16

Minkowski-like spacetimes

This chapter studies the existence and stability of Minkowski-like spacetimes, that is, solutions to the vacuum Einstein field equations with vanishing cosmological constant. The main result of this chapter is very similar in spirit to the main result concerning the global existence and stability of de Sitter-like spacetimes of Chapter 15. There is, however, a key difference: while the results in Chapter 15 are global in nature, the ones in the present chapter are ***semiglobal***. More precisely, the spacetimes to be discussed arise as the development of suitable initial data on hyperboloidal hypersurfaces – an examination of the Penrose diagram of the Minkowski spacetime in Figure 16.1 reveals that these hypersurfaces are not Cauchy hypersurfaces of the spacetime. Accordingly, only a portion of the whole spacetime can be recovered from this type of initial value problem. The main result of this chapter can be formulated as follows:

Theorem *(semiglobal existence and stability of Minkowski-like spacetimes).* *Small enough perturbations of hyperboloidal initial data for the Minkowski spacetime give rise to solutions to the vacuum Einstein field equations which exist globally towards the future and have an asymptotic structure similar to that of the Minkowski spacetime.*

This result was first proved in Friedrich (1986b) and subsequently extended to the Einstein-Yang-Mills equations in Friedrich (1991). The original proof of the result made use of the standard conformal Einstein field equations and is similar to the argument given for the de Sitter spacetime in Section 15.3. In this chapter a proof of the theorem is given which makes use of the extended conformal field equations and conformal Gaussian systems following the ideas in Lübbe and Valiente Kroon (2009). This approach allows for a more detailed and explicit discussion of the structure of the conformal boundary.

The restriction of the analysis of the present chapter to the hyperboloidal initial value problem may seem mysterious at first sight. As will be discussed in some detail in Chapter 20, the initial data for the conformal Einstein field

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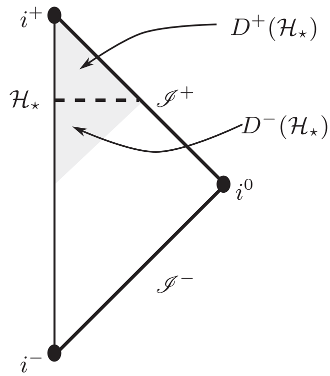

Figure 16.1 Penrose diagram of the Minkowski spacetime and the regions that can be recovered from data on the standard hyperboloid $\mathcal{H}_\star$. The future and past domains of dependence $D^+(\mathcal{H}_\star)$ and $D^-(\mathcal{H}_\star)$ are depicted in grey shading. Observe that $\mathcal{H}_\star$ is not a Cauchy hypersurface as there are portions of the conformal diagram that cannot be recovered from the data on $\mathcal{H}_\star$.

equations on an asymptotically Euclidean (Cauchy) hypersurface is generically singular at spatial infinity – the various issues associated to this singular behaviour are usually known as the ***problem of spatial infinity***.

Despite the above limitation, hyperboloidal initial value problems arise naturally in evolution problems in which the behaviour of gravitational radiation is the main concern; see, for example, Rinne and Moncrief (2013) or Zenginoglu (2008). While the ADM mass – which is computed on asymptotically flat hypersurfaces (see Section 11.6.1) – is a conserved quantity, the notion of mass associated to hyperboloidal hypersurfaces, the so-called ***Bondi mass***, shows a monotonic behaviour, and so it describes the process of mass loss due to gravitational radiation; see Section 10.4.

16.1 The Minkowski spacetime and the conformal field equations

The first step of the stability analysis is a study of the Minkowski spacetime in the gauge used to deduce the conformal evolution equations of Proposition 13.3.

