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PARAMETRIZED STRICT DEFORMATION QUANTIZATION OF C^*-BUNDLES AND HILBERT C^*-MODULES

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Abstract

In this paper, we review the parametrized strict deformation quantization of C^*-bundles obtained in a previous paper, and give more examples and applications of this theory. In particular, it is used here to classify H_3-twisted noncommutative torus bundles over a locally compact space. This is extended to the case of general torus bundles and their parametrized strict deformation quantization. Rieffel's basic construction of an algebra deformation can be mimicked to deform a monoidal category, which deforms not only algebras but also modules. As a special case, we consider the parametrized strict deformation quantization of Hilbert C^*-modules over C^*-bundles with fibrewise torus action.

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1. Introduction

Parametrized strict deformation quantization of C^*-bundles was introduced by the authors in an earlier paper [9], as a generalization of Rieffel's strict deformation quantization of C^*-algebras [21, 22]. The particular version of Rieffel's theory that was generalized in [9] was due to Kasprzak [13] and Landstad [14, 15]. The results in [9] were used to classify noncommutative principal torus bundles as defined by Echterhoff *et al.* [6], as parametrized strict deformation quantizations of principal torus bundles. These arise as special cases of the continuous fields of noncommutative tori that appear as T-duals to space-times with background H-flux in [16, 17].

In Section 3 we review the construction of the parametrized deformation quantization from our earlier paper [9] and give more examples of this construction in Sections 3 and 6 and applications in Section 4. For example, we generalize the main application of our results in [9] (as well as those in [6]). More precisely, suppose

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that $A(X)$ is a C^*-bundle over a locally compact space X with a fibrewise action of a torus T, and that $A(X) \rtimes T \cong CT(X, H_3)$, where $CT(X, H_3)$ is a continuous trace algebra with spectrum X and Dixmier–Douady class $H_3 \in H^3(X; \mathbf{Z})$. We call such C^*-bundles H_3-twisted NCPT bundles over X. Our first main result is that any H_3-twisted NCPT bundle $A(X)$ is equivariantly Morita equivalent to the parametrized deformation quantization of the continuous trace algebra

$$CT(Y, q^*(H_3))_\sigma,$$

where $q : Y \to X$ is a principal torus bundle with Chern class $H_2 \in H^2(X; H^1(T; \mathbf{Z}))$, and $\sigma \in C_b(X, Z^2(\widehat{T}, \mathbf{T}))$ a defining deformation such that $[\sigma] = H_1 \in H^1(X; H^2(T; \mathbf{Z}))$. This enables us to prove in Section 5 that the continuous trace algebra

$$CT(X \times T, H_1 + H_2 + H_3),$$

whose Dixmier–Douady class is of the form $H_1 + H_2 + H_3 \in H^3(X \times T; \mathbf{Z})$, where $H_j \in H^j(X; H^{3-j}(T; \mathbf{Z}))$, has an action of the vector group V that is the universal cover of the torus T, and covering the V-action on $X \times T$. Moreover, the crossed product can be identified up to T-equivariant Morita equivalence,

$$CT(X \times T, H_1 + H_2 + H_3) \rtimes V \cong CT(Y, q^*(H_3))_\sigma.$$

That is, the T-dual of $(X \times T, H_1 + H_2 + H_3)$ is the parametrized strict deformation quantization of $(Y, q^*(H_3))$ with deformation parameter σ, $[\sigma] = H_1$. From this we obtain the explicit dependence of the K-theory of $CT(Y, q^*(H_3))_\sigma$ in terms of the deformation parameter.

In Section 6 we extend this to the case of general torus bundles and their noncommutative parametrized strict deformation quantizations. It has proved a useful principle that deformation of an algebraic structure should be viewed within the context of a deformation of an appropriate category; see, for example, [2, Introduction]. Following this philosophy, we show in Section 7 how Rieffel deformations can be regarded as monoidal functors which allow us to deform modules as well as algebras. In the last section, we construct the parametrized strict deformation quantization of Hilbert C^*-modules over C^*-bundles with fibrewise torus action directly.

2. C^*-bundles and fibrewise smooth $*$-bundles

We begin by recalling the notion of C^*-bundles over X and then introduce the special case of H_3-twisted noncommutative principal bundles. Then we discuss the fibrewise smoothing of these, which is used in parametrized Rieffel deformation later on.

Let X be a locally compact Hausdorff space and let $C_0(X)$ denote the C^*-algebra of continuous functions on X that vanish at infinity. A *C^*-bundle* $A(X)$ over X in the sense of [6] is exactly a $C_0(X)$-algebra in the sense of Kasparov [12]. That is, $A(X)$ is a C^*-algebra together with a nondegenerate $*$-homomorphism

$$\Phi : C_0(X) \to ZM(A(X)),$$

called the *structure map*, where $ZM(A)$ denotes the centre of the multiplier algebra $M(A)$ of A. The *fibre* over $x \in X$ is then $A(X)_x = A(X)/I_x$, where

$$I_x = \{\Phi(f) \cdot a;\, a \in A(X) \text{ and } f \in C_0(X) \text{ such that } f(x) = 0\},$$

and the canonical quotient map $q_x : A(X) \to A(X)_x$ is called the *evaluation map* at x.

Note that this definition does not require local triviality of the bundle, or even that the fibres of the bundle be isomorphic to one another.

