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A COMPACTNESS PRINCIPLE FOR MAXIMISING SMOOTH FUNCTIONS OVER TOROIDAL GEODESICS

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Abstract

Let $f \in C^2(\mathbb{T}^2)$ have mean value 0 and consider

$$\sup_{\gamma \text{ closed geodesic}} \frac{1}{|\gamma|}\left|\int_\gamma f \, d\mathcal{H}^1\right|,$$

where γ ranges over all closed geodesics $\gamma : \mathbb{S}^1 \to \mathbb{T}^2$ and $|\gamma|$ denotes its length. We prove that this supremum is always attained. Moreover, we can bound the length of the geodesic γ attaining the supremum in terms of the smoothness of the function: for all $s \geq 2$,

$$|\gamma|^s \lesssim_s \left(\max_{|\alpha|=s} \|\partial_\alpha f\|_{L^1(\mathbb{T}^2)}\right)\|\nabla f\|_{L^2}\|f\|_{L^2}^{-2}.$$

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1. Introduction and main result

The purpose of this short note is to discuss an interesting phenomenon: let f be a smooth function with mean value 0 on $\mathbb{T}^2$ and suppose we are interested in finding the largest (absolute) average value f can assume on closed geodesics. More precisely, if $\gamma : \mathbb{S}^1 \to \mathbb{T}^2$ is a closed geodesic on $\mathbb{T}^2$, then we are interested in the maximal possible size of

$$\frac{1}{|\gamma|}\left|\int_\gamma f \, d\mathcal{H}^1\right|,$$

where $\mathcal{H}^1$ is the Hausdorff measure or, since everything is smooth, the usual arc-length measure and $|\gamma|$ is the length of the geodesic. Our result states that the maximal value is assumed for a geodesic of finite length and that length can be bounded by the smoothness of the function.

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THEOREM 1.1. *Let $f : \mathbb{T}^2 \to \mathbb{R}$ be at least $s \geq 2$ times differentiable and have mean value 0. Then*

$$\sup_{\gamma \text{ closed geodesic}} \frac{1}{|\gamma|} \left| \int_\gamma f \, d\mathcal{H}^1 \right|$$

is assumed for a closed geodesic $\gamma : \mathbb{S}^1 \to \mathbb{T}^2$ of length no more than

$$|\gamma|^s \lesssim_s \left(\max_{|\alpha|=s} \|\partial_\alpha f\|_{L^1(\mathbb{T}^2)} \right) \|\nabla f\|_{L^2} \|f\|_{L^2}^{-2}.$$

We believe this to be a rather interesting phenomenon. *A priori*, it might seem very implausible. Clearly we can take a closed geodesic γ as long as we like, define f to be, say, 1 on the geodesic and have it assume smaller values everywhere else (while balancing it in such a way that the mean value is 0). The main result states that this cannot be done without either introducing very large derivatives or having the function be so negative as to create shorter geodesics assuming larger extreme values somewhere else. Or, put differently, smoothness of the function f is enough to ensure simplicity of the extremising geodesic. We have no reason to believe that the inequality is sharp but the result is not arbitrarily far away from the truth. If we consider $f(x, y) = \sin(x + \ell y)$ for some $\ell \in \mathbb{N}$, then the extremising geodesic has length $|\gamma| \sim \ell$ while

$$\max_{|\alpha|=s} \|\partial_\alpha f\|_{L^1(\mathbb{T}^2)} \sim \ell^s, \quad \|\nabla f\|_{L^2} \sim \ell \quad \text{and} \quad \|f\|_{L^2}^{-2} \sim 1.$$

For examples of this type, with functions whose Fourier series is compactly supported, we can explicitly control the limit $s \to \infty$ and obtain a sharp result.

COROLLARY 1.2. *Let $N \in \mathbb{N}$ and let $f : \mathbb{T}^2 \to \mathbb{R}$ be of the form*

$$f(x) = \sum_{\|k\| \leq N} \widehat{f}(k) e^{2\pi i \langle k, x \rangle}.$$

Then the supremum

$$\sup_{\gamma \text{ closed geodesic}} \frac{1}{|\gamma|} \left| \int_\gamma f \, d\mathcal{H}^1 \right|$$

is attained by a closed geodesic γ with $|\gamma| \lesssim N$.