16.1.1 The basic representation

As discussed in Section 6.2, the Minkowski spacetime $(\mathbb{R}^4, \tilde{\boldsymbol{\eta}})$ can be conformally embedded into the *expanding Einstein cylinder* $(\mathbb{R} \times \mathbb{S}^3, \bar{\boldsymbol{g}}_{\mathscr{E}})$ where

$$\bar{\boldsymbol{g}}_{\mathscr{E}} \equiv \mathbf{d}\bar{\tau} \otimes \mathbf{d}\bar{\tau} - \left(1 + \frac{\bar{\tau}^2}{4}\right)^2 \boldsymbol{\hbar},$$

by means of the conformal rescaling

$$\bar{\boldsymbol{g}}_{\mathscr{E}} = \Theta^2_{\mathscr{M}} \tilde{\boldsymbol{\eta}}, \qquad \Theta_{\mathscr{M}} \equiv 2\cos^2 \frac{\psi}{2}\left(1 - \frac{1}{4}\tan^2 \frac{\psi}{2}\, \bar{\tau}^2\right). \tag{16.1}$$

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The coordinate $\bar{\tau}$ is an affine parameter of the conformal geodesics $(x_{\mathscr{E}}(\bar{\tau}), \bar{\boldsymbol{\beta}}_{\mathscr{E}}(\bar{\tau}))$ with

$$x_{\mathscr{E}}(\bar{\tau}) = (\bar{\tau}, x_{\star}^{\alpha}), \qquad \boldsymbol{x}'_{\mathscr{E}}(\bar{\tau}) = \boldsymbol{\partial}_{\bar{\tau}}, \qquad \bar{\boldsymbol{\beta}}_{\mathscr{E}}(\bar{\tau}) = \frac{2\bar{\tau}}{4+\bar{\tau}^2}\mathbf{d}\bar{\tau}.$$

The underlying geometry of the conformal representation of the Minkowski spacetime described in the previous paragraph is that of the expanding cylinder. Accordingly, the *geometric fields for this conformal representation of the Minkowski spacetime coincide with those of the conformal representation of the de Sitter spacetime* discussed in Section 15.1.2. That is, one has

$$\bar{e}^{\mathbf{0}} = 1, \qquad \bar{e}_{(\boldsymbol{AB})}{}^{\mathbf{0}} = 0, \tag{16.2a}$$

$$\bar{e}^{\boldsymbol{i}} = 0, \qquad \bar{e}_{(\boldsymbol{AB})}{}^{\boldsymbol{i}} = \frac{4}{4+\bar{\tau}^2}\sigma_{\boldsymbol{AB}}{}^{\boldsymbol{i}}, \tag{16.2b}$$

$$f_{\boldsymbol{AB}} = 0, \tag{16.2c}$$

$$\bar{\xi}_{\boldsymbol{ABCD}} = -\frac{4\mathrm{i}}{4+\bar{\tau}^2}h_{\boldsymbol{ABCD}}, \tag{16.2d}$$

$$\bar{\chi}_{(\boldsymbol{AB})\boldsymbol{CD}} = \frac{2\mathrm{i}}{4+\bar{\tau}^2}h_{\boldsymbol{ABCD}}, \tag{16.2e}$$

$$\Theta_{\boldsymbol{AB}} = 0, \qquad \Theta_{\boldsymbol{ABCD}} = -\frac{2}{4+\bar{\tau}^2}h_{\boldsymbol{ABCD}}, \tag{16.2f}$$

$$\phi_{\boldsymbol{ABCD}} = 0. \tag{16.2g}$$

It is important to emphasise, however, that the *conformal gauge fields* $\Theta_{\mathscr{M}}$ and $\boldsymbol{d}_{\mathscr{M}}$, relating the conformal representation to the physical Minkowski spacetime, are different from those of the de Sitter spacetime.