Let G be a locally compact group. One says that there is a *fibrewise action* of G on a C^*-bundle $A(X)$ if there is a homomorphism $\alpha : G \longrightarrow \mathrm{Aut}(A(X))$ which is $C_0(X)$-linear in the sense that

$$\alpha_g(\Phi(f)a) = \Phi(f)(\alpha_g(a)) \quad \forall g \in G,\, a \in A(X),\, f \in C_0(X).$$

This means that α induces an action α^x on the fibre $A(X)_x$ for all $x \in X$.

The first observation is that if $A(X)$ is a C^*-algebra bundle over X with a fibrewise action α of a *Lie group* G, then there is a canonical *smooth* $*$*-algebra bundle* over X. We recall its definition from [3]. A vector $y \in A(X)$ is said to be a *smooth vector* if the map

$$G \ni g \longrightarrow \alpha_g(y) \in A(X)$$

is a smooth map from G to the normed vector space $A(X)$. Then

$$\mathcal{A}^\infty(X) = \{y \in A(X) \mid y \text{ is a smooth vector}\}$$

is a $*$-subalgebra of $A(X)$ which is norm dense in $A(X)$. Since G acts *fibrewise* on $A(X)$, it follows that $\mathcal{A}^\infty(X)$ is again a $C_0(X)$-algebra which is *fibrewise smooth*.

Let T denote the torus of dimension n. We define an *H_3-twisted noncommutative principal T-bundle* (or *H_3-twisted NCP T-bundle*) over X to be a separable C^*-bundle $A(X)$ together with a fibrewise action $\alpha : T \to \mathrm{Aut}(A(X))$ such that there is a Morita equivalence,

$$A(X) \rtimes_\alpha T \cong CT(X,\, H_3),$$

as C^*-bundles over X, where $CT(X,\, H_3)$ denotes the continuous trace C^*-algebra with spectrum equal to X and Dixmier–Douady class equal to $H_3 \in H^3(X,\, \mathbf{Z})$.

If $A(X)$ is a H_3-twisted NCP T-bundle over X, then we call $\mathcal{A}^\infty(X)$ a *fibrewise smooth H_3-twisted noncommutative principal T-bundle* (or *fibrewise smooth H_3-twisted NCP T-bundle*) over X. In this paper, we are able to give a complete classification of fibrewise smooth H_3-twisted NCP T-bundles over X via a parametrized version of Rieffel's theory of strict deformation quantization as derived in [9].

3. Parametrized strict deformation quantization of C^*-bundles with fibrewise action of T

In a nutshell, parametrized strict deformation quantization is a functorial extension of Rieffel's strict deformation quantization from algebras A to $C(X)$-algebras $A(X)$, and in particular to C^*-bundles over X. Unlike Rieffel's deformation theory [21], the version in [9] starts with multipliers via the Landstad–Kasprzak approach. Here we review the theory for C^*-bundles over X.

Let $A(X)$ be a C^*-algebra bundle over X with a fibrewise action α of a torus T. Let $\sigma \in C_b(X, Z^2(\widehat{T}, \mathbf{T}))$ be a deformation parameter. Then we define the parametrized strict deformation quantization of $A(X)$, denoted $A(X)_\sigma$, as follows. We have the direct sum decomposition,

$$A(X) \cong \widehat{\bigoplus_{\chi \in \widehat{T}}} A(X)_\chi$$
$$\phi(x) = \sum_{\chi \in \widehat{T}} \phi_\chi(x)$$

for $x \in X$, where for $\chi \in \widehat{T}$,

$$A(X)_\chi := \{a \in A(X) \mid \alpha_t(a) = \chi(t) \cdot a \ \forall t \in T\}.$$

Since T acts by $\star$-automorphisms,

$$A(X)_\chi \cdot A(X)_\eta \subseteq A(X)_{\chi\eta} \quad \text{and} \quad A(X)^*_\chi = A(X)_{\chi^{-1}} \quad \forall \chi, \eta \in \widehat{T}. \tag{3.1}$$

Therefore the spaces $A(X)_\chi$ for $\chi \in \widehat{T}$ form a Fell bundle $A(X)$ over $\widehat{T}$ (see [8]); there is no continuity condition because $\widehat{T}$ is discrete. The completion of the direct sum is explained as follows. The representation theory of T shows that

$$\bigoplus_{\chi \in \widehat{T}} A(X)_\chi = A(X)^{\text{alg}}$$

is a T-equivariant dense subspace of $A(X)$, where T acts on $A(X)_\chi$ as follows:

$$\hat{\alpha}_t(\phi_\chi(x)) = \chi(t)\phi_\chi(x) \quad \forall t \in T, x \in X.$$

Then

$$\widehat{A(X)^{\text{alg}}} = \widehat{\bigoplus_{\chi \in \widehat{T}}} A(X)_\chi$$

is the completion in the C^*-norm of $A(X)$, and is isomorphic to $A(X)$.

The product of elements in $A(X)^{\text{alg}} \subset A(X)$ then also decomposes as

$$(\phi\psi)_\chi(x) = \sum_{\chi_1\chi_2 = \chi} \phi_{\chi_1}(x)\psi_{\chi_2}(x)$$

for $\chi_1, \chi_2, \chi \in \widehat{T}$. The product can be deformed by setting

$$(\phi \star_\sigma \psi)_\chi(x) = \sum_{\chi_1\chi_2 = \chi} \phi_{\chi_1}(x)\psi_{\chi_2}(x)\sigma(x; \chi_1, \chi_2)$$

which is associative because of the cocycle property of σ.

