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LOCALLY PRIMITIVE GRAPHS AND BIDIRECT PRODUCTS OF GRAPHS

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Abstract

We characterise regular bipartite locally primitive graphs of order $2p^e$, where p is prime. We show that either $p = 2$ (this case is known by previous work), or the graph is a binormal Cayley graph or a normal cover of one of the basic locally primitive graphs; these are described in detail.

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1. Introduction

Studying locally primitive graphs has been a central topic in algebraic graph theory for more than half a century. Giudici *et al.* [6] established a framework for studying locally primitive bipartite graphs, which reduces the study to 'basic' objects in terms of O'Nan–Scott types. In this paper, we will study locally primitive graphs based on bidirect products of graphs, defined below.

DEFINITION 1.1. Let Σ be a connected bipartite graph with biparts U and W. The *bidirect square* $\Sigma^{\times_{\rm bi}2}$ is defined to be the graph with vertex set $(U \times U) \cup (W \times W)$ such that $(u_1, u_2) \sim (w_1, w_2)$ if and only if both $u_1 \sim w_1$ and $u_2 \sim w_2$ in Σ (where $\sim$ denotes adjacency). Recursively, the *bidirect mth power* $\Sigma^{\times_{\rm bi}m}$ is defined as the graph with vertex set $U^m \cup W^m$ such that, if $u_1 \in U^{m-1}$, $w_1 \in W^{m-1}$, $u_2 \in U$ and $w_2 \in W$, then $(u_1, u_2) \sim (w_1, w_2)$ if and only if both $u_1 \sim w_1$ in $\Sigma^{\times_{\rm bi}(m-1)}$ and $u_2 \sim w_2$ in Σ.

We remark that the square $\Sigma^{\times_{\rm bi}2}$ is one of the connected component of the direct product $\Sigma \times \Sigma$, see Section 2 for details. Giudici *et al.* [5] used the bidirect product to study homogeneous factorisations of graphs, calling it the *bipartite product*. We think that 'bidirect product' is more appropriate.

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Let $\Gamma = (V, E)$ be a connected graph with vertex set V and edge set E. Denote by $\Gamma(v)$ the neighbourhood of the vertex v, that is, the set of vertices adjacent to v. For a subgroup X of $\mathrm{Aut}\,\Gamma$, the graph Γ is called *X-locally-primitive* if, for each vertex $v \in V$, the action of X_v, the stabiliser of v, on $\Gamma(v)$ is primitive.

Theorem 1.2. *Let Σ be a connected bipartite graph, and let $\Gamma = \Sigma^{\times_{\mathrm{bi}} m}$. Then Γ is G-locally-primitive for some subgroup G of a wreath product $X \wr S_m$, where $X \leq \mathrm{Aut}\,\Sigma$, if and only if Σ is X-locally-primitive.*

For the rich literature on locally primitive graphs, see the references in [6, 7]. In particular, a theory is established in [6] for the global action analysis of such graphs. The main point of the global action analysis is to take normal quotient and then to analyse 'basic objects', defined below.

Let $\Gamma = (V, E)$ be a connected G-locally-primitive graph. If Γ is bipartite, then let U and W be the biparts. We denote by G^+ the stabiliser of U and W, that is, $G^+ = G_U = G_W$. Let N be a normal subgroup of G. Denote by V_N the set of N-orbits in V. The *normal quotient* Γ_N of Γ induced by N is defined as the graph with vertex set V_N, and two vertices $B, C \in V_N$ are adjacent if there exist $u \in B$ and $v \in C$ that are adjacent in Γ. Then Γ is a *normal multicover* of Γ_N, and, further, if Γ and Γ_N have the same valency, then Γ is a *normal cover* of Γ_N, that is, the induced subgraph on $[B, C]$ for adjacent B and C is a perfect matching.

Let $M \lhd G$ be maximal subject to the condition that M has at least three orbits on V, and, further, M is intransitive on each of the biparts U and W if Γ is bipartite. Then Γ is a normal cover of the quotient Γ_M, and each minimal normal subgroup of G/M is transitive on V_M or one of U_M and W_M if Γ is bipartite. Therefore, with respect to this group G/M, the graph Γ_M has no further nontrivial normal quotient. Such graphs lie in the core of the class of locally primitive graphs, which are basic (or minimal) objects for the class of graphs.

A bipartite locally primitive graph Γ is called *basic* if there exists a subgroup G of $\mathrm{Aut}\,\Gamma$ which acts on Γ locally primitively and any nontrivial normal subgroup of G is transitive on at least on one of the biparts. This leads to the study of basic objects, as described in [6, 7], associated with a theory based on the O'Nan–Scott–Praeger theorem for quasiprimitive permutation groups, proved by Praeger in [14].

Locally primitive graphs of prime power order are characterised in [9, 10, 12]. Here we characterise the family of graphs that are regular, bipartite, locally primitive, and of order $2p^e$, where p is prime. Typical examples include:

(i) the complete bipartite graphs K_{p^e,p^e};
(ii) the graphs $\mathrm{K}_{p^e,p^e} - p^e\mathrm{K}_2$ obtained by deleting a 1-factor from K_{p^e,p^e};
(iii) the incidence graph $\mathrm{D}_2^1(11, 5)$ and the nonincidence graph $\overline{\mathrm{D}}_2^1(11, 5)$ of the 2-(11, 5, 1)-design;
(iv) the incidence graph $\mathrm{PH}(d, q)$ and the nonincidence graph $\overline{\mathrm{PH}}(d, q)$ of the projective geometry $\mathrm{PG}(d - 1, q)$, where $d \geq 3$; and
(v) the standard double cover of the Schläfli graph.

Other basic graphs are bidirect powers of these graphs. We remark that the *Schläfli graph* is the graph on isotropic lines in the U(4, 2) geometry, adjacent when disjoint, which is a strongly regular graph of valency 16, and is a locally primitive Cayley graph of the metacyclic group $\mathbb{Z}_9 \rtimes \mathbb{Z}_3$ (see [1] or [2] for more details).

