ON EXPERIENCE RATING AND OPTIMAL REINSURANCE

By JUKKA RANTALA

The Ministry of Social Affairs and Health
Helsinki, Finland

ABSTRACT

This paper presents applications of stochastic control theory in determining an insurer's optimal reinsurance and rating policy. Optimality is defined by means of variances of such variables as underwriting result of the insurer, solvency margins of the insurer and reinsurer and the premiums paid by policy-holders.

KEYWORDS

Optimal reinsurance; control theory; Kalman filter.

INTRODUCTION

The problem of optimal reinsurance has been widely discussed in risk-theoretical literature. This problem has several answers depending on the optimality criteria used and assumptions on random variables involved. However, from the theoretical point of view a marked simplification is possible. It has been shown e.g. by BORCH (see GERBER 1979) p. 95) that for every pair of concave utility functions of the cedant and reinsurer the optimal reinsurance arrangement can be found among those where the reinsurer's share of the claims s a function of the total claims amount only; dependence on individual risks or claim sizes is not needed. In PESONEN (1984), Theorem 10.5, a method for constructing an optimal reinsurance form is also presented when the utility functions are known but arbitrary. Usually the problem of optimal reinsurance is treated as a static one; i.e. the problem is to divide the total claims amount of a fixed time period, e.g. one year, into cedant's and reinsurer's components in an optimal way. In this paper a longer perspective is taken by assuming that

a) a reinsurance contract between two insurance companies (the cedant and reinsurer) has been made for a fairly long period and both parties will look for an arrangement which would be optimal (under some criterion) over a longer term.

This assumption justifies among other things the use of asymptotic methods.

Moreover, we assume that

b) the reinsurer's annual share of the total claims amount is a function of present and past annual total claim amounts only (i.e. reinsurance does not depend on individual risks);

and

c) the reinsurer's share is a linear function.

Assumption (b) is motivated by the above-mentioned theorem of BORCH. The linearity assumption (c) allows us to use the methods of linear stochastic control theory. It has been shown by PESONEN (1984), Theorem 10.13, that linear functions are optimal if the utility functions of the cedant and the reinsurer are linear functions of each other.

It is obvious that the three parties involved, the policy-holders, the cedant and the reinsurer, have conflicting interests. Each of them desires to have as small a share as possible of the total variation emerging from claims occurrences. It is in the interest of policy-holders that fluctuation in the premium rates be only moderate. The cedant and the reinsurer put value on smooth flows of underwriting results and solvency margins. In this paper we attempt to find a balance between these different interests by stating the optimality criteria in terms of the variances of the main variables. Examples are minimization of the variance of the total claims amount retained, subject to a constraint on the variance of the reinsurer's accumulated profit; or minimization of the variance of the premiums collected by the cedant, subject to a constraint on the sum of the variances of cedant's and reinsurer's accumulated profits.

The basic model is introduced in Section 1. Section 2 studies a simple case where both cedant's and reinsurer's premiums are assumed to be constants. In that section we use a technique of BOX-JENKINS (1976), Section 13.2; see also RANTALA (1984). In Section 3 a more general case is considered. It is then assumed that the premiums paid by policy-holders to the cedant company are also a controllable variable. This introduces an experience rating aspect into the model. The numerical solutions are relatively easy to find with the aid of the Kalman filter technique (see also RANTALA (1986)).

The main purpose of this paper is more to show a feasible way to attack the problems of reinsurance than to give explicit results directly applicable in practice. Related works are among others those by BOHMAN (1986), (who also considers the reinsurance contract on a long-term basis), GERBER (1984) and LEMAIRE-QUAIRIERE (1986) (who consider reinsurance chains).

1. The Basic Model

Consider two insurance companies. The variables relating to company $j (i = 1, 2)$ are labelled with the subscript j. Company 1 is called the *cedant* and company 2 the *reinsurer*. All variables are measured as proportions of a joint basic volume measure $V(t)$. This may be taken as e.g. the sum of insurance sums, payroll, a suitable monetary index multiplied by the number of policies,

https://doi.org/10.2143/AST.19.2.2014906 Published online by Cambridge University Press

or it may be some measure which is a basis for tariffication. Thus the variables may be termed *rates* (claims rate, premium rate etc.). Moreover, all variables refer to that part of the portfolio which is covered by the reinsurance agreement in question.

We assume that $V(t)$ progresses according to equation

$$V(t) = r_g(t)\, r_x(t)\, V(t-1). \tag{1.1}$$

In equation (1.1) the total growth of the volume $V(t)$ is attributed to two factors: the growth in number of policies or risks units described by $r_g(t)$ and the growth due to inflation described by $r_x(t)$.

Now the accumulated profit (rate) $u_j(t)$ of company j satisfies equation (see BEARD-PENTIKÄINEN-PESONEN (1984), Section 6.5)

$$u_j(t) = r_j(t)\, u_j(t-1) + p_j(t) - x_j(t), \tag{1.2}$$

where $p_j(t)$ is the rate of the premiums and $x_j(t)$ the rate of the total claims amount retained by company j, $r_j(t) = r_{ij}(t)/r_g(t)\, r_x(t)$ and $r_{ij}(t)$ is the interest coefficient of company j and $r_j(t)$ may be called the relative interest rate of company j. The nature of $r_j(t)$'s is stochastic, but for simplicity they are in the following taken as time-independent non-random constants $r_j (j = 1, 2)$.

Note that even if there is variation in $r_{ij}(t)$ and $r_x(t)$, coefficient $r_j(t)$ will be fairly stable if $r_{ij}(t)/r_x(t)$ and $r_g(t)$ are stable as can often be assumed. In general, values of r_j:*s* around 1.0 are perhaps the most usual.

In addition, $x_j(t)$'s and $p_j(t)$'s must satisfy the equations

$$\begin{cases} p(t) = p_1(t) + P_2(t) \\ x(t) = x_1(t) + x_2(t), \end{cases} \tag{1.3}$$

where $p(t)$ is the total premium rate paid by the policy-holders and $x(t)$ is the total claims rate.

Another form of (1.2) and (1.3) which better brings out the control-theoretic aspects is

$$\begin{cases} u_1(t) = r_1 u_1(t-1) + y_1(t) \\ u_2(t) = r_2 u_2(t-1) + p(t) - x(t) - y_1(t), \end{cases} \tag{1.4}$$

where $y_1(t) = p_1(t) - x_1(t)$ is the cedant's underwriting result in the year t. The controllable variables in (1.4) are $y_1(t)$ (both through $p_1(t)$ and $x_1(t)$) and $p(t)$.

We study first in Section 2 a simpler case where premium rates $p(t)$, $p_1(t)$ and $p_2(t)$ are kept as constants and the problem is only do divide $x(t)$ into cedant's and reinsurer's shares.

2. The case of constant premium rates

Assume that $Ex(t)$ is known and both the total premium rate $p(t)$ and the reinsurer's premium rate $p_2(t)$ are constants. *In order to prevent* $u_j(t)$*:s from* *unlimited asymptotic behaviour it has to be assumed that* $r_j < 1$ (which has generally been the case in many countries due to rapid growth in business volume and high inflation). This assumption can be relaxed when premium control is also introduced in Section 3. *Moreover, to simplify notation we consider only deviations from corresponding expectations and thus take* $Ex(t) = 0$. Hence the premium rates are in fact the corresponding safety loadings. Determination of their rational magnitude can be based on the variances of $u_j(t)$'s but is omitted here (see however Example in Section 2.1).

Thus the accumulated profits are governed by the equations

$$\begin{cases} u_1(t) = r_1 u_1(t-1) + p_1 - x_1(t) \\ u_2(t) = r_2 u_2(t-1) + p_2 - (x(t) - x_1(t)) . \end{cases} \tag{2.1}$$

In the following we briefly sketch the method for finding the optimal linear reinsurance policy

$$x_1(t) = a_0 x(t) + a_1 x(t-1) + \ldots, \tag{2.2}$$

when optimality is defined to mean

(a) minimization of Dx_1 when Du_2 is restricted to a given value (or vice versa)
(b) minimization of $D(\Delta x_1)$ when Du_2 is restricted to a given value (or vice versa),

where D denotes standard deviation (i.e. D^2 is the variance operator) and Δ is the difference operator: $\Delta x(t) = x(t) - x(t-1)$.

The former criterion aims at restricting the variation range (i.e. minimums and maximums) of the cedant's annual profit, whereas the latter stresses more its smooth flow from year to year. Variation in the reinsurer's accumulated profit can be controlled by the choice of the admissible value for Du_2. If the safety margin p_2 in ceded premiums is an increasing function of Du_2, criteria (a) and (b) also give the answers to the problem: minimize loading p_2 for given Dx_1 or $D\Delta x_1$.