We next describe the norm completion aspects. For $x \in X$, let $\mathcal{H}_x$ denote the universal Hilbert space representation of the fibre C^*-algebra $A(X)_x$ which one obtains via the Gelfand–Naimark–Segal theorem. Let $\mathcal{H}_1 = \int_X \mathcal{H}_x \, dx$ denote the direct integral, which is the universal Hilbert space representation of $A(X)$. By considering instead the Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes L^2(T) \otimes \mathcal{H}_2$, for an infinite-dimensional Hilbert space $\mathcal{H}_2$, where we note that every character of T occurs with infinite multiplicity in $L^2(T) \otimes \mathcal{H}_2$, we obtain a T-equivariant embedding $\varpi : A(X) \to B(\mathcal{H})$. The equivariance means that

$$\varpi(\phi(x)_\chi) = \varpi(\phi(x))_\chi.$$

Now consider the action of $A(X)^{\text{alg}}$ on $\mathcal{H}$ given by the deformed product $\star_\sigma$, that is, for $\phi \in A(X)^{\text{alg}}$ and $\Psi \in \mathcal{H}$,

$$(\phi \star_\sigma \Psi)_\chi(x) = \sum_{\chi_1 \chi_2 = \chi} \varpi(\phi_{\chi_1}(x)) \Psi_{\chi_2}(x) \sigma(x; \chi_1, \chi_2).$$

The operator norm completion of this action is the parametrized strict deformation quantization of $A(X)$, denoted by $A(X)_\sigma$.

We next consider a special case of this construction. Consider a smooth fiber bundle of smooth manifolds.

$$\begin{array}{ccc} Z & \longrightarrow & Y \\ & & \Big\downarrow \pi \\ & & X \end{array}$$

Suppose there is a fibrewise action of a torus T on Y. That is, assume that there is an action of T on Y satisfying

$$\pi(t \cdot y) = \pi(y) \quad \forall t \in T, \, y \in Y.$$

Let $\sigma \in C_b(X, Z^2(\widehat{T}, \mathbf{T}))$ be a deformation parameter. $C_0(Y)$ is a C^*-bundle over X, and as above, form the parametrized strict deformation quantization $C_0(Y)_\sigma$.

In particular, let Y be a principal G-bundle over X, where G is a compact Lie group such that $\text{rank}(G) \geq 2$ (for instance, $G = \text{SU}(n)$ where $n \geq 3$ or $G = \text{U}(n)$ where $n \geq 2$). Let T be a maximal torus in G and $\sigma \in C_b(X, Z^2(\widehat{T}, \mathbf{T}))$ be a deformation parameter. Then $C_0(Y)$ is a C^*-bundle over X, and as above, form the parametrized strict deformation quantization $C_0(Y)_\sigma$.

4. Classifying H_3-twisted NCPT bundles

Here we prove another application of parametrized strict deformation quantization (compare with Section 3 and [9]).

Let $A(X)$ be a H_3-twisted NCPT bundle over X. Suppose that $A(X) \otimes_{C_0(X)} CT(X, -H_3)$ is a C^*-bundle over X and has a fibrewise (diagonal) action of T, where

T acts trivially on $CT(X, -H_3)$. Then

$$\begin{aligned}(A(X) \otimes_{C_0(X)} CT(X, -H_3)) \rtimes T &\cong (A(X) \rtimes T) \otimes_{C_0(X)} CT(X, -H_3)\\ &\cong CT(X, H_3) \otimes_{C_0(X)} CT(X, -H_3)\\ &\cong C_0(X, \mathcal{K}).\end{aligned}$$

Therefore, $A(X) \otimes_{C_0(X)} CT(X, -H_3)$ is an NCPT bundle over X. By the classification [9, Theorem 5.1] (which in turn used the results of Echterhoff and Williams [7]), we deduce that

$$A(X) \otimes_{C_0(X)} CT(X, -H_3) \cong C_0(Y)_\sigma.$$

Therefore,

$$A(X) \cong C_0(Y)_\sigma \otimes_{C_0(X)} CT(X, H_3).$$

LEMMA 4.1. *With the notation above,*

$$C_0(Y)_\sigma \otimes_{C_0(X)} CT(X, H_3) \cong CT(Y, q^*(H_3))_\sigma.$$

PROOF. We use the explicit Fourier decomposition as in Example 6.2 and [9, Example 6.2] to deduce that both sides are naturally isomorphic as T-vector spaces, and also that products are compatible under the isomorphism. □

To summarize, we have the following main result, which follows from [9, Theorem 3.1, Section 4], Example 6.2, and the observations above.

THEOREM 4.2. *Given an H_3-twisted NCPT bundle $A(X)$, there exist a defining deformation $\sigma \in C_b(X, Z^2(\widehat{T}, \mathbf{T}))$ and a principal torus bundle $q : Y \to X$ such that $A(X)$ is T-equivariant Morita equivalent over $C_0(X)$ to the parametrized strict deformation quantization of $CT(Y, q^*(H_3))$ with respect to σ, that is,*

$$A(X) \cong CT(Y, q^*(H_3))_\sigma.$$

Conversely, by Example 6.2, the parametrized strict deformation quantization of $CT(Y, q^*(H_3))$ is the H_3-twisted NCPT bundle $CT(Y, q^*(H_3))_\sigma$.

This also easily follows from the well-known fact that, when G acts trivially on X, every element in the Brauer group $\mathrm{Br}_G(X)$ factors by a product of the trivial action on A and an action on $C_0(X, \mathcal{K})$ (see, for instance, [5]). Anyway, the arguments presented here are direct and quite simple.

