LETTER TO THE EDITOR

Dear Editor,

Characterizing the exponential distribution

In a recent paper, Grosswald, Kotz and Johnson (1980) proved that, under certain conditions, convolution and relevation are the same only when one of the distributions involved is the exponential. More precisely, suppose Y_1, Y_2 are non-negative variables with survivor functions $S_1(t)$, $S_2(t)$ ($t \geqq 0$), $S_i(0) = 1$. Then Grosswald et al. prove that

$$(1) \qquad \int_0^t S_2(t-x)S_1(dx) = \int_0^t \{S_2(t)/S_2(x)\}S_1(dx)$$

for all S_1 if and only if $S_2(t) = \exp(-at)$, provided S_2 has a power series expansion. They conjectured the result is still true when S_2 is merely assumed continuous. Their methods are entirely analytical.

It is always satisfying to prove probabilistic theorems by essentially probability arguments. What follows is a short probability-based proof of the above result assuming only that S_2 is continuous.

Consider a sequence $X_0, X_1, \cdots$ of non-negative i.i.d. variables with continuous survivor function S_2. Construct the sequence of upper record values $U_0, U_1, \cdots$ (see Shorrock (1972)), with $U_0 = X_0$, and survivor functions $G_0, G_1, \cdots$, say. Note that $G_0 = S_2$. Then the right side of (1), with $S_1 = G_{n-1}$, equals $G_{n-1} - G_n$. Because (1) is supposed true for all S_1, it follows by induction that G_n is also the survivor function for the nth point of a renewal process with common interval survivor function S_2. But the point process $\{U_i\}$ has independent increments (if this is not clear, a proof may be found in Shorrock (1972)), hence so does the renewal process which must therefore be Poisson. Thus S_2 is exponential. The converse result is trivial.

Clearly, a similar idea will handle the discrete case.

For a further example of a probabilistic method, see Isham et al. (1975).

I have recently learnt that L. A. Stefanski (University of North Carolina at Chapel Hill) has proved the characterization with no prior assumptions on S_2. He also gives an example of an S_1 and non-exponential S_2 for which (1) holds.

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Yours sincerely,

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References

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