The main purpose of this paper is to introduce the phenomenon and ask a simple question.

QUESTION 1.3. *Does such a principle exist on more general compact manifolds? Are there examples of other geometries where such a bound can be established?*

We are not aware of any results in this direction. Variants of our results can be established on $\mathbb{T}^d$ with $d \geq 3$ but the mechanism is the same and will not yield additional insight into whether this compactness phenomenon is true in a general context or on other nontoroidal geometries. A result on very general manifolds may either be false or is likely out of reach since already structural statements about closed geodesics are highly nontrivial (see Berger [1, Section 10.4] or Klingenberg [3]). Possible candidates for examples might be hyperbolic surfaces on which there exists a suitably accessible description of closed geodesics or groups on which Fourier analysis is well understood.

2. Proofs

Proof of Theorem 1.1. The main idea of the proof is rather simple: we show that there exists a relatively short geodesic for which the absolute value of the arising mean value is at least of a certain size. This is done with an averaging argument and is nonconstructive. The second part of the proof shows that all long geodesics are uniformly bounded. Throughout the paper, we use $A \lesssim B$ to denote $A \leq cB$ for some universal constant $c > 0$ and $A \lesssim_s B$ to denote that the implicit constant depends on s. We identify $\mathbb{T}^2 \cong [0,1]^2$ and write

$$f(x) = \sum_{k\in\mathbb{Z}^2} \widehat{f}(k) e^{2\pi i\langle x,k\rangle}.$$

Since we assume that $f \in C^s(\mathbb{T}^2)$ with $s \geq 2$, the Fourier series converges. Moreover, since f has mean value 0, we have $\widehat{f}(0,0) = 0$. Closed geodesics on $\mathbb{T}^2$ can be written as

$$\gamma(t) = (at, bt + c) \quad \text{where } (0,0) \neq (a,b) \in \mathbb{Z}^2 \text{ and } 0 < c < 1.$$

We assume henceforth that $\gcd(a,b) = 1$. This implies that the closed geodesic can be parametrised by $0 \leq t \leq 1$. Moreover, this induces a bijective mapping between closed geodesics and the set of coprime pairs of integers $(a,b) \times [0,1]$. We now argue that it is possible to assume without loss of generality that

$$\sum_{k\in\mathbb{Z}} |\widehat{f}(k,0)|^2 \leq \frac{\|f\|_{L^2}^2}{2}.$$

If this were false, then we consider the function $\tilde{f}(x,y) = f(y,x)$: geodesics over f correspond to geodesics over $\tilde{f}$ and Plancherel's theorem implies that the desired inequality is now satisfied. The integral over a closed geodesic can be written as

$$\begin{aligned}\frac{1}{|\gamma|}\int_\gamma f\, d\mathcal{H}^1 &= \int_0^1 f(\gamma(t))\,dt = \int_0^1 \sum_{k\in\mathbb{Z}^2} \widehat{f}(k) e^{2\pi i\langle\gamma(t),k\rangle} = \int_0^1 \sum_{k\in\mathbb{Z}^2} \widehat{f}(k) e^{2\pi i(k_1 at + k_2(bt+c))}\,dt \\ &= \int_0^1 \sum_{k\in\mathbb{Z}^2} \widehat{f}(k) e^{2\pi i k_2 c} e^{2\pi i t(k_1 a + k_2 b)}\,dt = \sum_{k_1 a + k_2 b = 0} \widehat{f}(k_1,k_2) e^{2\pi i k_2 c}.\end{aligned}$$

We now interpret this as a function in c. For any fixed $(a,b) \in \mathbb{Z}^2$ with coprime a,b and $(a,b) \neq (0,0)$, the only solutions to $k_1 a + k_2 b = 0$ are given by $(k_1,k_2) = (-db, da)$, where $d \in \mathbb{Z}$. We will only consider $(a,b) \neq (0,0)$ for which $a \neq 0$: if $a = 0$, then the representation for the integral over a closed geodesic does not give rise to a Fourier series but collapses to a number instead. This is not surprising considering