5. T-duality and K-theory

THEOREM 5.1. *With the notation above, $(X \times T, H_1 + H_2 + H_3)$ and the parametrized strict deformation quantization of $(Y, q^*(H_3))$ with deformation parameter σ, where $[\sigma] = H_1$, are T-dual pairs, and the first Chern class $c_1(Y) = H_2$. That is,*

$$CT(Y, q^*(H_3))_\sigma \rtimes V \cong CT(X \times T, H_1 + H_2 + H_3).$$

PROOF. As before, let V be the vector group that is the universal covering group of the torus group T, and the action of V on the spectrum factors through T. By [6, Lemma 8.1], the crossed product $CT(X \times T, H_1 + H_2) \rtimes_\beta V$ is Morita equivalent to $C_0(X, \mathcal{K}) \rtimes_\sigma \widehat{T}$, where, as before, the fibrewise action of the Pontryagin dual group $\widehat{T}$ on $C_0(X, \mathcal{K})$ is determined by the continuous family of $\mathbf{T}$-valued 2-cocycles $\sigma \in C_b(X, Z^2(\widehat{T}, \mathbf{T}))$. We remark that the notation used in [6, Lemma 8.1] is $CT(X \times T, \delta)$, where $\delta \in H^3(X \times T, \mathbf{Z})$ has the property that the restrictions

$$\delta|_{X\times\{1\}} = H_3 = 0 \quad \text{and} \quad \delta|_{\{\mathrm{pt}\}\times T} = H_0 = 0;$$

see the second sentence of the second paragraph of [6, Section 8]. Therefore $\delta = H_1 + H_2$.

Setting $A(X) = C_0(X, \mathcal{K}) \rtimes_\sigma \widehat{T}$, then it is a C^*-bundle over X with a fibrewise action of T and, by Takai duality, $A(X) \rtimes T \cong C_0(X, \mathcal{K})$. Therefore $A(X)$ is an NCPT bundle and, by [9, Theorem 5.1], we see that there is a T-equivariant Morita equivalence,

$$A(X) \sim C_0(Y)_\sigma,$$

where the notation is as in the statement of this theorem. By Lemma 4.1,

$$CT(Y, q^*(H_3))_\sigma \rtimes V \cong (C_0(Y)_\sigma \rtimes V) \otimes_{C_0(X)} CT(X, H_3).$$

By Takai duality, $C_0(Y)_\sigma \rtimes V \cong CT(X \times T, H_1 + H_2)$. Therefore

$$CT(Y, q^*(H_3))_\sigma \rtimes V \cong CT(X \times T, H_1 + H_2 + H_3),$$

proving the result. □

Using the Connes–Thom isomorphism theorem [4] and the result above, we obtain our next result.

COROLLARY 5.2. *The K-theory of $CT(Y, q^*(H_3))_\sigma$ depends on the deformation parameter in general. More precisely, with the notation above $[\sigma] = H_1$, $c_1(Y) = H_2$,*

$$K_\bullet(CT(Y, q^*(H_3))_\sigma) \cong K^{\bullet+\dim V}(X \times T, H_1 + H_2 + H_3),$$

where the right-hand side denotes the twisted K-theory.

6. Examples

EXAMPLE 6.1 (Noncommutative torus). We begin by recalling the construction by Rieffel [21] realizing the smooth noncommutative torus as a deformation quantization of the smooth functions on an n-dimensional torus $T = \mathbf{R}^n/\mathbf{Z}^n$.

Recall that any translation invariant Poisson bracket on T is just

$$\{a, b\} = \sum \theta_{ij} \frac{\partial a}{\partial x_i} \frac{\partial b}{\partial x_j},$$

for $a, b \in C^\infty(T)$, where (θ_{ij}) is a skew symmetric matrix. The action of T on itself is given by translation. The Fourier transform is an isomorphism between $C^\infty(T)$ and $\mathcal{S}(\widehat{T})$, taking the pointwise product on $C^\infty(T)$ to the convolution product on $\mathcal{S}(\widehat{T})$ and taking differentiation with respect to a coordinate function to multiplication by the dual coordinate. In particular, the Fourier transform of the Poisson bracket gives rise to an operation on $\mathcal{S}(\widehat{T})$ denoted in the same way. For $\phi, \psi \in \mathcal{S}(\widehat{T})$, define

$$\{\psi, \phi\}(p) = -4\pi^2 \sum_{p_1+p_2=p} \psi(p_1)\phi(p_2)\gamma(p_1, p_2),$$

where γ is the skew symmetric form on $\widehat{T}$ defined by

$$\gamma(p_1, p_2) = \sum \theta_{ij} p_{1,i} p_{2,j}.$$

For $\hbar \in \mathbf{R}$, define a skew bicharacter $\sigma_\hbar$ on $\widehat{T}$ by

$$\sigma_\hbar(p_1, p_2) = \exp(-\pi\hbar\gamma(p_1, p_2)).$$

Using this, define a new associative product $\star_\hbar$ on $\mathcal{S}(\widehat{T})$,

$$(\psi \star_\hbar \phi)(p) = \sum_{p_1+p_2=p} \psi(p_1)\phi(p_2)\sigma_\hbar(p_1, p_2).$$

This is precisely the smooth noncommutative torus $A^\infty_{\sigma_\hbar}$.

The norm $\|\cdot\|_\hbar$ is defined to be the operator norm for the action of $\mathcal{S}(\widehat{T})$ on $L^2(\widehat{T})$ given by $\star_\hbar$. Via the Fourier transform, carry this structure back to $C^\infty(T)$, to obtain the smooth noncommutative torus as a strict deformation quantization of $C^\infty(T)$ [21], with respect to the translation action of T.