A schematic representation of the conformal boundary associated to the above conformal representation of the Minkowski spacetime is given in Figure 16.2. It is observed that, as a consequence of the explicit time symmetry of the

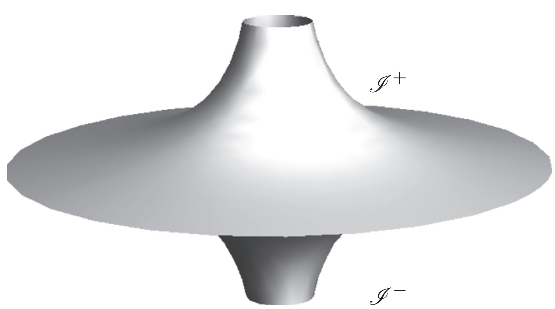

Figure 16.2 Plot of the conformal boundary of the Minkowski spacetime in the conformal Gaussian gauge given by Equation (16.1). This conformal representation is explicitly time symmetric and does not contain the points $i^{\pm}$ representing future and past null infinity. The image has been cropped. This figure is a coordinate plot, not a conformal diagram; thus, null geodesics do not have a slope of 45 degrees.

representation, the points $i^{\pm}$ representing future and past timelike infinity are not included. This representation is not the most convenient one to use in analysing a hyperboloidal initial value problem. A related, more convenient representation is given in the next subsection.

16.1.2 A conformal representation adapted to the standard hyperboloid

As discussed in Section 6.2, the Minkowski spacetime can be embedded into the Einstein cylinder using the conformal factor

$$\Xi_{\mathscr{M}} \equiv \cos\tau + \cos\psi.$$

In the following it will be convenient to shift the above standard embedding by $\pi/2$ to the past with the replacement $\tau \mapsto \check{\tau} + \pi/2$, so that the ***standard Minkowski hyperboloid*** $\mathcal{H}_\star$ which is given by the condition $\tau = \pi/2$ is now located at $\check{\tau} = 0$; see Equation (6.26). Accordingly, one obtains the shifted conformal factor

$$\check{\Xi}_{\mathscr{M}} \equiv \cos\left(\check{\tau} + \frac{\pi}{2}\right) + \cos\psi = \cos\psi - \sin\check{\tau}. \tag{16.3}$$

In particular, the conformal factor embedding the hyperboloidal 3-manifold into $\mathbb{S}^3$ is given by

$$\bar{\Omega} \equiv \cos\psi.$$

One has that $\bar{\Omega} = 0$ at $\psi = \pi/2$. Hence, it is natural to define

$$\partial\mathcal{H}_\star \equiv \left\{ p \in \mathbb{S}^3 \mid \psi(p) = \frac{\pi}{2} \right\}.$$

Observe that $\mathbf{d}\bar{\Omega} \neq 0$ at $\partial\mathcal{H}_\star$.

To relate the conformal representation of the Minkowski spacetime given by the conformal factor in Equation (16.3) to the so-called expanding Einstein cylinder discussed in Section 16.1.1, it is recalled that $\boldsymbol{g}_{\mathscr{E}} = \check{\Xi}^2_{\mathscr{M}}\tilde{\boldsymbol{\eta}}$ and $\bar{\boldsymbol{g}}_{\mathscr{E}} = \bar{\Theta}^2_{\mathscr{E}}\boldsymbol{g}_{\mathscr{E}}$ so that

$$\bar{\boldsymbol{g}}_{\mathscr{E}} = \check{\Theta}^2_{\mathscr{M}}\tilde{\boldsymbol{\eta}}, \qquad \check{\Theta} \equiv \bar{\Theta}_{\mathscr{E}}\check{\Xi}_{\mathscr{M}}.$$

The relation between the *shifted coordinate* $\check{\tau}$ and the affine parameter $\bar{\tau}$ of the conformal geodesics $\big(x_{\mathscr{E}}(\bar{\tau}), \boldsymbol{\beta}_{\mathscr{E}}(\bar{\tau})\big)$ in the Einstein cylinder is, *formally*, the same as the one between the original coordinate τ and $\bar{\tau}$; in particular, one has that $\bar{\tau} = 0$ if $\check{\tau} = 0$. Thus, using the conformal transformation properties of conformal geodesics as described in Section 5.5.2, one finds that the pair $\big(\check{x}_{\mathscr{M}}(\check{\tau}), \check{\boldsymbol{\beta}}_{\mathscr{M}}(\check{\tau})\big)$ with