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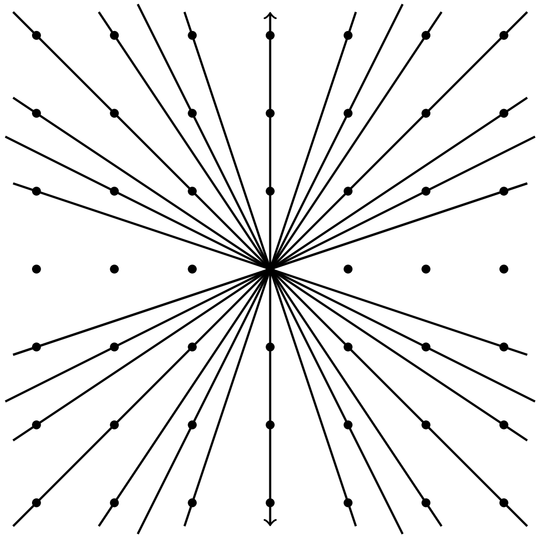

FIGURE 1. Covering lattice points outside the x-axis with lines.

that $\gamma(t) = (at, bt + c)$: if $a = 0$, then the closed geodesic has length 1 and is invariant under c (see Figure 1). An application of Plancherel's theorem yields

$$\sum_{\substack{k\in\mathbb{Z}^2\\ k_1 a+k_2 b=0}} |\widehat{f}(k_1,k_2)|^2 = \Big\| \sum_{k_1 a+k_2 b=0} \widehat{f}(k_1,k_2)e^{2\pi i k_2 c}\Big\|_{L^2[0,1]}^2 \le \max_{0\le c\le 1}\Big|\frac{1}{|\gamma_{a,b,c}|}\int_{\gamma_{a,b,c}} f\, d\mathcal{H}^1\Big|^2.$$

Now let $N \in \mathbb{N}$ be a large number to be fixed later. We will sum this inequality over all geodesics $\gamma(t) = (at, bt + c)$ with $|a|, |b| \le N$ and $a \ne 0$ in such a way that the lines cover all lattice points outside the x-axis in a radius $\sim N$. All these geodesics have length $\sqrt{a^2 + b^2} \le 2N$. There are $\sim N^2$ such lines (no improvement over the trivial bound is possible because asymptotically $6/\pi^2$ of all lattice points have coprime coordinates). Altogether, this implies that

$$\sum_{\|k\|\le N} |\widehat{f}(k)|^2 - \sum_{|k_1|\le N} |\widehat{f}(k_1,0)|^2 \lesssim N^2 \max_{|\gamma|\le 2N}\Big|\frac{1}{|\gamma|}\int_\gamma f\, d\mathcal{H}^1\Big|^2.$$

Using our assumption on the amount of L^2-mass on frequencies $\mathbb{Z}\times\{0\}$, we can derive the weaker inequality

$$\sum_{\|k\|\le N} |\widehat{f}(k)|^2 - \frac{\|f\|_{L^2}^2}{2} \lesssim N^2 \max_{|\gamma|\le 2N}\Big|\frac{1}{|\gamma|}\int_\gamma f\, d\mathcal{H}^1\Big|^2.$$

We now bound the left-hand side in a way that implies a natural choice for N by showing that

$$\sum_{\|k\|\ge \|\nabla f\|_{L^2}\|f\|_{L^2}^{-1}} |\widehat{f}(k)|^2 \le \frac{\|f\|_{L^2}^2}{4}.$$

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If the inequality was false, we could reach a contradiction by using

$$\|\nabla f\|_{L^2}^2 = 4\pi^2 \sum_{k\in\mathbb{Z}^2} |k|^2 |\widehat{f}(k)|^2 \geq 4\pi^2 \sum_{\|k\| \geq \|\nabla f\|_{L^2} \|f\|_{L^2}^{-1}} |k|^2 |\widehat{f}(k)|^2$$

$$\geq 4\pi^2 \frac{\|\nabla f\|_{L^2(\mathbb{T}^2)^2}^2}{\|f\|_{L^2(\mathbb{T}^2)^2}^2} \sum_{\|k\| \geq \|\nabla f\|_{L^2} \|f\|_{L^2}^{-1}} |\widehat{f}(k)|^2 \geq \pi^2 \|\nabla f\|_{L^2(\mathbb{T}^d)}^2.$$