EXAMPLE 6.2. We next generalize the first example above to the case of principal torus bundles $q : Y \to X$ of rank n, together with a algebra bundle $\mathcal{K}_P \to X$ whose fibre is the C^*-algebra of compact operators $\mathcal{K}$. Here $P \to X$ is a principal bundle whose structure group is the projective unitary group PU, which acts on $\mathcal{K}$ by conjugation. Note that fibrewise smooth sections of $q^*(\mathcal{K}_P)$ over Y decompose as a direct sum,

$$C^\infty_{\text{fibre}}(Y, q^*(\mathcal{K}_P)) = \widehat{\bigoplus_{\alpha\in\widehat{T}}} C^\infty_{\text{fibre}}(X, \mathcal{L}_\alpha \otimes \mathcal{K}_P)$$

$$\phi = \sum_{\alpha\in\widehat{T}} \phi_\alpha$$

where $C^\infty_{\text{fibre}}(X, \mathcal{L}_\alpha \otimes E)$ is defined as the subspace of $C^\infty_{\text{fibre}}(Y, q^*(E))$ consisting of sections which transform under the character $\alpha \in \widehat{T}$, and where $\mathcal{L}_\alpha$ denotes the associated line bundle $Y \times_\alpha \mathbf{C}$ over X. That is, $\phi_\alpha(yt) = \alpha(t)\phi_\alpha(y)$ for all $y \in Y$ and $t \in T$. The direct sum is completed in such a way that the function $\widehat{T} \ni \alpha \mapsto \|\phi_\alpha\|_\infty \in \mathbf{R}$ is in $\mathcal{S}(\widehat{T})$.

For $\phi, \psi \in C^\infty_{\text{fibre}}(Y, q^*(\mathcal{K}_P))$, define a deformed product $\star_\hbar$ as follows. For $y \in Y$, $\alpha, \alpha_1, \alpha_2 \in \widehat{T}$, let

$$(\psi \star_\hbar \phi)(y, \alpha) = \sum_{\alpha_1\alpha_2=\alpha} \psi(y, \alpha_1)\phi(y, \alpha_2)\sigma_\hbar(q(y); \alpha_1, \alpha_2), \tag{6.1}$$

using the notation $\psi(y, \alpha_1) = \psi_{\alpha_1}(y)$ and so on, and where $\sigma_\hbar \in C(X, Z^2(\widehat{T}, \mathbf{T}))$ is a continuous family of bicharacters of $\widehat{T}$ such that $\sigma_0 = 1$. The cocycle property of $\sigma_\hbar$ ensures that the product defined above is associative. The construction is clearly T-equivariant. We denote the deformed algebra as $C^\infty_{\text{fibre}}(Y, q^*(\mathcal{K}_P))_{\sigma_\hbar}$.

7. General torus bundles

We extend the results of the previous sections to the case of deformations of general torus bundles, not just principal torus bundles. That is, the noncommutative torus bundles (NCT bundles) of this section strictly include the noncommutative principal torus bundles (NCPT bundles) of [6, 9].

For any fibre bundle

$$F \to Y \xrightarrow{\xi} X$$

with structure group G, the action of G on F induces an action of $\pi_0(G)$ on the homology and cohomology of F. When X is a connected manifold, there is a well-defined homomorphism $\pi_1(X) \to \pi_0(G)$ that gives each homology or cohomology group of F the structure of a $\mathbf{Z}\pi_1(X)$-module. Now suppose that F is a torus T and let $G = \text{Diff}(T)$ denote the group of diffeomorphisms of the torus T. It is well-known that $\pi_0(G) \cong \text{GL}(n, \mathbf{Z})$, where $n = \dim(T)$, and that the $\pi_0(G)$-action on $H_1(T)$ may be identified with the natural action of $\text{GL}(n, \mathbf{Z})$ on $\widehat{T}$. Given any representation $\rho : \pi_1(X) \to \text{GL}(n, \mathbf{Z})$, we let $\mathbf{Z}^n_\rho$ denote the corresponding $\mathbf{Z}\pi_1(X)$-module. We will make use of the following proposition, which is well-known, and explicitly stated in the appendices of [10].

Proposition 7.1. *Assume that X is a compact, connected manifold, and choose any representation $\rho : \pi_1(X) \to \text{GL}(n, \mathbf{Z})$. Then there is a natural, bijective correspondence between the equivalence classes of torus bundles over X inducing the module structure $\mathbf{Z}^n_\rho$ on $H_1(T)$ and the elements of $H^2(X; \mathbf{Z}^n_\rho)$, the second cohomology group of X with local coefficients $\mathbf{Z}^n_\rho$.*

Remark 7.2. We call the cohomology class corresponding to the symplectic torus bundle ξ the *characteristic class* of ξ and denote it by $c(\xi) \in H^2(X; \mathbf{Z}^n_\rho)$. Then the characteristic class $c(\xi)$ vanishes if and only if ξ admits a section. When the representation ρ is trivial, then ξ is a principal torus bundle and the characteristic class reduces to the first Chern class. We have that $c(\xi) = c_1(\xi) = 0$ if and only if ξ is trivial.