$$\check{x}_{\mathscr{M}} \equiv (\bar{\tau}, x^\alpha_\star) = \left(\cos^2\frac{\check{\tau}}{2}, x^\alpha_\star\right),$$

$$\check{\boldsymbol{\beta}}_{\mathscr{M}} \equiv \boldsymbol{\beta}_{\mathscr{E}} + \check{\Xi}^{-1}_{\mathscr{M}}\mathbf{d}\check{\Xi}_{\mathscr{M}} = \tan\frac{\check{\tau}}{2}\,\mathbf{d}\check{\tau} + \frac{1}{\sin\check{\tau} - \cos\psi}\big(\cos\check{\tau}\,\mathbf{d}\check{\tau} + \sin\psi\,\mathbf{d}\psi\big)$$

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and

$$\check{\tau} = 2\arctan\frac{\bar{\tau}}{2} \tag{16.4}$$

gives rise to a congruence of conformal geodesics in the Minkowski spacetime adapted to the conformal factor $\check{\Xi}_{\mathscr{M}}$ in Equation (16.3). This congruence can be used to construct a conformal Gaussian gauge system for the Minkowski spacetime.

A calculation using standard trigonometric identities and the relation (16.4) between the parameters $\check{\tau}$ and $\bar{\tau}$ yields the expression

$$\check{\Theta} = \cos\psi\left(1 - \sec\psi\bar{\tau} + \frac{\bar{\tau}^2}{4}\right) \tag{16.5}$$

for the conformal factor associated to the new conformal Gaussian gauge system. This conformal factor vanishes whenever

$$\bar{\tau} = \frac{2 \pm \sin\psi}{\cos\psi}.$$

A plot of this conformal factor can be seen in Figure 16.3. Moreover, the components of the covector $\check{\boldsymbol{d}}_{\mathscr{M}} \equiv \check{\Theta}\check{\boldsymbol{\beta}}_{\mathscr{M}}$ with respect to a Weyl propagated frame $\{\bar{\boldsymbol{e}}_{\boldsymbol{a}}\}$ such that $\bar{\boldsymbol{e}}_{\boldsymbol{0}} = \dot{\boldsymbol{x}}_{\mathscr{M}}$ are given by

$$\check{\beta}_{\boldsymbol{0}} = \partial_{\bar{\tau}}\check{\Theta}_{\mathscr{M}}, \qquad \check{\beta}_{\boldsymbol{i}} = \bar{\boldsymbol{e}}_{\boldsymbol{i}}(\bar{\Omega}).$$

Finally, it follows from the discussion from the previous paragraphs that the geometry of the conformal representation of the Minkowski spacetime given by the conformal factor (16.5) is described by the fields (16.2a)–(16.2g); that is, *the geometry of this representation coincides with that of the expanding Einstein*

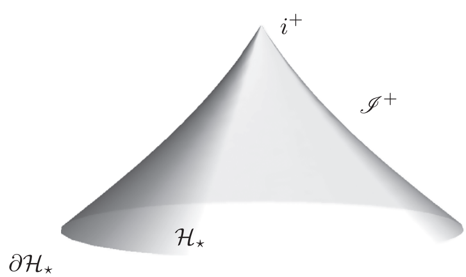

Figure 16.3 Plot of the conformal boundary for the Minkowski spacetime in a conformal Gaussian gauge adapted to the standard hyperboloid. In this particular representation timelike infinity i^+ is at a finite location. The set $\partial\mathcal{H}_\star$ denotes the intersection of the conformal boundary $\mathscr{I}^+$ with the initial hyperboloid $\mathcal{H}_\star$. As in the case of Figure 16.2, this plot is not a conformal diagram.