In what follows the derivation of the optimal coefficients $a_0, a_1, \ldots$ in (2.2) is limited in case (a) to autoregressive claims rates $x(t)$ of at most order two (abbreviated as AR(2) processes and in case (b) for AR(1) claims rates. An important special case of these, usually considered in traditional risk theory, is the white noise process of identically and independently distributed (abbreviated i.i.d.) random variables. The motivation for considering AR claims processes is the empirical observation (see BEARD-PENTIKÄINEN-PESONEN (1984), PENTIKÄINEN-RANTALA (1982), RANTALA (1988)) that claims processes are at least in some cases subject to cyclical variations. Such variations can be generated by AR(2) processes by a suitable choice of parameters. AR (or more generally ARMA processes) are also used in KREMER (1982) to find credibility premiums. A natural way to introduce the AR component into the claims process is to assume that the structure variation (see BEARD-PENTIKÄINEN-PESONEN (1984), Section 2.7) of the claims process is of autoregressive character and the process has also the usual Poisson "random noise". However, this decomposition is not used in this paper so as not to overcomplicate the model-structure and the better to extract the relevant features of the control problems.

In both cases (a) and (b) a modification of the method presented in BOX-JENKINS (1976), Section 13.2 is used to find the optimal rules. Also the Kalman filter technique to be presented in Section 3 could be used in Section 2.1, but not in Section 2.2.

2.1. *Minimization of $Dx_1(t)$ subject to a constraint on $Du_2(t)$*

The problem is (a): i.e. to minimize Dx_1 when $Du_2(t)$ is given. As stated above we restrict our considerations to autoregressive processes of at most order two. Solutions for more general processes could be found by solving the general difference equations (A1.12)-(A1.13) in Appendix 1. Thus the claims rate process is assumed to obey the difference equation

(2.1.1) $$x(t) = \phi_1 x(t-1) + \phi_2 x(t-2) + \varepsilon(t),$$

where $\varepsilon(t)$'s are uncorrelated random variables with mean zero and with variance σ_ε^2. To have finite variance for $x(t)$ coefficients ϕ_1 and ϕ_2 must satisfy the stationarity conditions

(2.1.2) $$\begin{cases} \phi_1 + \phi_2 < 1 \\ \phi_2 - \phi_1 < 1 \\ -1 < \phi_2 < 1 . \end{cases}$$

The formulas become more handy if the so-called backward shift operator B (e.g. $Bx(t) = x(t-1)$) is taken into use. With this notation (2.1.1) can be rewritten as

(2.1.3) $$\Phi(B)\, x(t) = \varepsilon(t),$$

where

(2.1.4) $$\Phi(B) = 1 - \phi_1 B - \phi_2 B^2.$$

It is shown in Appendix 1 that for this claims process the solution to problem (a) is (see equations (A1.25)-(A1.26) in Appendix 1)

(2.1.5) $$x_1(t) = [-(1-r_2 B)\, \mu(B)\, \Phi(B) + 1]\, x(t)$$

or equivalently

(2.1.6) $$x_1(t) = [-(1-r_2 B)\, \mu(B) + \Phi^{-1}(B)]\, \varepsilon(t),$$

where $^{-1}$ denotes the inverse operator and

(2.1.7) $$\mu(B) = A(1-z_0 B)^{-1} + (W_1 + W_2 B)\, \Phi^{-1}(B)$$

and coefficients A, W_1, and W_2 are given by equations (A1.14), (A1.21)-(A1.24) in Appendix 1 and z_0 is that solution of (A1.16) for which $|z_0| < 1$. Note that the formulas do not depend on σ_ε^2. The relevant parameters are ϕ_1, ϕ_2, r_2 and the parameter v in (A1.14) defining the ratio Du_2/Dx_1.

The reinsurance scheme (2.1.5) leads to the following equations for u_1 and u_2:

(2.1.8) $$(1-r_1 B)\, u_1(t) = -[-(1-r_2 B)\,\mu(B)\,\Phi(B)+1]\, x(t)+p_1$$

and

(2.1.9) $$\Phi^{-1}(B)\, u_2(t) = -\mu(B)\, x(t)+p_2/(1-\phi_1-\phi_2)\,(1-r_2).$$

The variances connected with these equations are fairly easy to calculate from the ARMA presentations containing $\varepsilon(t)$'s, which result when $x(t)$ is replaced by $\Phi^{-1}(B)\,\varepsilon(t)$ in (2.1.8) and in (2.1.9). The details are omitted here (see e.g. BOX-JENKINS (1976) Section 3.4.2).

EXAMPLE. Take the classical case of risk theory that $x(t)$:s are i.i.d. random variables: $\phi_j = 0$ for $j = 1, 2$. Then $K = D_j = W_j = 0$ $(j = 1, 2)$ in equations (A1.24), and thus

(2.1.10) $$\mu(B) = r_2^{-1} z_0 (1-z_0 B)^{-1},$$

where z_0 is that root of $r_2 z^2-(1+r_2^2+v)z+r_2 = 0$ whose modulus is less than one. Here v is the parameter fixing the ratio Du_2/Dx_1. The optimal reinsurance scheme is from (2.1.5) and (2.1.7)

(2.1.11) $$x_1(t) = (1-z_0 B)^{-1}(1-r_2^{-1} z_0)\, x(t)$$

or equivalently

(2.1.12) $$x_1(t) = z_0 x_1(t-1)+(1-r_2^{-1} z_0)\, x(t),$$

i.e. $x_1(t)$ is calculated according to the classical exponential smoothing formula of experience rating theory. The corresponding variance is

(2.1.13) $$D^2 x_1 = D^2 x \cdot (1-r_2^{-1} z_0)^2/(1-z_0^2).$$

The resulting solvency rate of the cedant is, from (2.1.8),

(2.1.14) $$(1-r_1 B)\,(1-z_0 B)\, u_1(t) = -(1-r_2^{-1} z_0)\, x(t)+p_1 (1-z_0)$$

with variance

(2.1.15) $$D^2 u_1 = \frac{(1+z_0 r_1)\,(1-r_2^{-1} z_0)^2}{(1-z_0 r_1)\,(1-r_1^2)\,(1-z_0^2)}\, D^2 x$$

The solvency rate of the reinsurer is

(2.1.16) $$u_2(t) = z_0 u_2(t-1)-r_2^{-1} z_0 x(t)+p_2 \cdot \frac{1-z_0}{1-r_2}.$$

and hence $u_2(t)$ is an AR(1) process with variance

https://doi.org/10.2143/AST.19.2.2014906 Published online by Cambridge University Press

(2.1.17) $$D^2 u_2 = D_2 x (r_2^{-2} z_0^2/(1-z_0^2)).$$

The following figure gives the optimal combinations of Du_1, Du_2, Dx_1 and the long-term safety loadings defined by $\lambda_1 = 3(1-r_1)Du_1$, $\lambda_2 = 3(1-r_2)Du_2$ and $\lambda = \lambda_1+\lambda_2$ as multiples of Dx when $r_1 = r_2 = 0.95$.

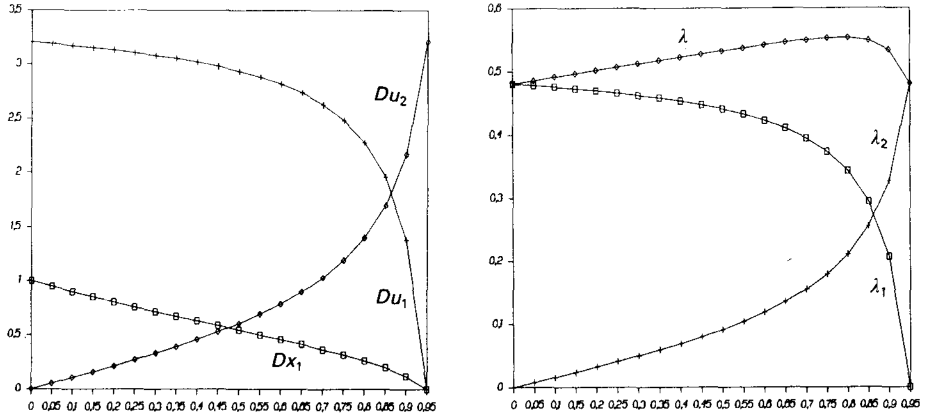

FIGURE 2.1.1. Optimal combinations of the main variables as multiples of D_x in Example 1 when $r_1 = r_2 = 0.95$.

Since an increase in z_0 means that the ceded share of the business increases it is quite natural that Dx_1 and Du_1 decrease and Du_2 increases when z_0 gets larger. Intuitively it is not so obvious that the sum of the safety loadings has its minimum when the whole risk is carried by one insurer only; i.e. if the risk is shared by two companies the safety loading is higher than without risk sharing. The reason is that in the case with reinsurance the total safety loading must maintain two solvency margins, both of which have with high probability to be positive: it is not sufficient that their sum is positive, as is in fact required in the case of no risk-sharing.

2.2. *Minimization of $D(\Delta x_1(t))$ subject to a constraint on $Du_2(t)$*

Now the problem is to minimize $D(\Delta x_1(t))$ when $Du_2(t)$ is given.

To simplify the formulas we restrict ourselves to AR(1) claims rate processes; i.e. coefficient ϕ_2 is zero in (2.1.1). Thus

(2.2.1) $$x(t) = \phi x(t-1)+\varepsilon(t),$$

where $|\phi| < 1$ and $\varepsilon(t)$'s are a series of uncorrelated random variables with mean zero and with variance σ_ε^2. Moreover, let $Eu_1(t) = Eu_2(t) = 0$.