Let X be compact and $T \to Y \xrightarrow{\xi} X$ be a torus bundle over X. Let

$$\Gamma \to \widehat{X} \xrightarrow{\eta} X$$

denote the universal cover of X, and consider the lifted torus bundle

$$T \to \eta^*(Y) \xrightarrow{\eta^*\xi} \widehat{X}.$$

Since $\widehat{X}$ is simply-connected, it is classified by a characteristic class in $H^2(\widehat{X}; \mathbf{Z}^n)$, so that

$$T \to \eta^*(Y) \xrightarrow{\eta^*\xi} \widehat{X}$$

is a principal torus bundle. Let $\sigma \in C_b(X, Z^2(\widehat{T}, \mathbf{T}))$ be a deformation parameter. Then $C_0(\eta^*(Y))$ is a C^*-bundle over $\widehat{X}$, and as before, we form the parametrized strict deformation quantization $C_0(\eta^*(Y))_{\widehat{\sigma}}$, where $\widehat{\sigma}$ is the lift of σ to $\widehat{X}$. Since $\eta^*(Y)$ is the total space of a principal Γ-bundle over the compact space Y, the action of Γ on $\eta^*(Y)$ is free and proper, so the action of Γ on $C_0(\eta^*(Y))_{\widehat{\sigma}}$ is also proper in the sense of Rieffel [20]. In particular, the fixed-point algebra $C_0(\eta^*(Y))^{\Gamma}_{\widehat{\sigma}}$ makes sense and is the parametrized strict deformation quantization of $C_0(Y)$. We define this to be a *noncommutative torus bundle* (NCT bundle) over X. In particular, they strictly include noncommutative principal torus bundles (NCPT bundles) over X. NCT bundles are classified by a representation $\rho : \pi_1(X) \to \mathrm{GL}(n, \mathbf{Z})$ together with a cohomology class with local coefficients, $H_2 \in H^2(X; \mathbf{Z}^n_\rho)$ and a deformation parameter $\sigma \in C_b(X, Z^2(\widehat{T}, \mathbf{T}))$.

Finally, we define a H_3-twisted NCT bundle over X to be a parametrized strict deformation quantization of $CT(Y, \xi^*(H_3))$ given in the notation above by $CT(\eta^*(Y), \eta^*\xi^*(H_3))^{\Gamma}_{\widehat{\sigma}}$.

8. Deformations of monoidal categories

Rieffel's strict deformation theory modifies the multiplication on an algebra. As noted in the introduction, it is a useful principle that deformation should involve not just an algebra, but a category in which the algebra is but one object. Fortunately Rieffel's strict C^*-algebra deformation can be extended to give a functor changing the tensor product in a monoidal category of Fréchet V-modules for a vector group V. Another motivation for this stems from the way in which nonassociative crossed products could be understood naturally in the context of a monoidal category, [1]. A similar interpretation of noncommutative crossed products could make it easier to unify the two examples. Let X be a locally compact Hausdorff space and V an abelian group. We start with Banach V-modules (in which V acts by isometries), with a commuting action of $C_0(X)$. Each module has a dense submodule of V-smooth vectors, on which V and $C_0(X)$ still act and which can be given the structure of a Fréchet space. Consider the strict symmetric monoidal category $C_0(X)$-V-mod of smooth $C_0(X) \rtimes V$-modules with tensor product $\otimes_0 = \otimes_{C_0(X)}$ as in [18], and unit object $C_0(X)$ (with the multiplication action of $C_0(X)$ and trivial action of V). We suppose also that we have a symmetric bicharacter $e = e^{iB} : V \times V \to C_0(X)$, and a B-skew-adjoint automorphism J of V. The monoidal functor $\mathcal{D}_J$ to a braided category $C_0(X)$-(V, J)-mod acts as the identity on objects and morphisms, but gives a braided tensor product $\otimes_J$ and (assuming the integral well-defined) with, for objects $\mathcal{A}$ and $\mathcal{B}$, the consistency map $c_J : \mathcal{A} \otimes_J \mathcal{B} \to \mathcal{A} \otimes \mathcal{B}$, taking $x \in \mathcal{A}$, $y \in \mathcal{B}$,

$$c_J(x \otimes_J y) = \int_{V \times V} e(u, v)((Ju) \cdot x) \otimes_0 (v \cdot y)\, du\, dv,$$

where e acts on the tensor product as an element of $C_0(X)$. (It follows from the discussion in [9, 21] that c_J is invertible.) Essentially all the technical estimates needed to show that this is well-defined, and to derive its properties, have already been provided by Rieffel in [21, Ch. I] using a simplified version of Hörmander's partial integration technique applied to smooth maps from a vector group to a Fréchet space. In a Hilbert V module M the smooth vectors M^∞ for the action provide a Fréchet space, and the rest is done as in [21]. (Within the tensor product of two V-modules M_1 and M_2 is the dense subspace of smooth vectors $(M_1 \otimes M_2)^\infty$, which contains the tensor product $M_1^\infty \otimes M_2^\infty$. We have just mimicked the constructions of [21, Ch. 2] to deform the tensor product, rather than an algebra product.) One can check that this is $C_0(X)$-linear (due to the triviality of the action of V on $C_0(X)$) and is compatible with strict associativity [21, Theorem 2.14], and $C_0(X)$ as unit object. (For nonvanishing H_0, we would instead need consistency with the new associativity map in the usual hexagonal diagram.) Since V is abelian it has the tensor product action Δ of V, (which changes to $c_J^{-1} \circ \Delta \circ c_J$ in the new monoidal category). Some care is needed because the asymmetry in c_J means that deformed category is braided. Assuming that one started with a trivial braiding given by the flip $\Psi_0 : m \otimes_0 n \to n \otimes_0 m$, one obtains $\Psi_J = c_J^{-1} \Psi_0 c_J : M \otimes_J N \to N \otimes_J M$. Since $\Psi_0^2 = 1$, we automatically have $\Psi_J^2 = 1$, so that Ψ_J is a symmetric braiding. More generally, Ψ_J and Ψ_0 satisfy the same polynomial identities: if Ψ_0 is of Hecke type then so is Ψ_J. If an object $\mathcal{A}$ has a multiplication morphism $\mu : \mathcal{A} \otimes_0 \mathcal{A} \to \mathcal{A}$, then the morphism property ensures that V acts by automorphisms, and the deformed multiplication is $\mu \circ c_J : \mathcal{A} \otimes_J \mathcal{A} \to \mathcal{A}$, or, using $\star_J$ and $\star$ for the deformed and undeformed multiplications,