A bipartite graph $\Gamma = (V, E)$ is called a *Cayley graph* of a group R if $\mathrm{Aut}\,\Gamma$ has a subgroup which is isomorphic to R and regular on V. If further $\mathrm{Aut}\,\Gamma$ has a subgroup R which is regular on V and $R \cap (\mathrm{Aut}\,\Gamma)^+$ is normal in $\mathrm{Aut}\,\Gamma$, then Γ is called a *binormal Cayley graph*. Binormal Cayley graphs have many interesting properties; see [11].

The main result of this paper is to present a classification of regular locally primitive graphs of order $2p^m$. The case where $p = 2$ is characterised in [12].

THEOREM 1.3. *Each regular bipartite locally primitive graph of order $2p^m$, where p is an odd prime number, is either a binormal Cayley graph, or a normal cover of one of the following basic G-locally-primitive graphs:*

(i) $\Gamma = \mathrm{K}_{p^e,p^e}$;

(ii) *the standard double cover of* $\Sigma^{\times m}$, *where* $m \geq 1$ *and* $\Sigma = \mathrm{K}_{p^e}$ *or the Schläfli graph;*

(iii) $\Gamma = \Sigma^{\times_{\mathrm{bi}} m}$, *where* $m \geq 1$ *and*

$$\Sigma = \mathrm{D}_2^1(11, 5), \overline{\mathrm{D}}_2^1(11, 5), \mathrm{PH}(d, q), \ \textit{or}\ \overline{\mathrm{PH}}(d, q).$$

We remark that a binormal Cayley graph is a normal cover of a biquasiprimitive graph of affine type, and basic locally primitive biquasiprimitive graphs of affine type are not yet completely determined, see [9].

For a positive integer s, an s-arc of a graph Γ is a sequence of $s + 1$ vertices $v_0, v_1, \ldots, v_s$ such that v_i is adjacent to v_{i+1} and $v_i \neq v_{i+2}$. A graph Γ is called *locally* (G, s)*-arc-transitive* if, for each vertex v, the stabiliser G_v is transitive on the t-arcs starting from v for all $t \leq s$.

COROLLARY 1.4. *Each regular bipartite locally 2-arc-transitive graph of order $2p^m$, where p is an odd prime number, is a normal cover of one of the following basic locally $(G, 2)$-arc-transitive graphs:*

(i) $\Gamma = \mathrm{K}_{p^e,p^e}, \mathrm{K}_{p^e,p^e} - p^e\mathrm{K}_2, \mathrm{D}_2^1(11, 5), \overline{\mathrm{D}}_2^1(11, 5)$, *or* $\mathrm{PH}(d, q)$;

(ii) $G^+ = \mathbb{Z}_p^e \rtimes G_v$, *where G^+ is a primitive affine group and $G_v \cong G_v^{\Gamma(v)}$ is a primitive permutation group.*

In particular, regular bipartite locally 4-arc-transitive graphs of order $2p^e$, where p is an odd prime number, are cycles or normal covers of $\mathrm{PH}(3, q)$.

It is shown in [10] that 4-arc-transitive graphs of order p^e are all cycles. To extend this result to the case of order $2p^e$ where p is an odd prime number, one needs to characterise the nonbipartite case, and so needs to extend the result of Guralnick [8] to classify almost simple groups which have subgroups of index $2p^e$.

2. Bidirect product

Let Σ_i be a connected bipartite graph with vertex set V_i and biparts U_i and W_i, where $i = 1$ or 2. Recall that the *direct product* $\Sigma_1 \times \Sigma_2$ is the graph with vertex set $V_1 \times V_2$

such that two vertices (v_1, v_2) and (v_1', v_2') are adjacent if and only if $v_i \sim v_i'$ in Σ_i for $i = 1$ and 2.

It is easily shown that $\Sigma_1 \times \Sigma_2$ is disconnected and has exactly two connected components, which are the induced subgraphs of $\Sigma_1 \times \Sigma_2$ whose vertex sets are either $(U_1 \times U_2) \cup (W_1 \times W_2)$ or $(U_1 \times W_2) \cup (U_2 \times W_1)$.

Let $X \leq \mathrm{Aut}\,\Sigma_1$, $Y \leq \mathrm{Aut}\,\Sigma_2$, and $G = X \times Y$. Then G acts on $V = V_1 \times V_2$ in the following way:

$$(v_1, v_2)^{(x,y)} = (v_1^x, v_2^y).$$

For all elements $(x, y) \in G$ and vertices (v_1, v_2) and (v_3, v_4) in $V_1 \times V_2$,

$$\begin{aligned}(v_1, v_2) \sim (v_3, v_4) &\Longleftrightarrow v_1 \sim v_3 \text{ in } \Sigma_1, \text{ and } v_2 \sim v_4 \text{ in } \Sigma_2\\ &\Longleftrightarrow v_1^x \sim v_3^x \text{ in } \Sigma_1, \text{ and } v_2^y \sim v_4^y \text{ in } \Sigma_2\\ &\Longleftrightarrow (v_1, v_2)^{(x,y)} = (v_1^x, v_2^y) \sim (v_3^x, v_4^y) = (v_3, v_4)^{(x,y)}.\end{aligned}$$

Thus, (x, y) is an automorphism of $\Sigma_1 \times \Sigma_2$, and

$$X \times Y \leq \mathrm{Aut}(\Sigma_1 \times \Sigma_2).$$

In particular, if Σ_1 and Σ_2 are both vertex-transitive, then so is $\Sigma_1 \times \Sigma_2$. However, the direct product of two edge-transitive graphs is not necessarily edge-transitive.