As is shown in Appendix 2 (formulas A2.18-A2.21), the solution is

(2.2.2) $$x_1(t) = [-(1-r_2 B)(1-\phi B)\mu(B)+1]\,x(t)$$

https://doi.org/10.2143/AST.19.2.2014906 Published online by Cambridge University Press

or

$$x_1(t) = [-(1-r_2 B)\,\mu(B) + (1-\Phi B)^{-1}]\,\varepsilon(t), \tag{2.2.3}$$

$$(1-r_1 B)\,u_1(t) = -x_1(t), \tag{2.2.4}$$

$$u_2(t) = -\mu(B)\,\varepsilon(t), \tag{2.2.5}$$

where $\mu(B)$ is given by (A2.15) in Appendix 2. Thus processes $u_1(t)$, $u_2(t)$ and $x_1(t)$ are ARMA processes, whose variances are easy to compute from the presentations containing $\varepsilon(t)$'s (see BOX-JENKINS (1976), Section 3.4.2).

As a limiting case when ϕ approaches 1 we obtain from (2.2.1) a random walk process. This process also follows as a special case of an ARIMA (0, 1, 1) process:

$$\Delta x(t) = (1-\theta B)\,\varepsilon(t) \tag{2.2.6}$$

with $\varepsilon(t)$'s uncorrelated and with $0 \leq \theta \leq 1$.

Equation (2.2.6) has the interpretation that every year a shock $\varepsilon(t)$ is added to the current "level" of the claims rate to produce a value $x(t)$. However, only a proportion $1-\theta$ of the shock is actually absorbed into the level to have lasting influence (see BOX-JENKINS (1976) Chapter 4).

In practice perhaps not every new shock changes the level; possible changes occur only occasionally. Thus (2.2.6) may be regarded as a cautious "upper limit)" for actual claims processes. Such changes in the claims level are to be expected e.g. due to changed policy conditions or changes in claims settlement practice. When $\theta \to 0$ we obtain a random walk process; i.e. every new shock is totally absorbed into the level, this being the most dangerous alternative. When θ is put to one we arrive at the traditional white noise claims process.

WHITE NOISE CASE $\theta = 1$. As is shown in Appendix 2 (see equation (A2.27)), the optimal reinsurance scheme is now

$$(1-k_0 B + k_1 B^2)\,x_1(t) = (1 - r_2^{-1} k_0 + r_2^{-2} k_1)\,x(t) \stackrel{\text{def}}{=} b_0 x(t)\,, \tag{2.2.7}$$

where k_0 and k_1 are given by the procedure I-III in Appendix 2. The variance of $x_1(t)$ is

$$D^2 x_1 = \frac{(1+k_1)(b_0^2+b_1^2) + 2 b_0 b_1 k_0}{(1-k_1)\,[(1+k_1)^2 - k_0^2]}\, D^2 x\,. \tag{2.2.8}$$

with $b_1 = 0$.

The accumulated process $u_1(t)$ is an ARMA process

$$(2.2.9)\quad (1-k_0B+k_1B^2)(1-r_1B)\,u_1(t) = -(1-r_2^{-1}k_0+r_2^{-2}k_1)\,x(t)\,,$$

whose variance is readily calculable. Moreover, $u_2(t)$ is an ARMA (2, 1) process

$$(2.2.10)\quad (1-k_0B+k_1B^2)\,u_2(t) = -[-r_2^{-2}k_1+r_2^{-1}k_0-r_2^{-1}k_1B]\,x(t) \overset{\text{def}}{=} (c_0+c_1B)\,x(t)\,,$$

whose variance is given by (2.2.8) when b's are replaced by c's.

The following Figure 2.2.1 shows Dx_1, Du, and Du_2 for different values of parameter v, when $r_1 = r_2 = 0.95$. The curves should be compared to those of figure 2.2.1. An increase in Dx_1 is reflected as an increase in Du_1 and as a decrease in Du_2. When $v \to \infty$ the total variation is shifted to u_1, the cedant then taking the whole risk. Naturally the minimum for Dx_1 and $D\Delta x_1$ is zero, which is achieved when $v = 0$. Then Du_2 has its maximum.

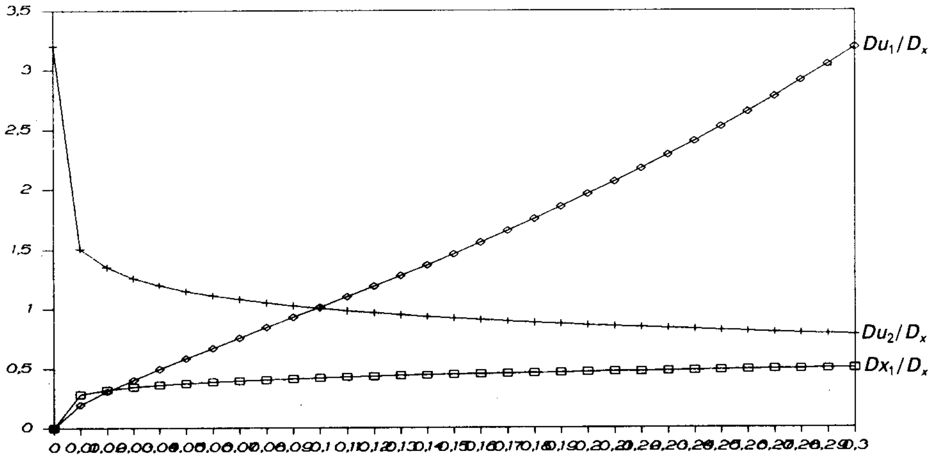

FIGURE 2.2.1. Dx_1, Du_1 and Du_2 as a functions of parameter v, when $r_1 = r_2 = 0.95$, $x(t)$ is a white noise process and $D\Delta x_1$ is minimized for given Du_2.

RANDOM WALK CASE $\theta = 0$. As is shown in Appendix 2, $u_2(t)$ corresponding to the optimal scheme is now an AR (2) process with variance (see (A2.27))

$$(2.2.11)\quad D^2u_2 = \frac{(1+k_1)(r_2^{-1}k_1)^2}{(1-k_1)[(1+k_1)^2-k_0^2]}\,\sigma_\varepsilon^2\,.$$

The optimal reinsurance scheme itself is

$$(2.2.12)\quad (1-k_0B+k_1B^2)\,x_1(\mathrm{t}) = [(1-r_2^{-1}k_1)+(r_2^{-1}k_1+k_1-k_0)B]\,x(t).$$

Thus $x_1(t)$ is a non-stationary process with infinite variance since the "driving" process $x(t)$ on the r.h.s. of (2.2.12) is such. The variance of Δx_1 is

$$(2.2.13)\qquad D^2(\Delta x_1) = \frac{(1+k_1)(w_0^2+w_1^2)+2k_0w_0w_1}{(1-k_1)[(1+k_1)^2-k_0^2]}\,\sigma_\varepsilon^2,$$

where $w_0 = (1-r_2^{-1}k_1)$ and $w_1 = r_2^{-1}k_1+k_1-k_0$.

The corresponding $u_1(t)$ process obeys equation

$$(2.2.14)\quad (1-r_1B)(1-k_0B+k_1B^2)\,u_1(t) = [1-r_2^{-1}k_1+(r_2^{-1}k_1+k_1-k_0)B]\,x(t)$$

and is thus non-stationary, since $x(t)$ is such a process.

Hence in the case of a random walk claims process the procedure produces finite $D(\Delta x_1)$ and Du_2 but with constant $p_1(t)$ Du_1 will be infinite. A finite Du_1 can be achieved if $p_1(t)$ is allowed to be non-stationary.

Although the cases considered in this section may be of some practical interest, their applicability may be rather limited since the premium rate $p(t)$ is unrealistically kept as a constant. In reality premiums are obviously also adjusted according to the observed claims experience. To obtain a more realistic model the variable premium rates should be incorporated into equations and the variation of the premium rate should also be regarded in optimality criteria.

Another limitation to the model above is that the relative interest rates r_j have to satisfy $|r_j| < 1$ in order not to have infinite variances for $u_j(t)$'s. If premium rate control is also introduced this assumption is not necessary.

3. The case where the premium rate may also vary

The technique of BOX-JENKINS used in the preceding section becomes rather messy when the number of the control variables or the complexity of the claims process increases. In the following the well-known Kalman filter is used instead. However, we then obtain only numerical solutions, not analytic expressions like (2.1.5) and (2.2.2). In addition, loss function (3.7) is not suitable for such optimization as envisaged in Section 2.2, since the order of the difference of $p(t)$ which occurs in (3.7) is the same as the smallest difference parameter d for the claims process (3.2) at which $\Delta^d x(t)$ is stationary.

Since the premiums are usually charged at the beginning of the insurance period, the optimal premium rate control scheme cannot utilize the most recent $x(t)$ to determine $p(t)$; i.e. $p(t)$ is a function $x(t-1)$, $x(t-2)$, ... In order to keep the formulas as simple as possible, we then assume that the same set of data is used to determine also the retained part $x_1(t)$ of the claims. In many cases it would also be more realistic to let the time delay be even longer.

