ (A)$ and $D \leq B\ (A)$.
(iv) if $A \cong A'$, $B \cong B'$ and $C \cong C'$, then $C \leq B\ (A)$ if and only if $C' \leq B'\ (A')$.

These properties are immediate consequences of the definition.

PROPOSITION 1.2. If $C \leq B\ (A)$ and ψ is a homomorphism into B, then $\psi^{-1}C \leq \psi^{-1}B\ (A)$.

Proof. Let ϕ be a partial homomorphism from $\psi^{-1}B$ into A such that $\psi^{-1}C \subseteq \ker\phi$. Then, for $d \in \psi^{-1}B$, $d \in C$ implies $\phi d = 0$. Thus we can define a partial homomorphism ϕ' from B into A with domain $C + \psi \operatorname{dom}\phi$ by $\phi'(c + \psi d) = \phi d$ ($c \in C$, $d \in \operatorname{dom}\phi$). Since $\phi' C = 0$ and $C \leq B\ (A)$, therefore $\operatorname{im}\phi' = 0$, and so $\operatorname{im}\phi = 0$. Thus $\psi^{-1}C \leq \psi^{-1}B\ (A)$, as was to be proved.

PROPOSITION 1.3.

(i) if $C \leq B\ (A)$ and $D \subseteq B$, then $C \cap D \leq D\ (A)$.
(ii) if $C \leq B\ (A)$ and $B \leq E\ (A)$, then $C \leq E\ (A)$.
(iii) if $C \leq B\ (A)$ and $D \leq B\ (A)$, then $C \cap D \leq B\ (A)$.

Proof. (i) is deduced from 1.2 by letting ψ be the injection of D into B, so that $\psi^{-1}C = C \cap D$.

(ii) Let ϕ be a partial homomorphism from E into A such that $C \subseteq \ker\phi$. By 1.1 (iii), $C \leq \operatorname{dom}\phi \cap B\ (A)$, and so $\phi(\operatorname{dom}\phi \cap B) = 0$. But by (i) $\operatorname{dom}\phi \cap B \leq \operatorname{dom}\phi(A)$; hence $\operatorname{im}\phi = 0$.

(iii) By (i), $C \cap D \leq D\ (A)$. Therefore by (ii), $C \cap D \leq B(A)$.

The relation $C \leq B\ (A)$ discussed in this section can also be defined in terms of elements rather than homomorphisms. Let I be the ring obtained from R by formally adjoining the integers. That is to say, if N is the ring of integers, $I = R \dotplus N$ as a direct sum of modules, with multiplication defined by the rule: $(r + n)(r' + n') = (rr' + nr' + n'r) + nn'$ ($r, r' \in R; n, n' \in N$).

PROPOSITION 1.4. $C \leq B\ (A)$ if and only if, for any $a \in A$ and $b \in B$, $a \neq 0$ implies the existence of an $i \in I$ such that

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$bi \in C$ and $ai \neq 0$.

We omit the proof of this result, as we shall make no use of it. While 1.4 could have been used in the proofs of many propositions in this paper, we have preferred to appeal to 1.2 instead. 1.4 generalizes the corresponding relation between rings given by Utumi.

2. Rational extensions. By an extension of a module C is meant a module B together with an isomorphism of C into B. However, there is no harm in assuming that B actually contains C. The isomorphism is then the inclusion mapping of C into B, and need not be specially mentioned.

If $C \subseteq B$ and $C \subseteq B'$, a homomorphism ϕ of B into B' is called a homomorphism over C if it induces the identity mapping of C.

Eckmann and Schopf [5] have called B an essential extension of C if $D \subseteq B$ and $C \cap D = 0$ imply $D = 0$. This is the same as saying that the identity mapping of C is an irreducible partial homomorphism from B into C.

For the second statement implies the first by 0.4. Conversely, let B be an essential extension of C and ϕ an irreducible partial homomorphism from B into C which extends the identity mapping of C. Then $C \cap \ker\phi = 0$, and so $\ker\phi = 0$. If $d \in \mathrm{dom}\,\phi$ then, for some $c \in C$, $\phi d = c = \phi c$, hence $\phi(d - c) = 0$ and therefore $d - c = 0$. It follows that $\mathrm{dom}\,\phi = C$, and so the identity mapping of C is irreducible.

PROPOSITION 2.1. Let A' be an essential extension of A. Then $C \leq B\,(A)$ if and only if $C \leq B\,(A')$.

Proof. Assume $C \leq B\,(A)$ and let ϕ be a partial homomorphism from B into A' such that $C \subseteq \ker\phi$. By 1.1(iii), $C \leq \phi^{-1}A\,(A)$, so that $\mathrm{im}\,\phi \cap A = \phi(\phi^{-1}A) = 0$, hence $\mathrm{im}\,\phi = 0$. Therefore $C \leq B\,(A')$.

The converse holds by 1.1 (ii).

We shall call an extension B of C a rational extension if $C \leq B\,(B)$.

PROPOSITION 2.2. B is a rational extension of C if and only if it is an essential extension and $C \leqq B\ (C)$.