Example 2.1. Let $\Sigma_1 = \mathrm{K}_{2,3}$ and $\Sigma_2 = \mathrm{K}_{3,4}$. Then the connected components of $\Sigma_1 \times \Sigma_2$ are $\mathrm{K}_{6,12}$ and $\mathrm{K}_{8,9}$. Because the two components are not isomorphic, $\Sigma_1 \times \Sigma_2$ is not edge-transitive.

Although the two components are not necessarily isomorphic, the following lemma shows that each of them is edge-transitive.

Lemma 2.2. *Assume that Σ_1 is X-edge-transitive, and Σ_2 is Y-edge-transitive. Then the connected components of $\Sigma_1 \times \Sigma_2$ are $(X \times Y)$-edge-transitive.*

Proof. Let Γ be the component of $\Sigma_1 \times \Sigma_2$ with vertex set $(U_1 \times U_2) \cup (W_1 \times W_2)$. Take two edges $\{(u_1, u_2), (w_1, w_2)\}$ and $\{(u_1', u_2'), (w_1', w_2')\}$. Then $\{u_1, w_1\}$ and $\{u_1', w_1'\}$ are two edges of Σ_1, and $\{u_2, w_2\}$ and $\{u_2', w_2'\}$ are two edges of Σ_2. Thus, relabelling if necessary, we may assume that there exist $x \in X$ and $y \in Y$ such that

$$u_1^x = u_1', \quad w_1^x = w_1', \quad u_2^y = u_2', \quad w_2^y = w_2'.$$

Therefore the element (x, y) maps the edge $\{(u_1, u_2), (w_1, w_2)\}$ to the edge $\{(u_1', u_2'), (w_1', w_2')\}$. Hence Γ is $(X \times Y)$-edge-transitive.

Similarly, the component with vertex set $(U_1 \times W_2) \cup (U_2 \times W_1)$ is also $(X \times Y)$-edge-transitive. □

Edge-transitive graphs are often described as coset graphs, which are defined as follows.

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DEFINITION 2.3. For a group G and subgroups L, R such that $L \cap R$ is *core-free* in G, that is, contains no nontrivial normal subgroup, let $[G : L] = \{Lx \mid x \in G\}$ and $[G : R] = \{Rx \mid x \in G\}$. Consider the G-edge-transitive graph with vertex set $[G : L] \cup [G : R]$ and edges $\{Lx, Ry\}$, where $yx^{-1} \in RL$. This graph is called a *coset graph*, and is denoted by $\mathrm{Cos}(G, L, R)$.

LEMMA 2.4. *Let $\Gamma = (V, E)$ and $G \leq \mathrm{Aut}\,\Gamma$ be such that G is transitive on E and intransitive on V. Then for an edge $\{u, w\}$, Γ is isomorphic to $\mathrm{Cos}(G, G_u, G_w)$.*

Coset graph representations for bidirect products are interesting and important.

LEMMA 2.5. *Let Σ_1 be an X-edge-transitive graph and Σ_2 be a Y-edge-transitive graph, both of which are connected and bipartite. Then*

$$\Sigma_1 = \mathrm{Cos}(X, X_{u_1}, X_{w_1}) \quad \textit{and} \quad \Sigma_2 = \mathrm{Cos}(Y, Y_{u_2}, Y_{w_2}),$$

where $\{u_1, w_1\}$ and $\{u_2, w_2\}$ are edges of Σ_1 and Σ_2 respectively, and the two connected components of $\Sigma_1 \times \Sigma_2$ are

$$\mathrm{Cos}(X \times Y, X_{u_1} \times Y_{u_2}, X_{w_1} \times Y_{w_2}) \quad \textit{and} \quad \mathrm{Cos}(X \times Y, X_{u_1} \times Y_{w_2}, X_{u_2} \times Y_{w_1}).$$

PROOF. Let $G = X \times Y$, and let v be a vertex of $\Sigma_1 \times \Sigma_2$. Then $v = (v_1, v_2)$, where v_1 and v_2 are vertices of Σ_1 and Σ_2. An element $g = (x, y) \in G$ lies in G_v if and only if

$$(v_1, v_2) = v = v^g = (v_1, v_2)^{(x,y)} = (v_1^x, v_2^y),$$

or, equivalently, $v_1^x = v_1$ and $v_2^y = v_2$, that is, $x \in X_{v_1}$ and $y \in Y_{v_2}$. Thus, $G_v = X_{v_1} \times Y_{v_2}$. By Lemma 2.2, each of the components of $\Sigma_1 \times \Sigma_2$ is $(X \times Y)$-edge-transitive.

Let $\{u_1, w_1\}$ and $\{u_2, w_2\}$ be edges of Σ_1 and Σ_2. Then both $\{(u_1, u_2), (w_1, w_2)\}$ and $\{(u_1, w_2), (u_2, w_1)\}$ are edges of $\Sigma_1 \times \Sigma_2$, and they lie in different components. By Lemma 2.4, the two components of $\Sigma_1 \times \Sigma_2$ can be represented as coset graphs. □

The case where $\Sigma_1 = \Sigma_2$ is especially interesting. Recall from Definition 1.1 that, given a connected bipartite graph Σ with biparts U and W, we write $\Sigma^{\times_{\mathrm{bi}} 2}$ for the connected component of $\Sigma \times \Sigma$ with vertex set $(U \times U) \cup (W \times W)$, and $\Sigma^{\times_{\mathrm{bi}} m}$ for the connected component of $\Sigma^{\times_{\mathrm{bi}} (m-1)} \times \Sigma$ with vertex set $(U^{m-1} \times U) \cup (W^{m-1} \times W)$ when $m \geq 3$.

The other component of $\Sigma \times \Sigma$ has vertex set $(U \times W) \cup (W \times U)$, and is arc-transitive provided that Σ is edge-transitive.

EXAMPLE 2.6. Let $X = \mathrm{PSL}(2, p)$, where p is a prime congruent to 5 modulo 8. Then X has subgroups $L \cong A_4$ (the alternating group) and $R \cong D_{12}$ (the dihedral group) such that $L \cap R \cong \mathbb{Z}_2^2$ and $\langle L, R \rangle = X$. Thus, the coset graph $\Sigma = \mathrm{Cos}(X, L, R)$ is a connected cubic graph and X-locally-primitive, but not X-arc-transitive.