RANTALA (1986) illustrates the incorporation of a time delay in a simple case.

Take the model in the form (1.4); i.e.

$$(3.1) \qquad \begin{cases} u_1(t) = r_1 u_1(t-1) + y_1(t) \\ u_2(t) = r_2 u_2(t-1) + p(t) - y_1(t) - x(t). \end{cases}$$

The control variables are the underwriting result $y_1(t)$ of the cedant and the total premiums $p(t)$. It is clear that the optimality criterion must include each of $u_1(t)$ (or alternatively $y_1(t)$), $u_2(t)$ and $p(t)$ if a solution is sought where none of these variables is identically constant: if the variation of only two variables is restricted the total variation produced by $x(t)$ can be directed to the remaining third variable by letting the other variables be constant.

We make the general assumption that the claims rate is an ARIMA (s, d, q) process

$$(3.2) \qquad \Phi(B)\, \Delta^d x(t) = \Theta(B)\, \varepsilon(t),$$

where

$$(3.3) \qquad \begin{cases} \Phi(B) = 1 - \phi_1 B - \phi_2 B^2 - \ldots - \phi_s B^s \\ \Theta(B) = 1 - \theta_1 B - \theta_2 B^2 - \ldots \theta_q B^q \\ \varepsilon(t) = \text{a sequence of uncorrelated random variables with mean zero and with variance } \sigma_\varepsilon^2. \end{cases}$$

If $d > 0$, then the $x(t)$ process defined by (3.2) is non-stationary, but if the roots of equation

$$(3.3) \qquad \Phi(B) = 0$$

lie outside the unit circle the d-th difference $\Delta^d x(t)$ of $x(t)$ is stationary. Note that for $d > 0$ the variances of $\Delta^i u_j(t)$ and $\Delta^i p(t)$ for $i < d$ and $j = 1, 2$ cannot all be finite. A natural demand is that $Du_j(t)$ ($j = 1, 2$) and $D\Delta^d p(t)$ should be finite, i.e. the accumulated profits have finite variances and the "stationarity order" of the premium process is the same as that of the claims process.

Next (3.1) and (3.2) are transformed to a state-space model. Equations (3.1) can be rewritten as

$$(3.4) \qquad \begin{cases} (1 - r_1 B)\, \Delta^d u_1(t) = \Delta^d y_1(t) \\ (1 - r_2 B)\, \Phi(B)\, \Delta^d u_2(t) = \Phi(B)\, [\Delta^d p(t) - \Delta^d y_1(t)] - \Theta(B)\, \varepsilon(t). \end{cases}$$

Let $n_1 = d+1$, $n_2 = \max\{s+d+1, q+1\}$ and $n = n_1 + n_2$.

Introduce n state variables $Z(i, t)$ ($i = 1, 2, \ldots, N$) obeying equation

$$(3.5) \qquad Z(t+1) = AZ(t) + G \begin{pmatrix} \Delta^d y_1(t) \\ \Delta^d p(t) \end{pmatrix} - M\, \varepsilon(t),$$

where

$$
(3.6)\quad
\begin{cases}
A = \left[\begin{array}{c|c|cc}
\begin{matrix} a_1 \\ \vdots \end{matrix} & I_{n_1}-1 & \multicolumn{2}{c}{\multirow{2}{*}{O_{n_2}}} \\
\hline
a_{n_1} & 0\ \ 0\ldots 0 & & \\
\hline
\multicolumn{2}{c|}{\multirow{2}{*}{O_{n_1}}} & \begin{matrix} \beta_1 \\ \vdots \end{matrix} & I_{n_2}-1 \\
\cline{3-4}
& & \beta_{n_2} & 0\ \ 0\ldots 0
\end{array}\right] \\[2ex]
I_n = \text{identity matrix of order } n, \\
O_n = nxn \text{ matrix of zeroes}, \\[1ex]
G = \begin{pmatrix} 1 & \overbrace{0\ldots 0}^{n_1-1} & \vdots & -1, & \phi_1,\ldots, & \phi_{n_2} \\ 0 & 0\ldots 0 & \vdots & 1, & -\phi_1,\ldots, & -\phi_{n_2} \end{pmatrix}', \\[1ex]
M = (\underbrace{0\ldots 0}_{n_1} \ \vdots\ 1, \quad -\theta_1,\ldots, -\theta_{n_2})', \\[1ex]
a(B) = (1-r_1 B)\Delta^d \overset{\text{def}}{=} 1 - a_1 B - a_2 B^2 - \ldots - a_{n_1} B^{n_1}, \\[1ex]
\beta(B) = (1-r_2 B)\Delta^d \Phi(B) \overset{\text{def}}{=} 1 - \beta_1 B - \beta_2 B^2 - \ldots - \beta_{n_2} B^{n_2}
\end{cases}
$$

with $\phi_i = 0$ for $i > s$ and $\theta_i = 0$ for $i > q$ and $'$ denoting transpose.

The accumulated profits $u_1(t)$ and $u_2(t)$ are given by $Z(1, t+1)$ and $Z(n_1+1, t+1)$.

Let the loss function to be minimized be

$$
(3.7)\qquad E\left\{ Z(N)' Q_0 Z(N) + \sum_{j=1}^{N} (Z(j)' Q_1 Z(j)) + Y(j)' Q_2 Y(j) \right\},
$$

where Q_0, Q_1 and Q_2 are symmetric positive definite matrices, $Y(j) = (\Delta^d y_1(j), \Delta^d p(j))'$ and $\{1, \ldots, N\}$ is the planning horizon (a suitable choice for which is the duration of the reinsurance agreement). According to our assumption at the beginning of this section $Y(t)$ can depend on $Z(t)$, $Z(t-1), \ldots$ but not on $Z(t+1)$.

https://doi.org/10.2143/AST.19.2.2014906 Published online by Cambridge University Press

The optimal linear control rule giving the minimum for this loss function is (see e.g. ÅSTRÖM (1970): Theorem 4.1 in Section 8.4):

$$Y(t) = -L(t)\,Z(t), \tag{3.8}$$

where $Y(t)$ is the vector of the cedant's optimal profit and premium setting to be applied at time t. $L(t)$ is a $(2\times n)$ matrix of constants given by

$$L(t) = [Q_2+G'\,S(t+1)\,G]^{-1}\,G'\,S(t+1)\,A, \tag{3.9}$$

where $S(t+1)$ is obtained from

$$S(t) = A'\,S(t+1)\,A+Q_1-A'\,S(t+1)\,GL(t) \tag{3.10}$$

with the initial condition

$$S(N) = Q_0. \tag{3.11}$$

Thus the optimal procedure is quite easy to reach from recurrence equations (3.8)-(3.11). However, it depends on the initial values of the state vector Z; i.e. on the immediate past of the accumulated profits $u_j(t)$. It can be shown that as the planning horizon $N\to\infty$, matrix $S(t)$ will converge to a unique steady-state positive definite value S. Denote the corresponding limit of $L(t)$ by L. Numerical calculation by computer of this *steady-state solution* is quite easy from equations (3.9) and (3.10) by successive iteration. (Note also that the results of Section 2 are in fact steady-state solutions.) The steady-state feedback rating and ceding formula is

$$Y(t) = -LZ(t). \tag{3.12}$$

This equation is quite easy to translate into a more traditional form involving only past $p(t)$'s and $u_j(t)$'s or $x(t)$'s. An example is given later.

The corresponding steady-state covariance matrix C_Z of the state vector $Z(t)$ can be obtained by iteration from equation

$$C_Z = (A-GL)\,C_Z\,(A-GL)'+\sigma_\varepsilon^2 MM'. \tag{3.13}$$

The corresponding variance of $Y(t)$ is

$$\text{Var}\;Y(t) = C_Y = LC_ZL'. \tag{3.14}$$

The steady-state variances of the accumulated profits and $\Delta^d y_1$ and $\Delta^d p$ can be found as the appropriate elements of matrices C_Z and C_Y.

Note that when $d>0$ the variance of the premiums (as that of $x(t)$) is infinite but the variances of the accumulated profits and cedant's profit $y_1(t)$ are finite. Note also that the KALMAN filter technique can easily be extended to more than one reinsurer.