Proof. Let B be a rational extension of C and suppose $D \subseteq B$ and $C \cap D = 0$. The projection π of $C + D$ onto D is a partial endomorphism of B such that $\pi C = 0$. Hence $D = \operatorname{im} \pi = 0$. Therefore B is an essential extension. Moreover, $C \leqq B\ (C)$, by 1.1 (ii).

The converse follows from 2.1.

PROPOSITION 2.3. If $A \leqq B\ (B)$ and $B \leqq C\ (C)$, then $A \leqq C\ (C)$.

Proof. We have $A \leqq B\ (B)$ and $B \leqq C\ (C)$. By 2.1, $A \leqq B\ (C)$. Hence by 1.3 (ii), $A \leqq C\ (C)$.

PROPOSITION 2.4. If A and B are rational extensions of C, then there exists exactly one irreducible partial homomorphism $\sigma_{A,B}$ over C from A into B. Moreover, if also D is a rational extension of C,

(i) $\sigma_{A,A}$ is the identity mapping of A,

(ii) $\sigma_{B,D}\sigma_{A,B}$ can be extended to $\sigma_{A,D}$,

(iii) $\sigma_{A,B}{}^{-1} = \sigma_{B,A}$.

Proof. Since $C \leqq A\ (A)$ we have $C \leqq A\ (C)$ by 2.2, hence $C \leqq A\ (B)$ by 2.2 and 2.1. The identity mapping of C is therefore extendible to a unique irreducible partial homomorphism $\sigma_{A,B}$ from A into B.

Since the identity mapping of A extends the identity mapping of C, we have (i). Since $\sigma_{B,D}\sigma_{A,B}$ is a partial homomorphism from A into D inducing the identity mapping of C, its unique irreducible extension coincides with $\sigma_{A,D}$, that is (ii). It follows from (i) and (ii) that $\sigma_{A,B}\sigma_{B,A}$ and $\sigma_{B,A}\sigma_{A,B}$ can be extended to the identity mappings of B and A respectively. From this it is easy to deduce (iii).

Note in particular that $\sigma_{A,B}$ is an isomorphism in consequence of (iii).

PROPOSITION 2.5. If $A \to B$ means that $\sigma_{A,B}$ is a full homomorphism, then $\to$ is a quasi-ordering of the rational extensions of C. If we identify two rational extensions which are isomorphic over C, $\to$ becomes a partial ordering.

Proof. In view of 2.4 we have

(i) $A \to A$,

(ii) if $A \to B$ and $B \to D$, then $A \to D$,

(iii) $A \to B$ and $B \to A$ if and only if A and B are isomorphic over C.

THEOREM 2.6. Every module C has a maximal rational extension.

Proof. In the partially ordered system of 2.5, consider any ascending sequence of rational extensions of C:

$$B_1 \to B_2 \to B_3 \to \dots\dots\dots\dots$$

Now the homomorphisms $\sigma_{B_i, B_{i+1}}$ are actually isomorphisms by 2.4. Hence, without loss of generality, we may assume

$$B_1 \subseteq B_2 \subseteq B_3 \subseteq \dots\dots\dots$$

Let B be the union of these B_i. Then B is also a rational extension of C. For let ϕ be a partial endomorphism of B such that $C \subseteq \ker \phi$. Given $b \in \operatorname{dom} \phi$, we want to show that $\phi b = 0$. Now $b \in B_i$ and $\phi b \in B_i$ for some i, and ϕ induces the partial endomorphism ϕ_i of B_i with domain $B_i \cap \phi^{-1} B_i$. Since $\phi_i C = \phi C = 0$, therefore $\phi b = \phi_i b = 0$. Thus $C \leq B\,(B)$.

The existence of a maximal rational extension M now follows by Zorn's Lemma [2, p.42].

Here "maximal" means that for any rational extension B of C, $M \to B$ implies $B \to M$. In view of 2.5 we can say that $M \to B$ implies that B and M are isomorphic over C. Hence M is also maximal in the sense that no proper extension of M is a rational extension of C. We shall show later (4.3) that M is unique up to isomorphism over C.

It is known that there exists a maximal essential extension

E of C, unique up to isomorphism over C, which contains a submodule isomorphic over C to each essential extension of C. Moreover E is injective, that is, every partial homomorphism from any module into E can be extended to a full homomorphism. This has been shown when R contains a unity by Eckmann and Schopf [5], in general by Johnson [8].

PROPOSITION 2.7. The intersection of all kernels of endomorphisms of E which contain C is a maximal rational extension of C.

We omit the proof of this proposition as we shall make no use of it. While many of the results in this paper could have been deduced from it, we have preferred to keep the paper self-contained.

One might perhaps think that the maximal essential extension of a module is itself a rational extension, and therefore a maximal rational extension. While this is often the case, Utumi [12, (1.1)] has given an example to show that an essential extension need not be rational. We shall present this example here:

Let F be any field, S the ring of polynomials in the indeterminate x over F modulo x^4. Consider the subring R of S generated by 1, x^2 and x^3. It is easy to verify that, regarded as an R-module, S is an essential extension of R but not a rational extension.

(To be continued)

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