Let $U = [X : L]$ and $W = [X : R]$. Then the component of $\Sigma \times \Sigma$ whose vertex set is $(U \times W) \cup (W \times U)$ is $X \wr S_2$-arc-transitive. The other component is $\Sigma^{\times_{\mathrm{bi}} 2}$, which is $X \wr S_2$-locally-primitive but not arc-transitive.

In terms of coset graphs, the mth power $\Sigma^{\times_{\mathrm{bi}} m}$ has a simple form.

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Lemma 2.7. *Suppose that* $\Sigma = \mathrm{Cos}(X, X_\alpha, X_\beta)$. *Then*

$$\Sigma^{\times_{\mathrm{bi}} m} = \mathrm{Cos}(X^m, X_\alpha^m, X_\beta^m).$$

If Σ *is of valency* $\{k_1, k_2\}$, *then* $\Sigma^{\times_{\mathrm{bi}} m}$ *has valency* $\{k_1^m, k_2^m\}$. *Further,* $\mathrm{Aut}\,\Sigma^{\times_{\mathrm{bi}} m}$ *has a subgroup* $X \wr S_m$, *that is,*

$$\mathrm{Aut}\,\Sigma^{\times_{\mathrm{bi}} m} \geq X \wr S_m.$$

Proof. The bipartition of the vertex set of Σ is $[X : X_\alpha] \cup [X : X_\beta]$, and the vertex set of $\Sigma^{\times_{\mathrm{bi}} m}$ is $[X : X_\alpha]^m \cup [X : X_\beta]^m$, which is equal to $[X^m : X_\alpha^m] \cup [X^m : X_\beta^m]$. We may inductively assume that

$$\Sigma^{\times_{\mathrm{bi}}(m-1)} = \mathrm{Cos}(X^{m-1}, X_\alpha^{m-1}, X_\beta^{m-1}).$$

By Lemma 2.5, $\Sigma^{\times_{\mathrm{bi}} m}$ is a component of $\mathrm{Cos}(X^{m-1}, X_\alpha^{m-1}, X_\beta^{m-1}) \times \mathrm{Cos}(X, X_\alpha, X_\beta)$. It follows that $\Sigma^{\times_{\mathrm{bi}} m} = \mathrm{Cos}(X^m, X_\alpha^m, X_\beta^m)$.

Let $\Gamma = \Sigma^{\times_{bi} m}$. Let $u = (\alpha, \alpha, \ldots, \alpha)$ and $w = (\beta, \beta, \ldots, \beta)$ be vertices of Γ. Then

$$\begin{aligned} |\Gamma(u)| &= |X_\alpha^m : X_\alpha^m \cap X_\beta^m| = |X_\alpha : X_\alpha \cap X_\beta|^m, \\ |\Gamma(w)| &= |X_\beta^m : X_\alpha^m \cap X_\beta^m| = |X_\beta : X_\alpha \cap X_\beta|^m. \end{aligned}$$

Thus, if Σ is of valency $\{k_1, k_2\}$, then Γ has valency $\{k_1^m, k_2^m\}$.

Write $g = (1, \ldots, 1; \pi) \in X \wr S_m$ where $\pi \in S_m$. For all elements $g_1 = (x_1, \ldots, x_m)$, $g_2 = (y_1, \ldots, y_m) \in X^m$, the following is true:

$$\begin{aligned} X_\alpha^m g_1 \sim X_\beta^m g_2 &\iff g_2^{-1} g_1 \in X_\alpha^m X_\beta^m \\ &\iff (y_1^{-1} x_1, \ldots, y_m^{-1} x_m) \in (X_\alpha X_\beta \times \cdots \times X_\alpha X_\beta) \\ &\iff (y_1^{-1} x_1, \ldots, y_m^{-1} x_m)^g \in (X_\alpha X_\beta \times \cdots \times X_\alpha X_\beta)^g \\ &\iff (g_1^{-1} g_2)^g \in (X_\alpha^m X_\beta^m)^g = X_\alpha^m X_\beta^m \\ &\iff (X_\alpha^m g_1)^g \sim (X_\beta^m g_2)^g. \end{aligned}$$

Thus, $g = (1, \ldots, 1; \pi)$ is an automorphism of $\Sigma^{\times_{\mathrm{bi}} m}$, and $X \wr S_m \leq \mathrm{Aut}\,\Sigma^{\times_{\mathrm{bi}} m}$. □

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. Let Σ be a connected bipartite graph with biparts U and W, and let $\Gamma = \Sigma^{\times_{\mathrm{bi}} m}$ with $m \geq 2$. Then Γ is a bipartite graph with biparts U^m and W^m. In this proof, H will denote the symmetric group S_m.

Assume that $\Gamma = \Sigma^{\times_{\mathrm{bi}} m}$ is G-locally-primitive, where $G \leq X \wr S_m$ with $X \leq \mathrm{Aut}\,\Sigma$. Let $Y = X \wr H$. By Lemma 2.7, $Y \leq \mathrm{Aut}\,\Gamma$, and thus Γ is Y-locally-primitive. So, if $\{u, w\}$ is an edge of Γ, then $Y_{uw} = Y_u \cap Y_w$ is a maximal subgroup of both Y_u and Y_w. Now $Y_u = X_\alpha \wr H$ and $Y_w = X_\beta \wr H$, and $Y_{uw} = Y_u \cap Y_w = (X_\alpha \cap Y_\beta) \wr H$. If it were true that $X_\alpha \cap X_\beta < A < X_\alpha$, then it would follow that $Y_{uw} = (X_\alpha \cap X_\alpha) \wr H < A \wr H < X_\alpha \wr H$, which would be a contradiction. Thus, $X_\alpha \cap X_\beta$ is a maximal subgroup of X_α; similarly, $X_\alpha \cap X_\beta$ is a maximal subgroup of X_β. Therefore Σ is X-locally-primitive.