EXAMPLE 1. Take first the white noise $x(t)$ process of traditional risk theory. This case was considered in the examples of Sections 2.1 and 2.2. Now the state-space equation (3.5) is simply

$$\text{(3.15)} \quad \left\{ \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix} \begin{pmatrix} u_1(t-1) \\ u_2(t-1) \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} y_1(t) \\ p(t) \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} x(t) \right.$$

and $MM' = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

Choose the matrices Q_0, Q_1 and Q_2 in loss function (3.7) as

$$\text{(3.16)} \qquad Q_0 = 0_2, Q_1 = \begin{pmatrix} w_1 & 0 \\ 0 & w_2 \end{pmatrix}, \qquad Q_2 = \begin{pmatrix} w_3 & 0 \\ 0 & w_4 \end{pmatrix}.$$

By varying w_i's different optimum combinations can be produced. As an example we take $r_1 = r_2 = 1.0$, $w_1 = 0.1$, $w_2 = 0.025$, $w_3 = 0.0001$ and $w_4 = 1$. Since w_3 is negligible this in fact means that the variance of premiums is minimized subject to $w_1 D^2 u_1 + w_2 D^2 u_2 =$ a given value. Furthermore, an increase in $D^2 p$ is ten times "worse" than in $D^2 u_1$ and forty times "worse" than in $D^2 u_2$ and an increase in $D^2 u_1$ four times "worse" than in $D^2 u_2$. This choice of weights reflects the thinking that the reinsurer should carry most of the fluctuations and the policy-holder the least.

With these parameters the steady-state optimal scheme turns out to be

$$\text{(3.17)} \qquad \begin{cases} y_1(t) = -0.826 \cdot u_1(t-1) + 0.173 \cdot u_2(t-1) \\ p(t) = -0.132 \cdot u_1(t-1) - 0.132 \cdot u_2(t-1) \end{cases}$$

with corresponding variances

$$\text{(3.18)} \qquad \begin{cases} D^2 y_1 = 0.0322\,\sigma_\varepsilon^2 \\ D^2 u_1 = 0.122\,\sigma_\varepsilon^2 \\ D^2 p = 0.0705\,\sigma_\varepsilon^2 \\ D^2 u_2 = 2.96\,\sigma_\varepsilon^2 . \end{cases}$$

Using equations (3.1) it can be shown that (3.17) is equivalent to

$$\text{(3.19)} \qquad \begin{cases} (1-2.652\,B+1.652\,B^2)\,y_1(t) = (0.173-0.173\,B)\,B(p(t)-x(t)) \\ (1-1.868\,B+0.868\,B^2)\,p(t) = (0.264-0.264\,B)\,By_1(t) + \\ \qquad + (0.132-0.132\,B)\,Bx(t) . \end{cases}$$

Figures 3.1 and 3.2 show the steady-state standard deviations of the main variables in the optimal schemes as a function of w_1, where loss matrices (3.16) are used with $w_3 = 0.0001$, $w_4 = 1$ and with two constant ratios $w_1/w_2 = 4$ and $w_1/w_2 = 1$.

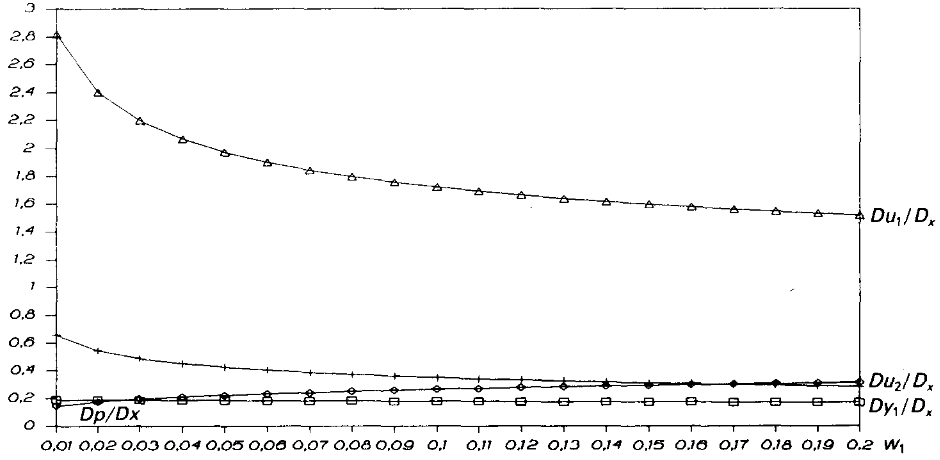

FIGURE 3.1. Steady-state Du_1, Dy_1, Du_2 and Dp of the optimal schemes as functions of w_1 when $w_3 = 0.0001$, $w_4 = 1$, $w_1/w_2 = 4$ and $r_1 = r_2 = 1.0$.

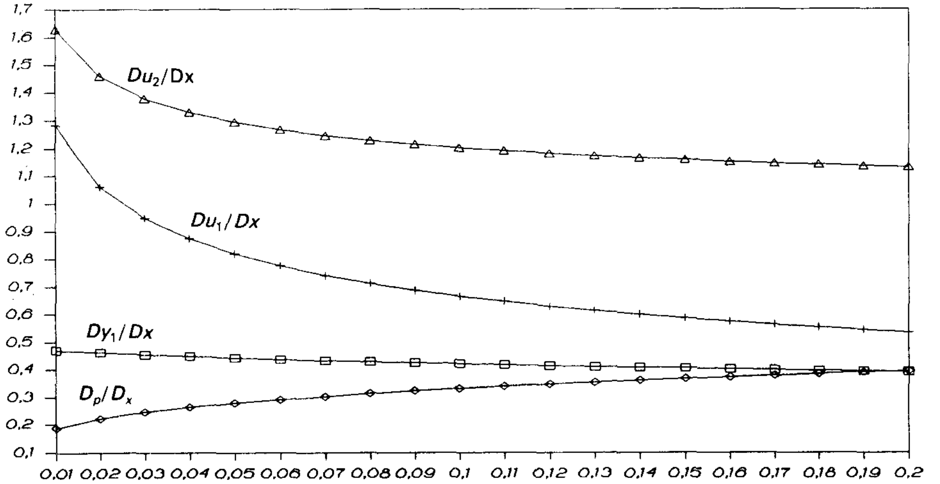

FIGURE 3.2. As Figure 3.1 but $w_1/w_2 = 1$.

In both cases Du_1, Du_2 and Dy_1 are decreasing functions of w_1, whereas Dp increases with w_1. For Du_1 and Dy_1 this is natural since the increasing w_1 means that an increase Du_1 is considered more serious and a smoother flow of u_1 is achieved by a smoother y_1. The decrease in Du_2 obviously emerges from the constancy of the ratio w_1/w_2; i.e. when w_1 increases w_2 also increases.

EXAMPLE 2. Assume that $s = q = 0$ and $d = 1$; i.e. $x(t)$ is a random walk process. As noted above, this case can be viewed as a cautious approximation which in a way constitutes an "upper limit" for actual claims processes. Now transformation (3.5) reads

$$\begin{pmatrix} Z(1,t+1) \\ Z(2,t+1) \\ Z(3,t+1) \\ Z(4,t+1) \end{pmatrix} = \begin{pmatrix} r_1+1 & 1 & 0 & 0 \\ -r_1 & 0 & 0 & 0 \\ 0 & 0 & r_2+1 & 1 \\ 0 & 0 & -r_2 & 0 \end{pmatrix} \begin{pmatrix} Z(1,t) \\ Z(2,t) \\ Z(3,t) \\ Z(4,t) \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ -1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \Delta y_1(t) \\ \Delta p(t) \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \varepsilon(t) \tag{3.20}$$

Choose $Q_0 = 0_4$, $Q_1 = \begin{pmatrix} w_1 & 0 & 0 & 0 \\ 0 & 0.0001 & 0 & 0 \\ 0 & 0 & w_2 & 0 \\ 0 & 0 & 0 & 0.0001 \end{pmatrix}$ and Q_2 as in (3.16).

Thus, instead of Dy_1 and Dp we now consider $D(\Delta y_1)$ and $D(\Delta p)$. Note also that Dp has now to be infinite if Du_1 and Du_2 are to be finite. Take $r_1 = r_2 = 1.0$ and $w_1 = 0.01$, $w_2 = 0.05$, $w_3 = 0.5$ and $w_4 = 1.0$. The two elements on the diagonal of Q_1 other than w_1 and w_2 cannot be taken as zero, since they must be positive in order to obtain a positive definite matrix. However, they are so small that their effect on the results is insignificant. Then the steady-state solution is in the feedback form

$$\begin{cases} \Delta y_1(t) = -0.433\,u_1(t-1) - 0.352\,u_1(t-2) + 0.294\,u_2(t-1) + 0.172\,u_2(t-2) \\ \Delta p(t) = 0.374\,u_1(t-1) - 0.317\,u_1(t-2) - 0.521\,u_2(t-1) - 0.403\,u_2(t-2) \end{cases} \tag{3.21}$$

with corresponding variances

$$\begin{cases} D^2 u_1 = 6.02\,\sigma_\varepsilon^2 \\ D^2(\Delta y_1) = 0.14\,\sigma_\varepsilon^2 \\ D^2 u_2 = 4.19\,\sigma_\varepsilon^2 \\ D^2(\Delta p) = 0.43\,\sigma_\varepsilon^2 \end{cases} \tag{3.22}$$

Figures 3.3-3.4 show the steady-state standard deviations Du_1, $D(\Delta y_1)$, Du_2 and $D(\Delta p)$ of the optimal schemes as a functions of w_3 when $w_1 = 0.01$, $w_4 = 1$, $w_3/w_2 = 10$ or $= 1$.