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cylinder. In particular, a suitable Jacobi field $z_{\boldsymbol{AB}}$ measuring the deviation of the curves of the congruence of conformal geodesics is given by

$$z = 0, \qquad z_{(\boldsymbol{AB})} = \left(1 + \frac{\bar{\tau}^2}{4}\right) z_{\star(\boldsymbol{AB})},$$

where $z_{\star(\boldsymbol{AB})}$ is some fiduciary initial value at the standard hyperboloid $\mathcal{H}_\star$; compare Equation (15.14).

16.1.3 Initial data for the Minkowski spacetime on the standard hyperboloid

The congruence of conformal geodesics giving rise to the conformal Gaussian system is specified on the standard hyperboloid $\mathcal{H}_\star$ by the data

$$\check{\Theta}_\star = \cos\psi, \qquad \check{d}_{\boldsymbol{0}\star} = -1, \qquad \check{d}_{\boldsymbol{i}\star} = \boldsymbol{c_i}(\bar{\Omega}).$$

On $\mathcal{H}_\star$, the conformal fields satisfy the conditions

$$\bar{e}^{\boldsymbol{0}} = 1, \qquad \bar{e}_{(\boldsymbol{AB})}{}^{\boldsymbol{0}} = 0, \tag{16.6a}$$

$$\bar{e}^{\boldsymbol{i}} = 0, \qquad \bar{e}_{(\boldsymbol{AB})}{}^{\boldsymbol{i}} = \sigma_{\boldsymbol{AB}}{}^{\boldsymbol{i}}, \tag{16.6b}$$

$$f_{\boldsymbol{AB}} = 0, \qquad \bar{\xi}_{\boldsymbol{ABCD}} = -\mathrm{i} h_{\boldsymbol{ABCD}}, \qquad \bar{\chi}_{(\boldsymbol{AB})\boldsymbol{CD}} = 0, \tag{16.6c}$$

$$\bar{\Theta}_{\boldsymbol{AB}} = 0, \qquad \bar{\Theta}_{\boldsymbol{ABCD}} = -\frac{1}{2} h_{\boldsymbol{ABCD}}, \tag{16.6d}$$

$$\phi_{\boldsymbol{ABCD}} = 0. \tag{16.6e}$$

16.2 Perturbations of hyperboloidal data for the Minkowski spacetime

In what follows, it is assumed that one has a solution $(\mathcal{H}, \boldsymbol{h}, \boldsymbol{K}, \Omega, \Sigma)$ to the conformal Hamiltonian and momentum constraints, Equations (11.15a) and (11.15b), with hyperboloidal boundary conditions as discussed in Section 11.7. It is convenient to regard the hyperboloidal manifold $\mathcal{H}$ as a region of a 3-manifold $\mathcal{S} \approx \mathbb{S}^3$. Following the conventions of the previous chapters, when regarding the 3-manifolds $\mathcal{H}$ and $\mathcal{S}$ as hypersurfaces of a four-dimensional spacetime one writes $\mathcal{H}_\star$ and $\mathcal{S}_\star$, respectively. One can use coordinates (x^α) on $\mathbb{S}^3$ as coordinates on $\mathcal{H}$ and introduce reference frame and coframe fields $\{\boldsymbol{c_i}\}$ and $\{\boldsymbol{\alpha^i}\}$ by requiring the identification between $\mathcal{S}$ and $\mathbb{S}^3$ to be a harmonic map; see the discussion in Section 15.2.1.

Initial data for the conformal evolution equations can be obtained from the basic initial data $(\mathcal{H}, \boldsymbol{h}, \boldsymbol{K}, \Omega, \Sigma)$ using the procedure described in Section 11.4.3. It will be assumed that the data can be written on the initial hyperboloid $\mathcal{H}_\star$ in the form

$$e^{\mathbf{0}} = 1, \qquad e_{(\boldsymbol{AB})}{}^{\mathbf{0}} = 0, \tag{16.7a}$$

$$e^{\boldsymbol{i}} = 0, \qquad e_{(\boldsymbol{AB})}{}^{\boldsymbol{i}} = \sigma_{\boldsymbol{AB}}{}^{\boldsymbol{i}} + \breve{e}_{(\boldsymbol{AB})}{}^{\boldsymbol{i}}, \tag{16.7b}$$