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Conversely, assume that Σ is X-locally-primitive. Let $\{\alpha, \beta\}$ be an edge of Σ. Since X_α is primitive on $[X_\alpha : X_\alpha \cap X_\beta]$ and X_β is primitive on $[X_\beta : X_\alpha \cap X_\beta]$, we see that $X_\alpha \cap X_\beta$ is a maximal subgroup of both X_α and X_β. Let $G = X \wr H$. By Lemma 2.7, G is a subgroup of $\mathrm{Aut}\,\Gamma$, and $G_u = X_\alpha \wr H$ and $G_w = X_\beta \wr H$. Hence the arc stabiliser G_{uw} is equal to $G_u \cap G_w = (X_\alpha \cap X_\beta) \wr H$. It follows from [4, Lemma 2.7A] that $G_u \cap G_w$ is maximal in both G_u and G_w. Thus, $\Sigma^{\times_{\mathrm{bi}} m}$ is G-locally-primitive. □

We end this section with an example.

Example 2.8. Let $X = \mathrm{PSL}(2, p)$, where p is a prime congruent to 5 modulo 8. Let Σ be the connected cubic X-locally-primitive graph defined in Example 2.6.

Then the bidirect mth-power $\Sigma^{\times_{\mathrm{bi}} m}$ is a G-locally-primitive graph of valency 3^m, where $G = X \wr S_m$.

3. Order twice a prime power

Let $\Gamma = (V, E)$ be an X-locally-primitive graph with biparts U and W. Assume that X is not transitive on V. Assume further that each minimal normal subgroup is transitive on U or on W. Then Γ is a 'basic' graph in the terminology of [6], and one of the following properties hold.

(i) Γ is a complete bipartite graph.
(ii) X is faithful on both U and W, and quasiprimitive on U.

The O'Nan–Scott–Praeger theorem [13] classifies quasiprimitive permutation groups into eight types. Let X act faithfully and quasiprimitively on U, of degree p^e, and N be a minimal normal subgroup of X. If N is a nonabelian simple group, then X is *almost simple*; we write X is *of type AS*. If N is abelian, then $N = \mathbb{Z}_{p^e}$ and $X \le N \rtimes \mathrm{GL}(e, p)$; then X is called *affine* or *of type HA*. Assume now that N is neither simple nor abelian; then X is primitive of *product action type*; we write X is *of type PA*. More precisely, let H be a group acting on Δ, and P be a subgroup of the symmetric group S_l. Let $G = H \wr P$. Then G acts naturally on Δ^l by the so-called product action, as follows: for $(\delta_1, \ldots, \delta_l) \in \Delta^l$, $x = (h_1, \ldots, h_l) \in H^l$, and $\sigma \in P$,

$$(\delta_a, \ldots, \delta_l)^{(h_1,\ldots,h_l)\sigma} = (\epsilon_1, \ldots, \epsilon_l),$$

where $\epsilon_i = \delta_{i'}$ and $i' = i^{\sigma^{-1}}$. It is known that G is primitive on Δ^l if and only if H acts primitively but not regularly on Δ and P is a transitive subgroup of S_l; see [4, Lemma 2.7A].

Noticing that $|U| = |W| = p^e$, we quote a theorem of Guralnick.

Theorem 3.1 (Guralnick [8]). *Let T be a nonabelian simple group with a subgroup H of index p^e with p prime. Then $T = A_{p^e}$, or $\mathrm{PSL}(m, q)$ with $(q^m - 1)/(q - 1) = p^e$, or $\mathrm{PSL}(2, 11)$ with $p^e = 11$, or M_{11} with $p^e = 11$, or M_{23} with $p^e = 23$, or $\mathrm{PSU}(4, 2)$ with $p^e = 27$. In particular, either T is 2-transitive on $[T : H]$ or $T = \mathrm{PSU}(4, 2)$.*

We first consider the case where X acts faithfully on both U and W.

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LEMMA 3.2. *Assume that X acts faithfully on both U and W, and quasiprimitively on U. Then X acts primitively on both U and W; further, the actions of X on U and W are permutationally isomorphic, and X is affine, or almost simple, or of product action type.*

PROOF. Since $|U| = p^e$, the quasiprimitive group X has degree p^e. By Theorem 3.1, X is primitive on U, and X is almost simple, affine or is of product action type. Moreover, since $|W| = |U|$, X is also primitive on W, and X^U and X^W are permutationally isomorphic. □

3.1. Almost simple groups. We need to introduce a special type of cover. Let $\Sigma = (V, E)$ be a graph. The *standard double cover* $\tilde{\Sigma}$ of Σ is defined as the bipartite graph with biparts $U = \{(u, 0) \mid u \in V\}$ and $W = \{(w, 1) \mid w \in V\}$, such that two vertices $(u, 0)$ and $(w, 1)$ are adjacent if and only if u, w are adjacent in Σ.

It is easily shown that $\tilde{\Sigma}$ is connected if and only if Σ is connected and nonbipartite. Moreover, if Σ is G-locally-primitive, then $\tilde{\Sigma}$ is also G-locally-primitive, with the natural actions of G on U and W. Thus, for each locally primitive graph Σ of order p^e with p prime, the standard double cover $\tilde{\Sigma}$ is an example satisfying our condition.

Now let $\Gamma = (V, E)$ be a connected bipartite graph with biparts U and W. Assume that $X \le \mathrm{Aut}\,\Gamma$ is transitive on E and intransitive on V. Assume further that X_u and X_w are conjugate in X, where $u \in U$ and $w \in W$. Then the action of X_u on $W \setminus \{w\}$ is equivalent to the action of X_w on $W \setminus \{w\}$. Hence, $\Gamma(u)$ is an orbit of X_w on $W \setminus \{w\}$. It follows that Γ is the standard double cover of the orbital graph Σ of X acting on W, where the arc set of Σ is equal to $(w, v)^X$ with $v \in \Gamma(u)$.