4. Concluding remarks

The results of the paper should not be seen as suggestions for explicit solutions to be used in reinsurance treaties. In practical situations there are many factors to be taken into account, which however cannot easily be included in a mathematical model. The main emphasis of the paper is on demonstrating an

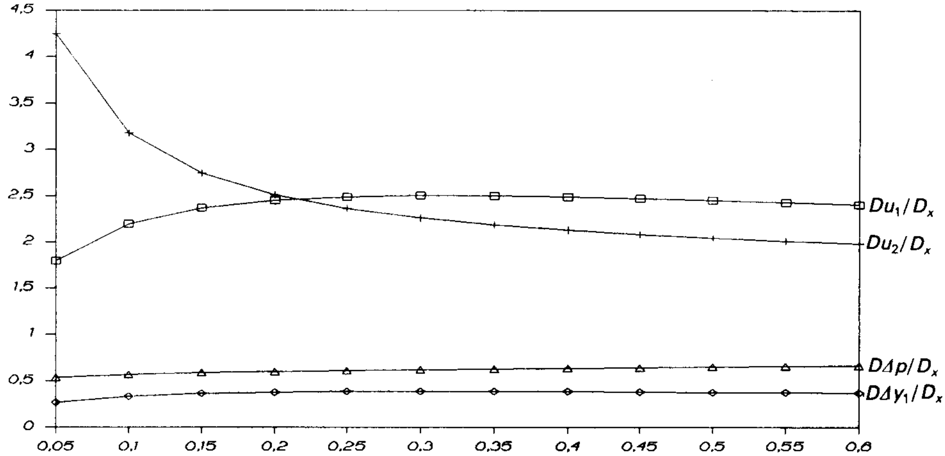

FIGURE 3.3. Steady-state Du_1, $D(\Delta y_1)$, Du_2 and $D(\Delta p)$ of the optimal schemes as functions of w_3 when $w_1 = 0.01$, $w_4 = 1$ and $w_3/w_2 = 10$ and $r_1 = r_2 = 1.0$.

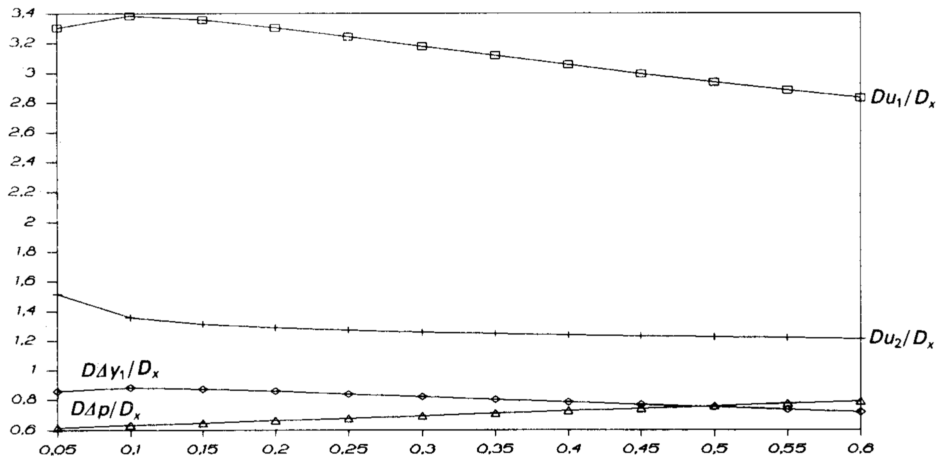

FIGURE 3.4. As figure 3.3 but $w_3/w_2 = 1$.

approach which would be considered as a rational means of tackling reinsurance problems. That is

1) cedant's and reinsurer's share of the claims are functions of the total claims amount in the reinsured part of the portfolio (i.e. they do not depend on individual risks)
2) the agreement is made on a long-term basis
3) an explicit definiton of the goals and criteria of both parties involved (such as acceptable variations in accumulated profits and in annual profits,

profitability in the long run, the rating procedure of the cedant etc.) (compare also BOHMAN (1986) and GERATHEWOHL-NIERHAUS (1986)).

In this way one may succeed in giving more weight to the most relevant factors related to a reinsurance treaty than in a heuristic approach.

This paper concentrates on point (3): how methods of stochastic control theory might be used in a search for the optimal reinsurance formulas (in Section 3 also for the rating formla), when the goals and criteria are expressed in terms of the variances of certain important variables. These rules could be applied if a sufficient consensus on the criteria and on the stochastic properties of the claims process is achieved. If there is considerable uncertainty about those properties then the formula candidates should be tested against various claims process alternatives.

APPENDIX 1

MINIMIZATION OF $Dx_1(t)$ SUBJECT TO A CONSTRAINT ON $Du_2(t)$ WITH CONSTANT PREMIUM RATES

It is assumed that the claims rate process $x(t)$ is a weakly stationary process given by equation

(A1.1) $$x(t) = \Psi(B)\,\varepsilon(t) = \varepsilon(t)+\psi_1\varepsilon(t-1)+\psi_2\varepsilon(t-2)+\ \dots,$$

where $\varepsilon(t)$ is the noise process of uncorrelated random variables with mean zero and with variance σ_ε^2, and ψ_j's are the weights of past $\varepsilon(t)$'s such that $\Sigma\psi_j^2 < \infty$ and B is the backward shift operator: $B\varepsilon(t) = \varepsilon(t-1)$. However, the explicit solution is given only for the case where ψ_j's are generated by an AR(2) claims process.

It is assumed that $x(t)$, $x(t-1)$, ... are used to determine $x_1(t)$. Thus the optimal scheme can be written as the output of a linear filter $L(B)$:

(A1.2) $$x_1(t) = L(B)\,\varepsilon(t),$$

or equivalently

(A1.3) $$x_1(t) = L(B)\,\Psi^{-1}(B)\,x(t),$$

where $^{-1}$ denotes the inverse operator. If $x_1(t)$ should be a function of delayed $x(t)$'s: $x(t-d)$, $x(t-1-d)$, ... with $d < 0$ then $L(B)$ should be replaced by $B^d L(B)$ and the formulas and equations to be presented below should be correspondingly modified (see RANTALA (1984), Appendices I and II).

Let $-\mu(B)$ be the linear filter corresponding to (A1.3) and transforming $\varepsilon(t)$ into $u_2(t)$; i.e.

(A1.4) $$u_2(t) = -\mu(B)\,\varepsilon(t) = -\mu(B)\,\Psi^{-1}(B)\,x(t),$$

where we have temporarily assumed that $p = p_1 = p_2 = 0$.

Thus $\mu(B)$ and $L(B)$ are connected via equation

(A1.5) $$L(B) = -(1-r_2B)\,\mu(B)+\Psi(B).$$

https://doi.org/10.2143/AST.19.2.2014906 Published online by Cambridge University Press

Obviously the minimum possible variance of $u_2(t)$ is zero, which results with the reinsurance scheme $L(B) = \Psi(B)$; i.e. the total business is taken over by the cedant.

The optimization problem stated in the title can be solved by finding the unrestricted minimum of

$$\frac{D^2 x_1(t)}{\sigma_\varepsilon^2} + v \cdot \left[\frac{D^2 u_2(t)}{\sigma_\varepsilon^2} - w \right], \tag{A1.6}$$

where v is the Lagrange multiplier and $w\sigma_\varepsilon^2$ the value allowed for $D^2 u_2(t)$.

The autocovariance-generating function for the autocovariances γ_k $(k = \ldots, -2-1, 0, 2, \ldots)$ is defined by (see BOX-JENKINS (1976)),

$$\gamma(B) = \sum_{k=-\infty}^{\infty} \gamma_k B^k, \tag{A1.7}$$

where B now is a complex variable.

If $x(t) = \Psi(B)\,\varepsilon(t)$, it is easy to see that the autocovariances of $x(t)$ are generated by

$$\gamma(B) = \Psi(B)\,\Psi(F), \tag{A1.8}$$

where $F = B^{-1}$.

Applying this technique to the minimization of (A1.6) we can equivalently require an unrestricted minimum of the coefficient of $B^0 = 1$ in the expression

$$G(B) = L(B)\,L(F) + v\mu(B)\,\mu(F). \tag{A1.9}$$

Regarding (A1.5) we obtain

$$\begin{aligned} G(B) = {} & [(1-r_2 B)(1-r_2 F)+v]\,\mu(B)\,\mu(F) - \\ & -(1-r_2 B)\,\mu(B)\,\Psi(F)-(1-r_2 F)\,\mu(F)\,\Psi(B)+\Psi(B)\,\Psi(F). \end{aligned} \tag{A1.10}$$

By differentiating $G(B)$ with respect to each μ_i $(i = 0, 1, 2, \ldots)$, we obtain

$$\begin{aligned} \frac{\partial}{\partial \mu_i} G(B) = {} & [1+r_2^2+v-r_2 B-r_2 F]\,[B^i \mu(F)+F^i \mu(B)] - \\ & - \Psi(F)\,[B^i - r_2 B^{i+1}] - \Psi(B)\,[F^i - r_2 F^{i+1}]. \end{aligned} \tag{A1.11}$$

After selecting the coefficients of $B^0 = 1$, and equating them to zero, we obtain the following equations:

$$r_2\mu_1 - b\mu_0 = r_2\psi_1 - 1 \qquad (i = 0) \tag{A1.12}$$

$$r_2\mu_{i+1} - b\mu_i + r_2\mu_{i-1} = r_2\psi_{i+1} - \psi_i \qquad (i \geq 1), \tag{A1.13}$$

where

(A1.14) $$b = 1 + r_2^2 + v.$$

REMARK. From (A1.12) and (A1.13) we obtain a relation for the characteristic function of μ which—if μ_0 is known—determines μ:

$$\mu(z)(r_2 + r_2 z^2 - bz) = \psi(z)(r_2 - z) - r_2 + r_2 \mu_0.$$

The solution of (A1.12)-(A1.13) is the sum of the solution of the corresponding homogeneous equation and any particular solution of the homogeneous equation.