$$x \star_J y = \int_{V \times V} e(u, v)((Ju) \cdot x) \star (v \cdot y)\, du\, dv,$$

which is the Rieffel deformed product [21]. Due to the braiding a commutative algebra product μ (satisfying $\mu \circ \Psi_0 = \mu$) turns into a braided commutative, but still noncommutative, product with $\mu_J \circ \Psi_J = \mu_J$, as one would expect for a map transforming classical to quantum theory. We can similarly deform $\mathcal{A}$-modules and bimodules. For example, an action $\alpha : \mathcal{A} \otimes_0 M \to M$ can be deformed to $\alpha_J : \mathcal{A} \otimes_J M \to M$,

$$\alpha_J(a)[m] = \int_{V \times V} e(u, v)\alpha((Ju) \cdot a)[v \cdot m]\, du\, dv.$$

One can follow the same strategy as in [1] and study the effects on compact operators, crossed products and so on, but we shall content ourselves with a discussion of the modules. As in [21, Theorem 2.15] we can show that the functors for different J satisfy $\mathcal{D}_K \circ \mathcal{D}_J = \mathcal{D}_{J+K}$.

9. Parametrized strict deformation quantization of Hilbert C^*-modules over C^*-bundles

As an application of the general procedure outlined in Section 8, we extend the parametrized strict deformation quantization of C^*-bundles $A(X)$ to Hilbert C^*-modules over $A(X)$, but somewhat more directly.

Recall that Hilbert C^*-modules generalize the notion of a Hilbert space, in that they endow a linear space with an inner product which takes values in a C^*-algebra. They were developed by Rieffel in [18], which used Hilbert C^*-modules to construct a theory of induced representations of C^*-algebras. In [19], Rieffel used Hilbert C^*-modules to extend the notion of Morita equivalence to C*-algebras. In [11], Kasparov used Hilbert C^*-modules in his formulation of bivariant K-theory. Hermitian vector bundles are examples of Hilbert C^*-modules over commutative C^*-algebras.

Let $A(X)$ be a C^*-algebra bundle over X, and $E(X)$ a Hilbert C^*-module over $A(X)$ respecting the fibre structure. That is, $E(X)$ has an $A(X)$-valued inner product,

$$\langle\cdot,\cdot\rangle : E(X) \times E(X) \longrightarrow A(X),$$

and the fibre $E(X)_x : E(X)/I_x$ is a (left) Hilbert C^*-module over $A(X)_x$ for all $x \in X$.

Suppose now that $A(X)$ has a fibrewise action α of a torus T and let $E(X)$ have a compatible fibrewise action of T.

We have the direct sum decomposition,

$$E(X) \cong \widehat{\bigoplus_{\chi\in\widehat{T}}} E(X)_\chi$$

$$\psi(x) = \sum_{\chi\in\widehat{T}} \psi_\chi(x)$$

for $x \in X$, where for $\chi \in \widehat{T}$,

$$E(X)_\chi := \{\psi \in E(X) \mid \alpha_t(\psi) = \chi(t)\cdot\psi \ \forall t \in T\}.$$

Since T acts by $\star$-automorphisms,

$$A(X)_\chi \cdot E(X)_\eta \subseteq E(X)_{\chi\eta} \quad \text{and} \quad E(X)_\chi^* = E(X)_{\chi^{-1}} \quad \forall \chi, \eta \in \widehat{T}. \tag{9.1}$$

Therefore the spaces $E(X)_\chi$ for $\chi \in \widehat{T}$ form a Fell bundle $E(X)$ over $\widehat{T}$ (see [8]); there is no continuity condition because $\widehat{T}$ is discrete. The completion of the direct sum is explained as before. The representation theory of T shows that $\bigoplus_{\chi\in\widehat{T}} E(X)_\chi$ is a T-equivariant dense subspace of $E(X)$, where T acts on $E(X)_\chi$ as follows:

$$\widehat{\alpha}_t(\phi_\chi(x)) = \chi(t)\phi_\chi(x) \quad \forall t \in T, x \in X.$$

Then $\widehat{\bigoplus}_{\chi\in\widehat{T}} E(X)_\chi$ is the completion in the Hilbert C^*-norm of $E(X)$.

The action of $A(X)$ on $E(X)$ then also decomposes as

$$(\phi\psi)_\chi(x) = \sum_{\chi_1\chi_2=\chi} \phi_{\chi_1}(x)\psi_{\chi_2}(x)$$

for $\chi_1, \chi_2, \chi \in \widehat{T}$, $\phi \in A(X)$ and $\psi \in E(X)$.

Let $\sigma \in C_b(X, Z^2(\widehat{T}, \mathbf{T}))$ be a deformation parameter. Then, as in Section 2, the parametrized strict deformation quantization of $A(X)$ can be defined, and is denoted by $A(X)_\sigma$.

The action of $A(X)_\sigma$ on $E(X)$ can be defined by deforming the action of $A(X)$ on $E(X)$,

$$(\phi \star_\sigma \psi)_\chi(x) = \sum_{\chi_1\chi_2=\chi} \phi_{\chi_1}(x)\psi_{\chi_2}(x)\sigma(x;\, \chi_1, \chi_2)$$

for $\chi_1, \chi_2, \chi \in \widehat{T}$, $\phi \in A(X)_\sigma$ and $\psi \in E(X)$. This is an action because $\sigma(x :, \cdot, \cdot)$ is a 2-cocycle for all $x \in X$.