LEMMA 3.3. *Let Γ be a connected bipartite graph with biparts U and W. Assume that $X \le \mathrm{Aut}\,\Gamma$ is transitive on E and intransitive on V such that X_u and X_w are conjugate (in X), where $u \in U$ and $w \in W$. Then Γ is the standard double cover of an orbital graph Σ of X acting on W. Furthermore, Γ is X-locally-primitive if and only if Σ is.*

We now consider the almost simple group case. We analyse the candidates for the simple groups T appearing in the list of Guralnick in Theorem 3.1. Recall that the socle $\mathrm{soc}(X)$ of a group X is the subgroup generated by all its minimal normal subgroups.

LEMMA 3.4. *Let X be almost simple, and assume that X_u and X_w are conjugate in X for some vertices $u \in U$ and $w \in W$. Then either $\Gamma = \mathrm{K}_{p^e,p^e} - p^e\mathrm{K}_2$, or Γ is the standard double cover of the Schläfli graph.*

PROOF. Let $T = \mathrm{soc}(X)$. By Lemma 3.3, Γ is the standard double cover of an orbital graph of X on W. Then, by Theorem 3.1, either X is 2-transitive on both U and W, or $T = \mathrm{PSU}(4, 2)$ and $|U| = |W| = 27$. In the former case, Γ is the standard double cover of a complete graph K_{p^e}, and so $\Gamma = \mathrm{K}_{p^e,p^e} - p^e\mathrm{K}_2$. In the latter case, by [12], the only locally primitive orbital graph of $T = \mathrm{PSU}(4, 2)$ on W is the Schläfli graph; so Γ is the standard double cover of the Schläfli graph. □

EXAMPLE 3.5. Suppose that $X = \mathrm{PSL}(2, 11)$ acts on 11 points. Then $X_u \cong X_w \cong A_5$, and $|U| = |W| = 11$. Assume that X_u and X_w are not conjugate (in X). Then X_u does not fix

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any point of W. The permutation degrees of $X_u \cong A_5$ that are less than 11 are 5 and 6. It follows that X_u acting on W has exactly two orbits, of sizes 5 and 6. Therefore Γ has valency 5 or 6.

The graph of valency 5, denoted by $\mathrm{D}_2^1(11,5)$, is the incidence graph of the well-known 2-$(11,5,1)$-design; that of valency 6 is the complement of $\mathrm{D}_2^1(11,5)$ in $\mathrm{K}_{11,11}$, denoted by $\overline{\mathrm{D}}_2^1(11,5)$. Both $\mathrm{D}_2^1(11,5)$ and $\overline{\mathrm{D}}_2^1(11,5)$ are 2-arc-transitive.

Example 3.6. Let $X = \mathrm{PSL}(d,q)$ act on $(q^d-1)/(q-1)$ points. Assume that X_u and X_w are not conjugate in X. Then

$$X_u \cong X_w \cong [q^{d-1}] \rtimes \frac{1}{(d,q-1)}\mathrm{GL}(d-1,q) \quad \text{and} \quad |U| = |W| = \frac{q^d-1}{q-1}.$$

Without loss of generality, we may assume that u is a 1-subspace, and w is a hyperplane. Now X_w does not fix any 1-subspace, point of U, and

$$X_w \cong [q^{d-1}] \rtimes \frac{1}{(d,q-1)}\mathrm{GL}(d-1,q).$$

Assume that u is contained in w. Now w is a space of dimension $d-1$, and contains exactly $(q^{d-1}-1)/(q-1)$ subspaces of dimension one. Moreover, $\mathrm{GL}(d-1,q)$ is 2-transitive on these 1-subspaces. It follows that X_w is 2-transitive on $\Gamma(w)$. This graph is actually the incidence graph of projective points and hyperplanes in the projective geometry, denoted by $\mathrm{PH}(d,q)$. The graph $\mathrm{PH}(d,q)$ is 2-arc-transitive.

Assume that u is not contained in w. The number of 1-subspaces that are not contained in w is equal to

$$\frac{q^d-1}{q-1} - \frac{q^{d-1}-1}{q-1} = q^{d-1}.$$

It follows that X_w is 2-transitive on $\Gamma(w)$. This graph, denoted by $\overline{\mathrm{PH}}(d,q)$, is the complement of $\mathrm{PH}(d,q)$ in $\mathrm{K}_{n,n}$ where $n = (q^d-1)/(q-1)$. It is not 2-arc-transitive.

Lemma 3.7. *Let X be almost simple, and assume that X_u and X_w are not conjugate in X for all $u \in U$ and all $w \in W$. Then $\Gamma = \mathrm{D}_2^1(11,5)$, $\overline{\mathrm{D}}_2^1(11,5)$, $\mathrm{PH}(d,q)$, or $\overline{\mathrm{PH}}(d,q)$.*

Proof. Let $T = \mathrm{soc}(X)$. It follows from Theorem 3.1 that either $T = \mathrm{PSL}(2,11)$ and $p^e = 11$, or $T = \mathrm{PSL}(d,q)$ and $p^e = (q^d-1)/(q-1)$.

In the former case, $\Gamma = \mathrm{D}_2^1(11,5)$ or $\overline{\mathrm{D}}_2^1(11,5)$, as in Example 3.5. These two graphs are locally $(T,2)$-arc-transitive.

In the latter case, $\Gamma = \mathrm{PH}(d,q)$ or $\overline{\mathrm{PH}}(d,q)$, as in Example 3.6. □

3.2. Product action type. Assume that the actions of X on U and W are both of product action type. Let $\{u,w\}$ be an edge of Γ, where $u \in U$ and $w \in W$. Let $N = \mathrm{soc}(X) = T^l$, where $l \geq 2$ and T is one of the simple groups that appear in Theorem 3.1.