First the solution of the homegeneous difference equation

(A1.15) $$r_2 \mu_{i+2} - b\mu_{i+1} + r_2 \mu_i = 0 \qquad (i = 0, 1, 2, \ldots)$$

is sought. The characteristic equation is

(A1.16) $$r_2 z^2 - bz + r_2 = 0;$$

i.e.

(A1.17) $$r_2 z + r_2 z^{-1} = b.$$

Thus if z_0 is a solution so is z_0^{-1} and the general solution of (A1.15) is

(A1.18) $$\mu_i = A z_0^i + A' z_0^{-i} \qquad (i = 0, 1, 2, \ldots).$$

Now, if z_0 has a modulus less than or equal to one, then z_0^{-1} has a modulus greater than or equal to one, and since $u_2(t)$ in the optimal solution must have finite variance, A' must be zero. Because of the property (A1.17) it is easy to see that z must be real. Thus the general solution of (A1.15) is $\mu(B) = A(1 - z_0 B)^{-1}$.

In deriving the particular solution of (A1.12)-(A1.13) we confine ourselves to autoregressive processes of at most order two; i.e. we assume that the weights are given by

(A1.19) $$\Psi(B) = (1 - \phi_1 B - \phi_2 B^2)^{-1}$$

and ϕ_1 and ϕ_2 are constants satisfying stationary conditions (2.1.2).

It can be shown (see RANTALA (1984), Appendix II) and is easy to check that the solution of (A1.12)-(A1.13) is then

(A1.20) $$\mu(B) = A(1 - z_0 B)^{-1} + (W_1 + W_2 B)(1 - \phi_1 B - \phi_2 B^2)^{-1},$$

where the second term on the r.h.s. is a particular solution. Coefficients A, W_1 and W_2 are given by equations

(A1.21)
$$\begin{cases} W_1 = \sqrt{-\phi_2}\,(D_1 \cos\theta + D_2 \sin\theta) \\ W_2 = -\phi_1 W_1 - \phi_2 (D_1 \cos 2\theta + D_2 \sin 2\theta) \\ \tan\theta = \sqrt{\dfrac{-\phi_1^2 - 4\phi_2}{\phi_1}} \qquad (0 \le \theta \le \pi) \\ D_1 = \dfrac{C_1 E_1 + C_2 E_2}{E_1^2 + E_2^2} \sqrt{-\phi_2} \\ D_2 = \dfrac{C_2 E_1 - C_1 E_2}{E_1^2 + E_2^2} \sqrt{-\phi_2} \\ E_1 = \dfrac{r_2 \phi_1}{2\sqrt{-\phi_2}} (1-\phi_2) - b\sqrt{-\phi_2} \\ E_2 = r_2 \sqrt{1 + \phi_1^2/4\phi_2} \cdot (1+\phi_2) \\ C_1 = r_2 \phi_1 - 1 \\ C_2 = \dfrac{(r_2\phi_1 - 1)\phi_1 + 2 r_2 \phi_2}{\sqrt{-\phi_1^2 - 4\phi_2}} \\ A = r_2^{-1} z_0 \cdot [D_1 (r\sqrt{-\phi_2}\cos\theta - b) + D_2 r \sqrt{-\phi_2} \sin\theta - r\phi_1 + 1] \end{cases}$$

when the roots of

(A1.22)
$$z^2 - \phi_1 z + \phi_2 = 0$$

are complex, and

(A1.23)
$$\begin{cases} W_1 = D_1 K_1 + D_2 K_2 \\ W_2 = -K_1 K_2 (D_1 + D_2) \\ D_1 = \dfrac{C_1 K_1}{r_2 K_1^2 - b K_1 + r_2} \\ D_2 = \dfrac{C_2 K_2}{r_2 K_2^2 - b K_2 + r_2} \\ C_1 = \dfrac{K_1 (1 - r_2 K_1)}{K_2 - K_1} \\ C_2 = -\dfrac{K_2 (1 - r_2 K_2)}{K_2 - K_1} \\ A = r_2^{-1} z_0 \cdot [D_1 (r K_1 - b) + D_2 (r K_2 - b) - r\phi_1 + 1] \end{cases}$$

when the roots K_1 and K_2 of (A1.22) are real and distinct.

When $K_1 = K_2 = K$ the following equations are obtained

$$\text{(A1.24)} \quad \begin{cases} C_1 = 2r_2 K - 1 \\ C_2 = r_2 K - 1 \\ D_2 = \dfrac{C_2 K}{r_2 K^2 - bK + r_2} \\ D_1 = \dfrac{C_1 K + r_2 D_2 (1 - K^2)}{r_2 K^2 - bK + r_2} \\ W_1 = (D_1 + D_2) K \\ W_2 = -D_1 K^2 \\ A = r_2^{-1} z_0 \cdot [(D_1 + D_2)(rK - b) - r\phi_1 + 1] . \end{cases}$$

Now the optimal reinsurance scheme may be found by substituting (A1.20) into (A1.5). As can be seen from equations (2.1), (A1.2)-(A1.5), the resulting difference equations for x_1, u_1 and u_2 are

$$\text{(A1.25)} \qquad x_1(t) = [-(1 - r_2 B)\,\mu(B)\,\Phi(B) + 1]\, x(t)$$

or equivalently

$$\text{(A1.26)} \qquad x_1(t) = [-(1 - r_2 B)\,\mu(B) + \Phi^{-1}(B)]\, \varepsilon(t) ,$$

$$\text{(A1.27)} \qquad (1 - r_1 B)\, u_1(t) = -[-(1 - r_2 B)\,\mu(B)\,\Phi(B) + 1]\, x(t) + p_1$$

and

$$\text{(A1.28)} \qquad \Phi^{-1}(B)\, u_2(t) = -\mu(B)\, x(t) + p_2/(1 - \phi_1 - \phi_2)(1 - r_2) .$$

In (A1.27) and (A1.28) the effects of non-zero premium rates are taken into account. Processes $x_1(t)$, $u_1(t)$ and $u_2(t)$ are ARMA processes whose variances are easy to compute from the presentations based on the noise process $\varepsilon(t)$.

APPENDIX 2

MINIMIZATION OF $D(\Delta x_1(t))$ SUBJECT TO A CONSTRAINT ON $Du_2(t)$ WITH CONSTANT PREMIUM RATES

Assume again that the total claims rate $x(t)$ is given by (A1.1). Moreover, in order to shorten the notations assume that $p = p_1 = p_2 = 0$.

By defining the change in the retained claims rate in the optimal linear scheme as

https://doi.org/10.2143/AST.19.2.2014906 Published online by Cambridge University Press

(A2.1) $$\Delta x_1(t) = (1-B)\, x_1(t) = L(B)\, \varepsilon(t)$$

we can proceed analogously to Appendix 1. The resulting difference equations are

(A2.2) $$(i = 0): r_2\mu_2-(r_2+1)^2\mu_1+c\mu_0 = r_2\psi_2-(2r_2+1)\,\psi_1+(r_2+2)\,,$$

(A2.3) $$(i = 1): r_2\mu_3-(r_2+1)^2\mu_2+c\mu_1-(r_2+1)^2\mu_0$$
$$= r_2\psi_3-(2r_2+1)\,\psi_2+(r_2+2)\,\psi_1-1$$

(A2.4) $$(i \geq 2): r_2\mu_{i+2}-(r_2+1)^2\mu_{i+1}+c\mu_i-(r_2+1)^2\mu_{i-1}+r_2\mu_{i-2}$$
$$= r_2\psi_{i+2}-(2r_2+1)\,\psi_{i+1}+(r_2+2)\,\psi_i-\psi_{i-1}\,,$$

where

(A2.5) $$c = 2(1+r_2+r_2^2)+v\,.$$

Thus we have to solve a difference equation of order four. The homogeneous equation is solvable by the methods presented in BOX-JENKINS (1976), Section 13.2.

The characteristic equation corresponding to difference equation (A2.4) is

(A2.6) $$r_2z^4-(r_2+1)^2z^2+cz^2-(r_2-1)^2z+r_2 = 0\,.$$

Hence, if z is a solution so is z^{-1}. Let the roots be K_1, K_1^{-1}, K_2 and K_2^{-1} with $|K_1| < 1$ and $|K_2| < 1$. If $v = 0$ then the roots of (A2.6) are 1, r_2 and r_2^{-1}. Then the modulus of only one root is less than 1. To rule out this case we assume that $v > 0$.