This is the parametrized strict deformation quantization of $E(X)$, denoted by $E(X)_\sigma$, which is a Hilbert C^*-module over $A(X)_\sigma$.

EXAMPLE 9.1. Let $E \to X$ be a complex vector bundle over X. Consider a smooth fiber bundle of smooth manifolds.

$$\begin{array}{ccc} Z & \longrightarrow & Y \\ & & \big\downarrow \pi \\ & & X \end{array}$$

Suppose there is a fibrewise action of a torus T on Y. That is, assume that there is an action of T on Y satisfying

$$\pi(t \cdot y) = \pi(y) \quad \forall t \in T,\ y \in Y.$$

Let $\sigma \in C_b(X, Z^2(\widehat{T}, \mathbf{T}))$ be a deformation parameter. $C_0(Y)$ is a C^*-bundle over X, and, as in Section 2, form the parametrized strict deformation quantization $C_0(Y)_\sigma$.

Then the space of sections $C_0(Y, \pi^*(E))$ is a Hilbert C^*-module over the C^*-bundle $C_0(Y)$ over X, with a fibrewise T-action compatible with the fibrewise action of T on $C_0(Y)$. Therefore, as above, we can construct a parametrized strict deformation quantization of $C_0(Y, \pi^*(E))$, denoted by $C_0(Y, \pi^*(E))_\sigma$, which is a Hilbert C^*-module over $C_0(Y)_\sigma$.

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References

[1] P. Bouwknegt, K. C. Hannabuss and V. Mathai, 'C*-algebras in tensor categories', *Clay Math. Proc.* **12** (2011), 127–165.

[2] S. Brain and G. Landi, 'Moduli spaces of noncommutative instantons: gauging away noncommutative parameters', *Oxford Quart. J.*, doi: 10.1093/qmath/haq036.

[3] A. Connes, 'C*-algèbres et géometrie différentielle', *C. R. Acad. Sci. Paris Ser. A–B* **290**(13) (1980), 599–604.

[4] A. Connes, 'An analogue of the Thom isomorphism for crossed products of a C^*-algebra by an action of **R**', *Adv. Math.* **39**(1) (1981), 31–55.

[5] D. Crocker, A. Kumjian, I. Raeburn and D. P. Williams, 'An equivariant Brauer group and actions of groups on C^*-algebras', *J. Funct. Anal.* **146**(1) (1997), 151–184.

[6] S. Echterhoff, R. Nest and H. Oyono-Oyono, 'Principal non-commutative torus bundles', *Proc. Lond. Math. Soc.* **99** (2009), 1–31.

[7] S. Echterhoff and D. P. Williams, 'Crossed products by $C_0(X)$-actions', *J. Funct. Anal.* **158**(1) (1998), 113–151.

[8] J. M. G. Fell and R. S. Doran, *Representations of $*$-Algebras, Locally Compact Groups, and Banach $*$-Algebraic Bundles* (Academic Press, New York, 1988).

[9] K. C. Hannabuss and V. Mathai, 'Noncommutative principal torus bundles via parametrised strict deformation quantization', *AMS Proc. Symp. Pure Math.* **81** (2010), 133–148.

[10] P. J. Kahn, 'Symplectic torus bundles and group extensions', *New York J. Math.* **11** (2005), 35–55.

[11] G. Kasparov, 'Hilbert C*-modules: theorems of Stinespring and Voiculescu', *J. Operator Theory* **4** (1980), 133–150.

[12] G. Kasparov, 'Equivariant K-theory and the Novikov conjecture', *Invent. Math.* **91** (1988), 147–201.

[13] P. Kasprzak, 'Rieffel deformation via crossed products', *J. Funct. Anal.* **257** (2009), 1288–1332.

[14] M. B. Landstad, 'Duality theory for covariant systems', *Trans. Amer. Math. Soc.* **248** (1979), 223–267.

[15] M. B. Landstad, 'Quantization arising from abelian subgroups', *Internat. J. Math.* **5** (1994), 897–936.

[16] V. Mathai and J. Rosenberg, 'T-duality for torus bundles via noncommutative topology', *Comm. Math. Phys.* **253**(3) (2005), 705–721.

[17] V. Mathai and J. Rosenberg, 'T-duality for torus bundles with H-fluxes via noncommutative topology, II: the high-dimensional case and the T-duality group', *Adv. Theor. Math. Phys.* **10**(1) (2006), 123–158.

[18] M. A. Rieffel, 'Induced representations of C*-algebras', *Adv. Math.* **13** (1974), 176–257.

[19] M. A. Rieffel, 'Applications of strong Morita equivalence to transformation group C^*-algebras', *AMS Proc. Symp. Pure Math.* **38** (1982), 299–310, part I.

[20] M. A. Rieffel, 'Proper actions of groups on C *-algebras', in: *Mappings of Operator Algebras (Philadelphia, PA, 1988)*, Progress in Mathematics, 84 (Birkhäuser, Boston, 1990), pp. 141–182.

[21] M. A. Rieffel, 'Deformation quantization for actions of $\mathbf{R}^d$', *Mem. Amer. Math. Soc.* **106**(506) (1993), 93.

[22] M. A. Rieffel, 'Quantization and C*-algebras', *Contemp. Math.* **167** (1994), 67–97.

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