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Lemma 3.8. *Suppose that N_u and N_w are conjugate in N. Then Γ is the standard double cover of a graph with form $\Sigma^{\times m}$ with $m \geq 2$, where $\Sigma = \mathrm{K}_{p^r}$ or Σ is the Schläfli graph.*

Proof. Since N is transitive on both U and W, we see that X_u and X_w are conjugate in X. By Lemma 3.3, Γ is the standard double cover of a locally primitive orbital graph Γ_0 of X acting U. Such a graph Γ_0 is characterised in [12], which shows that $\Gamma_0 = \Sigma^{\times m}$, where $\Sigma = \mathrm{K}_{p^r}$ or Σ is the Schläfli graph. □

Next we consider the case where N_u and N_w are not conjugate.

Lemma 3.9. *Assume that N_u and N_w are not conjugate in N. Then $\Gamma = \Sigma^{\times_{\mathrm{bi}} m}$, where $m \geq 2$ and*

$$\Sigma = \mathrm{D}_2^1(11, 5), \overline{\mathrm{D}}_2^1(11, 5), \mathrm{PH}(d, q), \textit{ or } \overline{\mathrm{PH}}(d, q).$$

Proof. In this case, it follows from Theorem 3.1 that either $T = \mathrm{PSL}(2, 11)$ and $N_u \cong N_w = A_5^m$, or $T = \mathrm{PSL}(d, q)$, associated with the actions on 1-subspaces and hyperplanes. Thus, G^U and G^W are both primitive of product action type.

We may write the biparts U and W as $U = [N : N_u] = \Delta^m$ and $W = [N : N_w] = \Pi^m$, where $\Delta = [T : T_\delta]$ and $\Pi = [T : T_\pi]$, and $u = (\delta, \delta, \ldots, \delta)$ and $w = (\pi, \pi, \ldots, \pi)$. Then $N_u = T_\delta^m$ and $N_w = T_\pi^m$. Since Γ is G-locally-primitive, Γ is N-edge-transitive. Thus,

$$\Gamma = \mathrm{Cos}(N, N_u, N_w) = \mathrm{Cos}(T^m, T_\delta^m, T_\pi^m).$$

By Lemma 2.5, we see that $\Gamma = \Sigma^{\times_{\mathrm{bi}} m}$, where $\Sigma = \mathrm{Cos}(T, T_\delta, T_\pi)$. By Theorem 1.2, Σ is T-locally-primitive. By Lemma 3.7, we conclude that

$$\Sigma = \mathrm{D}_2^1(11, 5), \overline{\mathrm{D}}_2^1(11, 5), \mathrm{PH}(d, q), \text{ or } \overline{\mathrm{PH}}(d, q),$$

as required. □

Proof of Theorem 1.3. Let $\Gamma = (V, E)$ be a connected bipartite G-locally-primitive graph of order $2p^e$. Assume that G has exactly two orbits on V, which are the biparts of Γ, denoted by U and W.

Let $M \lhd G$ be maximal subject to the condition that M has at least three orbits on each of U and W. Let U_M and W_M be the sets of M-orbits on U and W, respectively. Then each minimal normal subgroup of G/M is transitive on U_M or W_M. By [6], Γ is a normal cover of Γ_M, M is semiregular on V, and either $\Gamma_M = \mathrm{K}_{p^e,p^e}$, or G/M is faithful and quasiprimitive on both U_M and W_M.

The former case fits Theorem 1.3(i). For the latter case, by Lemma 3.2, the actions of G/M on U_M and W_M are equivalent and of type HA, AS or PA. If these actions are of type HA, then G/M has a normal subgroup, N say, which is regular on both U_M and W_M, and hence MN is a normal subgroup of G which is regular on both U and W. Thus, Γ is a binormal Cayley graph.

Assume finally that the actions of G/M on U_M and W_M are of type AS or PA. Then by Lemmas 3.4 and 3.7 to 3.9, the quotient graph Γ_M fits either part (ii) or (iii) of Theorem 1.3. □

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In order to prove Corollary 1.4, we need to quote a result about the vertex stabiliser of s-arc-transitive graphs. For a graph Γ and an arc (u, w), denote by $G_u^{[1]}$ the kernel of G_u acting on $\Gamma(u)$. Then the induced permutation group $G_u^{\Gamma(u)}$ is isomorphic to $G_u/G_u^{[1]}$. Let $G_{uw}^{[1]} = G_u^{[1]} \cap G_w^{[1]}$, the kernel of the arc stabiliser G_{uw} acting on the double star $\Gamma(u) \cup \Gamma(w)$. Then the well-known Thompson–Wielandt theorem tells us that $G_{uw}^{[1]}$ is a r-group for some prime r. Moreover, the following result is already known.

Theorem 3.10 (Weiss [15]). *Let Γ be a connected (G, s)-arc-transitive graph where $s \geq 2$. Then, for an arc (u, w), either $G_{uw}^{[1]} = 1$ and $s \leq 3$, or $G_{uw}^{[1]}$ is a nontrivial r-group and $\mathrm{soc}(G_u^{\Gamma(u)}) = \mathrm{PSL}(d, q)$, where $|\Gamma(u)| = (q^d - 1)/(q - 1)$ and $r \mid q$; furthermore, if $s \geq 4$, then $d = 2$.*

Observe that G_u is an extension of the kernel $G_a^{[1]}$ by the factor group $G_u^{\Gamma(u)}$, and, furthermore, $G_u^{[1]}$ is an extension of $G_{uw}^{[1]}$ by $(G_u^{[1]})^{\Gamma(w)}$, that is,

$$G_u \cong G_u^{[1]} G_u^{\Gamma(u)} \cong (G_{uw}^{[1]} (G_u^{[1]})^{\Gamma(w)}) G_u^{\Gamma(u)}.$$

From the above theorem, the following statements follow.