For the choice $N = \|\nabla f\|_{L^2} / \|f\|_{L^2}$, we thus derive

$$\sum_{\|k\| \leq N} |\widehat{f}(k)|^2 - \frac{\|f\|_{L^2}^2}{2} \geq \frac{\|f\|_{L^2}^2}{4}.$$

Altogether, we have shown the existence of a relatively short geodesic satisfying

$$\max_{|\gamma| \leq \|\nabla f\|_{L^2} \|f\|_{L^2}^{-1}} \left| \frac{1}{|\gamma|} \int_\gamma f \, d\mathcal{H}^1 \right| \gtrsim \frac{\|f\|_{L^2}^2}{\|\nabla f\|_{L^2}}.$$

This concludes the first part of the proof.

The second part of the proof shows that sufficiently large geodesics yield integrals that are always smaller than the lower bound we just obtained. The combination of those two facts then yields the desired outcome. Consider the 'long' closed geodesic $\gamma(t) = (at, bt + c)$. Here, 'long' means that $\sqrt{a^2 + b^2}$ is large, as embedded in a one-parameter family γ_c indexed by c (we can assume that $a \neq 0 \neq b$ since the geodesic has length 1 otherwise). We will assume without loss of generality that $a \geq b$: if that is not the case, then we run the subsequent argument on $\tilde{f}(x, y) = f(y, x)$. (The subsequent argument only uses bounds on the Fourier coefficients of f that are radial and has no preferred directions.) We know that, averaged over c, the mean value of the average value over γ_c is 0 and we want to show that it can never be very large. Moreover, since

$$\gamma(t) = (at, bt + c),$$

there is a periodicity in c with period $\lesssim |\gamma|^{-1}$ because the geodesic crosses the x-axis at least $\sim |\gamma|$ times at equally spaced intervals because $a \geq b$. We now use a standard estimate for periodic functions $g : \mathbb{T} \to \mathbb{R}$ with mean value 0: there exists at least one point for which $g(x_0) = 0$. Then

$$g(x)^2 = g(x)^2 - g(x_0)^2 = \int_{x_0}^x \frac{d}{dy} g(y)^2 \, dy = 2 \int_{x_0}^x g(y) g'(y) \, dy \leq 2\|g\|_{L^2(x_0,x)} \|g'\|_{L^2(x_0,x)}.$$

The periodicity in c implies that we can pick the point x_0 at distance $\lesssim |\gamma|^{-1}$ from x and thus

$$2\|g\|_{L^2(x_0,x)} \|g'\|_{L^2(x_0,x)} \lesssim \frac{1}{|\gamma|} \|g\|_{L^2(\mathbb{T})} \|g'\|_{L^2(\mathbb{T})}.$$

Therefore, since x was completely arbitrary,

$$\|g\|_{L^\infty(\mathbb{T})} \lesssim \frac{1}{|\gamma|^{1/2}} \|g\|_{L^2(\mathbb{T})}^{1/2} \|g'\|_{L^2(\mathbb{T})}^{1/2}.$$

This implies that

$$\max_{0\le c\le 1} \frac{1}{|\gamma_{a,b,c}|}\left|\int_{\gamma_{a,b,c}} f\, d\mathcal{H}^1\right| = \left\|\sum_{k_1a+k_2b=0} \widehat{f}(k)e^{2\pi i k_2 c}\right\|_{L^\infty[0,1]}$$

$$\lesssim \frac{1}{|\gamma|^{1/2}}\left(\sum_{k_1a+k_2b=0} |\widehat{f}(k_1,k_2)|^2\right)^{1/4}$$

$$\times\left(\sum_{k_1a+k_2b=0} \|k_2\|^2|\widehat{f}(k_1,k_2)|^2\right)^{1/4}.$$

We bound these quantities by invoking uniform bounds on the Fourier coefficients of C^s functions: for any $s \ge 1$,

$$|\widehat{f}(k)| \lesssim_s \frac{\max_{|\alpha|=s}\|\partial_\alpha f\|_{L^1}}{\|k\|^s}.$$