In subsequent applications we need only coefficients $k_0 = K_1+K_2$ and $k_1 = K_1K_2$. They can be found by the following procedure (see BOX-JENKINS (1976)):

(I) Compute $M = (1+r_2)^2/r_2$ and $N = [(1+r_2)^2+(1+r_2^2)+v]/r_2$

for a series of values of v chosen to provide a suitable range for Du_2 and $D\Delta x_1$.

(II) Compute $z_1 = 0.5(N-2) + \sqrt{0.25(N-2)^2+2N-M^2}$

and $z_2 = 0.5(N-2) - \sqrt{0.25(N-2)^2+2N-M^2}$.

(III) Compute $k_1 = 0.5z_1 - \sqrt{(0.5z_1)^2-1}$

and $k_0 = \sqrt{k_1(z_2+2)}$.

The general solution of the homogeneous equation is

(A2.7) $$\mu_i = A_1K_1^i+A_1'K_1^{-1}+A_2'K_2^i+A_2'K_2^{-i} \qquad (i = 0, 1, 2, \ldots)\,.$$

In this solution A_1' and A_2' must be zero because in the optimal solution the solvency rate cannot have infinite variance. Hence

(A2.8) $\mu_i = A_1 K_1^i + A_2 K_2^i, \quad |K_1| < 1, \quad |K_2| < 1 \quad (i = 0, 1, 2, \ldots)$.

This solution is the same, apart from coefficients A_1 and A_2, for every $x(t)$ process. The exact solution contains features which are specific to individual $x(t)$ processes; i.e. it depends on the particular solution of (A2.2)-(A2.4).

For the case $\Psi(B) = (1-\phi B)^{-1}$ with $|\phi| < 1$ a particular solution of (A2.2)-(A2.4) is easy to find. In fact, a particular solution is given by

(A2.9) $$\mu_i = D\phi^i \qquad (i = 1, 2, \ldots),$$

where

(A2.10) $$D/\phi = \frac{r_2(\phi-1)^2(\phi-r_2^{-1})}{r_2\phi^4-(r_2+1)^2\phi^3+c\phi^2-(r_2+1)^2\phi+r_2}.$$

Constants A_1 and A_2 can be determined from initial conditions (A2.2) and (A2.3), giving

(A2.11) $$\begin{cases} A_1 = \dfrac{K_1^2\left(\dfrac{r_2 D K_2}{\phi^2}+\dfrac{K_2}{\phi}-\dfrac{r_2 D}{\phi}\right)}{r_2(K_1-K_2)} \\[2ex] A_2 = \dfrac{K_2^2\left(\dfrac{r_2 D K_1}{\phi^2}+\dfrac{K_1}{\phi}-\dfrac{r_2 D}{\phi}\right)}{r_2(K_2-K_1)}. \end{cases}$$

In deriving $\mu(B)$ and $L(B)$ it is useful to observe that

(A2.12) $$A_1+A_2 = Dk_1/\phi^2+k_1/r_2\phi-Dk_0/\phi$$

and

(A2.13) $$A_1K_2+A_2K_1 = -k_1 D/\phi.$$

The final solution is

(A2.14) $$\mu_i = A_1K_1^i+A_2K_2^i+D\phi^i \qquad (i = 0, 1, 2, \ldots)$$

or equivalently

(A2.15) $$\mu(B) = \frac{\mu_0+\mu_1 B}{1-k_0B+k_1B^2}+\frac{D}{1-\phi B},$$

where (see (A2.12) and (A2.13))

(A2.16) $$\mu_0 = A_1+A_2$$

and

(A2.17) $$\mu_1 = -(A_1 K_2 + A_2 K_1).$$

Thus the final formulas are:

(A2.18) $$x_1(t) = [-(1-r_2 B)(1-\phi B)\mu(B)+1]\, x(t)$$

or

(A2.19) $$x_1(t) = [-(1-r_2 B)\mu(B)+(1-\phi B)^{-1}]\,\varepsilon(t),$$

(A2.20) $$(1-r_1 B)\, u_1(t) = -x_1(t),$$

(A2.21) $$u_2(t) = -\mu(B)\,\varepsilon(t).$$

The necessary coefficients can be found from equations (A2.5), procedure I-III, (A2.10), (A2.11)-(A2.13) and (A2.15)-(A2.17).

The corresponding variances can most easily be calculated from the presentations containing $\varepsilon(t)$'s. Note that the effect of the constant premium rates p, p_1 and p_2 is not shown in equations (A2.18)-(A2.21), since we assumed the rates to be identically zero.

Next, the random walk claims process is considered. For this purpose we take a slightly more general process by assuming that

(A2.22) $$\Delta x(t) = (1-\theta B)\,\varepsilon(t)$$

with $\varepsilon(t)$'s uncorrelated; i.e. $x(t)$ is an ARIMA (0, 1, 1) process.

When looking for the solution we can proceed analogously with the considerations earlier in this Appendix. Now the following difference equations are obtained:

(A2.23) $$r_2\mu_2-(r_2+1)^2\mu_1+c\mu_0 = 1+(r_2+1)\theta \qquad (i = 0)$$

(A2.24) $$r_2\mu_3-(r_2+1)^2\mu_2+c\mu_1-(r_2^2+1)^2\mu_0 = -\theta \qquad (i=1)$$

(A2.25) $$r_2\mu_{i+2}-(r_2+1)^2\mu_{i+1}+c\mu_i-(r_2+1)^2\mu_{i-1}+r_2\mu_{i-2} = 0 \quad (i \geq 2)$$

The solution of this difference equation is exactly the same as that of the homogeneous equation above; i.e.

(A2.26) $$\mu_i = A_1 K_1^i + A_2 K_2^i, \quad |K_1| < 1, \quad |K_2| < 1 \quad (i = 0, 1, 2, \ldots)$$

and K_1 and K_2 are the solutions of equation (A2.6). Constants A_1 and A_2 can be computed from intial conditions (A2.23) and (A2.24).

For all θ $\mu(B)$ is of the form

(A2.27) $$\mu(B) = \frac{\mu_0+\mu_1 B}{1-k_0 B+k_1 B^2},$$

where

$$\mu_0 = A_1 + A_2 = r_2^{-2}[r_2-r_2\theta-\theta]\,k_1 + r_2^{-1}\theta k_0$$

and

$$\mu_1 = -(A_1 K_2 + A_2 K_1) = -r_2^{-1} k_1 \theta .$$

White noise case $\theta = 1$ gives $\mu_0 = -r_2^{-2} k_1 + r_2^{-1} k_0$ and $\mu_1 = -r_2^{-1} k_1$ and the random walk case $\theta = 0$ gives $\mu = r_2^{-1} k_1$ and $\mu_1 = 0$.

REFERENCES

BEARD, R.E., PENTIKÄINEN, T. and PESONEN, E. (1984) *Risk Theory*. Chapman and Hall, London.

BOHMAN, H. (1986) Reinsurance, Profitability and Insurability. *The Geneva Papers on Risk and Insurance* **11**, 126–129.

BOX, G.P.E. and JENKINS, G.M. (1976) *Time-series analysis*. Holden Day, New York.

GERATHEWOHL, K. and NIERHAUS, F. (1986) Princing Proportional Reinsurance. *Nordisk Försäkringstidsskrift* **66**, 274–288.

GERBER, H.U. (1979) *An Introduction to Mathematical Risk Theory*. S.S. Huebner Foundation, University of Pennsylvania.

GERBER, H.U. (1984) Chains of Reinsurance. *Insurance: Mathematics and Economics* **3**, 43–48.

KREMER, E. (1982) Credibility Theory for Some Evolutionary Models. *Scandinavian Actuarial Journal*, 129–142.

LEMAIRE, J. and QUAIRIERE, J.-P. (1986) Chains of Reinsurance Revisited. *ASTIN Bulletin* **16**, 77–88.

PENTIKÄINEN, T. and RANTALA, J. (ed.) (1982) *Solvency of Insurers and Equalization Reserves*, Vol. I and II. The Insurance Publishing Company, Helsinki.

PESONEN, M. (1984) Optimal Reinsurance. *Scandinavian Actuarial Journal*, 65–90.

RANTALA, J. (1984) *An application of stochastic control theory to insurance business*. Ph.D. Thesis, University of Tampere.

RANTALA, J. (1986) Experience rating of ARIMA claims processes by using Kalman filter. *ASTIN Bulletin* **16**, 19–31.

RANTALA, J. (1988) Fluctuations in Insurance Business Results: Some Control-Theoretical Aspects. General lecture on Subject 1. The 23rd International Congress of Actuaries, Helsinki.

ÅSTRÖM, K.J. (1970) *Introduction to Stochastic Control Theory*. Academic Press, New York.

DR. JUKKA RANTALA
Ministry of Social Affairs and Health, P.O. Box 267, SF-00171 Helsinki, Finland.