Lemma 3.11. *The stabiliser G_u has at most two insoluble composition factors, and, further, one of the following results holds.*

(i) *G_u is soluble, and $G_u^{\Gamma(u)} \leq \mathrm{A\Gamma L}(1, r^f)$, the 1-dimensional affine semilinear group.*
(ii) *G_u has only one insoluble composition factor, namely, $\mathrm{soc}(G_u^{\Gamma(u)})$.*
(iii) *G_u has exactly two insoluble composition factors, one of which is $\mathrm{soc}(G_u^{\Gamma(u)})$ and the other is the unique insoluble composition factor of $G_{uw}^{\Gamma(u)}$.*
(iv) *$G_u^{\Gamma(u)}$ is affine of degree r^d and G_u has exactly two insoluble composition factors that are isomorphic to the unique insoluble composition factor of $G_{uw}^{\Gamma(u)}$.*

Proof of Corollary 1.4. Let Γ be a 2-arc-transitive graph of order $2p^e$, where p is an odd prime. Then Γ is G-locally-primitive and satisfies Theorem 1.3. We may assume that each nontrivial normal subgroup of G has at most two orbits on the vertex set. If Γ is a binormal Cayley graph, then it is easy to see that $G^+ = \mathbb{Z}_p^e \rtimes G_u$.

Assume now that Γ is a standard double cover of $\Sigma^{\times m}$ where $\Sigma = \mathrm{K}_{p^e}$ or the Schläfli graph. Then $\Sigma^{\times m}$ is 2-arc-transitive. If $m \geq 2$ and $\Sigma = \mathrm{K}_{p^e}$, then Γ is a Hamming graph, and is known not to be 2-arc-transitive. If Σ is the Schläfli graph, then $G_u \rhd (\mathbb{Z}_2^4 \rtimes A_5)^m$. It follows from Lemma 3.11 that $m = 1$, and hence Γ itself is the Schläfli graph and has valency 16. So $G_u = \mathbb{Z}_2^4 \rtimes A_5$ or $\mathbb{Z}_2^4 \rtimes S_5$, which is not possible since neither $\mathbb{Z}_2^4 \rtimes A_5$ nor $\mathbb{Z}_2^4 \rtimes S_5$ has a representation of degree 16. Therefore $\Sigma = \mathrm{K}_{p^e}$ and $\Gamma = \mathrm{K}_{p^e,p^e} - p^e\mathrm{K}_2$.

Suppose that $\Gamma = \Sigma^{\times_{\mathrm{bi}} m}$, where

$$\Sigma = \mathrm{D}_2^1(11, 5), \overline{\mathrm{D}}_2^1(11, 5), \mathrm{PH}(d, q), \text{ or } \overline{\mathrm{PH}}(d, q).$$

Then $m = 1$ by Lemma 3.11. Obviously, $\overline{\mathrm{PH}}(d, q)$ is not 2-arc-transitive. Therefore

$$\Gamma = \Sigma = \mathrm{D}_2^1(11, 5), \overline{\mathrm{D}}_2^1(11, 5), \text{ or } \mathrm{PH}(d, q).$$

Finally, it is clear that neither K_{p^e,p^e} nor $K_{p^e,p^e} - p^e K_2$ is 4-arc-transitive. By Theorem 3.10, we conclude that if Γ is 4-arc-transitive, then $\Gamma = \mathrm{PH}(3, q)$. □

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References

[1] A. E. Brouwer, 'Descriptions of various graphs', available at http://www.win.tue.nl/~aeb/graphs.
[2] A. E. Brouwer and H. A. Wilbrink, 'Ovoids and fans in the generalized quadrangle GQ(4, 2)', *Geom. Dedicata* **36** (1990), 121–124.
[3] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and P. A. Wilson, *Atlas of Finite Groups* (Oxford University Press, London, 1985).
[4] J. D. Dixon and B. Mortimer, *Permutation Groups* (Springer, New York, 1996).
[5] M. Giudici, C. H. Li, P. Potočnik and C. E. Praeger, 'Homogeneous factorisations of graph products', *Discrete Math.* **308** (2008), 3652–3667.
[6] M. Giudici, C. H. Li and C. E. Praeger, 'Analysing locally s-arc transitive graphs', *Trans. Amer. Math. Soc.* **356** (2004), 291–317.
[7] M. Giudici, C. H. Li and C. E. Praeger, 'On finite locally s-arc transitive graphs with a star normal quotient', *J. Group Theory* **9** (2006), 641–658.
[8] R. Guralnick, 'Subgroups of prime power index in a simple group order', *J. Algebra* **246** (1983), 304–311.
[9] A. A. Ivanov and C. E. Praeger, 'On finite affine 2-arc transitive graphs', *European J. Combin.* **14** (1993), 421–444.
[10] C. H. Li, 'Finite s-arc transitive graphs of prime power order', *Bull. Lond. Math. Soc.* **33** (2001), 129–137.
[11] C. H. Li, 'Finite edge-transitive Cayley graphs and rotary Cayley maps', *Trans. Amer. Math. Soc.* **358** (2006), 4605–4635.
[12] C. H. Li, J. M. Pan and L. Ma, 'Locally primitive graphs of prime-power order', *J. Aust. Math. Soc.* **86** (2009), 111–122.
[13] C. E. Praeger, 'Primitive permutation groups with a doubly transitive subconstituent', *J. Aust. Math. Soc. (Series A)* **45** (1988), 66–77.
[14] C. E. Praeger, 'An O'Nan–Scott theorem for finite quasiprimitive permutation groups and an application to 2-arc transitive graphs', *J. Lond. Math. Soc.* (2) **47** (1993), 227–239.
[15] R. Weiss, 's-transitive graphs', in: *Algebraic Methods in Graph Theory,* vol. II, Colloquia Mathematica Societatis Janos Bolyai, 25 (eds. L. Lovasz and V. T. Sos) (North-Holland, Amsterdam–New York, 1981), pp. 827–847.

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