Using this uniform estimate, we obtain the following bound for the first term:

$$\sum_{k_1a+k_2b=0} |\widehat{f}(k_1,k_2)|^2 = \sum_{\substack{d\in\mathbb{Z}\\ d\neq 0}} |\widehat{f}(-db,da)|^2$$

$$\lesssim_s \left(\max_{|\alpha|=s}\|\partial_\alpha f\|_{L^1}\right)^2 \sum_{\substack{d\in\mathbb{Z}\\ d\neq 0}} \frac{1}{d^{2s}}\frac{1}{(a^2+b^2)^s}$$

$$\lesssim_s \frac{(\max_{|\alpha|=s}\|\partial_\alpha f\|_{L^1})^2}{(a^2+b^2)^s}.$$

The same kind of estimate can be applied to the second term and results in

$$\sum_{k_1a+k_2b=0} |k_2|^2|\widehat{f}(k_1,k_2)|^2 = \sum_{\substack{d\in\mathbb{Z}\\ d\neq 0}} d^2a^2|\widehat{f}(-db,da)|^2$$

$$\lesssim_s \left(\max_{|\alpha|=s}\|\partial_\alpha f\|_{L^1}\right)^2 \sum_{\substack{d\in\mathbb{Z}\\ d\neq 0}} \frac{1}{d^{2(s-1)}}\frac{1}{(a^2+b^2)^{s-1}}$$

$$\lesssim_s \frac{(\max_{|\alpha|=s}\|\partial_\alpha f\|_{L^1})^2}{(a^2+b^2)^{s-1}},$$

where the last step requires $s \ge 2$. For $s \ge 2$, these two bounds imply that

$$\left\|\sum_{k_1a+k_2b=0} \widehat{f}(k)e^{2\pi i k_2 c}\right\|_{L^\infty[0,1]} \lesssim_s \frac{\max_{|\alpha|=s}\|\partial_\alpha f\|_{L^1}}{(a^2+b^2)^{s/2}}.$$

Since a, b have no common divisors, the length of a closed geodesic indexed by $\gamma(t) = (at, bt + c)$ is $\sqrt{a^2+b^2}$. Altogether, this implies the uniform estimate

$$\left|\frac{1}{|\gamma|}\int_\gamma f\, d\mathcal{H}^1\right| \lesssim_s \frac{\max_{|\alpha|=s}\|\partial_\alpha f\|_{L^1}}{|\gamma|^s}.$$

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This allows us to find a critical length beyond which geodesics are bound to be suboptimal by solving for

$$\frac{\max_{|\alpha|=s} \|\partial_\alpha f\|_{L^1}}{|\gamma|^s} \lesssim_s \frac{\|f\|_{L^2}^2}{\|\nabla f\|_{L^2}},$$

which yields the desired result. □

Proof of Corollary 1.2. The main result states that

$$|\gamma|^s \lesssim_s \Big(\max_{|\alpha|=s} \|\partial_\alpha f\|_{L^1(\mathbb{T}^2)}\Big)\|\nabla f\|_{L^2}\|f\|_{L^2}^{-2}.$$

For functions

$$f(x) = \sum_{\|k\|\leq N} \widehat{f}(k)e^{2\pi i\langle k,x\rangle},$$

the quantity $\max_{|\alpha|=s} \|\partial_\alpha f\|_{L^1}$ can be controlled fairly well. Assume that $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_1 + \alpha_2 = s$. Then, using the triangle inequality and the Cauchy–Schwarz inequality,

$$\begin{aligned}\|\partial_\alpha f\|_{L^1} = \Big\| \sum_{\|k\|\leq N} k_1^{\alpha_1} k_2^{\alpha_2} \widehat{f}(k)e^{2\pi i\langle k,x\rangle}\Big\|_{L^1} &\leq \sum_{\|k\|\leq N} |k_1|^{\alpha_1}|k_2|^{\alpha_2}|\widehat{f}(k)| \\ &\leq N^s \sum_{\|k\|\leq N} |\widehat{f}(k)| \lesssim N^{s+1}\|f\|_{L^2}.\end{aligned}$$

It remains to compute the implicit constant in $\lesssim_s$, which is known [2, Theorem 3.2.9] to grow at most like c^s for a universal constant $c > 0$. Letting $s \to \infty$ then implies the result. In fact, a stronger statement is true (and would yield a simple alternative proof). The main algebraic identity shows that all longer closed geodesics are actually orthogonal to f and always yield 0. □

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