$$f_{\boldsymbol{AB}} = 0, \tag{16.7c}$$

$$\xi_{\boldsymbol{ABCD}} = \bar{\xi}_{\boldsymbol{ABCD}} + \breve{\xi}_{\boldsymbol{ABCD}}, \tag{16.7d}$$

$$\chi_{(\boldsymbol{AB})\boldsymbol{CD}} = \bar{\chi}_{(\boldsymbol{AB})\boldsymbol{CD}}, \tag{16.7e}$$

$$\Theta_{\boldsymbol{AB}} = \breve{\Theta}_{\boldsymbol{AB}}, \qquad \Theta_{\boldsymbol{ABCD}} = \bar{\Theta}_{\boldsymbol{ABCD}} + \breve{\Theta}_{\boldsymbol{ABCD}}, \tag{16.7f}$$

$$\phi_{\boldsymbol{ABCD}} = \breve{\phi}_{\boldsymbol{ABCD}}, \tag{16.7g}$$

with

$$\bar{\xi}_{\boldsymbol{ABCD}}, \qquad \bar{\chi}_{(\boldsymbol{AB})\boldsymbol{CD}}, \qquad \bar{\Theta}_{\boldsymbol{ABCD}}$$

as given by Equations (16.6a)–(16.6e), while the fields

$$\breve{e}_{(\boldsymbol{AB})}{}^{\boldsymbol{i}}, \qquad \breve{\xi}_{\boldsymbol{ABCD}}, \qquad \breve{\chi}_{(\boldsymbol{AB})\boldsymbol{CD}}, \qquad \breve{\Theta}_{\boldsymbol{AB}}, \qquad \breve{\Theta}_{\boldsymbol{ABCD}}, \qquad \breve{\phi}_{\boldsymbol{ABCD}}$$

describe the perturbation from standard hyperboloidal Minkowski data and $\sigma_{\boldsymbol{AB}}{}^{\boldsymbol{i}}$ are the spatial Infeld-van der Waerden symbols.

While the ***background fields*** $\sigma_{\boldsymbol{AB}}{}^{\boldsymbol{i}}$, $\bar{\xi}_{\boldsymbol{ABCD}}$, $\bar{\chi}_{(\boldsymbol{AB})\boldsymbol{CD}}$, $\bar{\Theta}_{\boldsymbol{ABCD}}$ are defined on the whole of $\mathcal{S} \approx \mathbb{S}^3$, the perturbation fields $\breve{e}_{(\boldsymbol{AB})}{}^{\boldsymbol{i}}$, $\breve{\xi}_{\boldsymbol{ABCD}}$, $\breve{\chi}_{(\boldsymbol{AB})\boldsymbol{CD}}$, $\breve{\Theta}_{\boldsymbol{AB}}$, $\breve{\Theta}_{\boldsymbol{ABCD}}$, $\breve{\phi}_{\boldsymbol{ABCD}}$ are defined only on $\mathcal{H}$. To apply the basic existence and stability result, Theorem 12.4, to the present situation one extends the hyperboloidal initial data set on $\mathcal{H}$ to data on $\mathcal{S}$. Using the *extension theorem*, Proposition 12.2, and given $m \geq 4$ there exists a linear operator $\mathscr{E} : H^m(\mathcal{H}, \mathbb{C}^N) \to H^m(\mathcal{S}, \mathbb{C}^N)$ such that for $\breve{\mathbf{u}}_\star \in H^m(\mathcal{H}, \mathbb{C}^N)$ then $(\mathscr{E}\breve{\mathbf{u}}_\star)(x) = \breve{\mathbf{u}}_\star(x)$ almost everywhere in $\mathcal{H}$ and

$$\parallel \mathscr{E}\breve{\mathbf{u}}_\star \parallel_{m,\mathcal{S}} \leq K \parallel \breve{\mathbf{u}}_\star \parallel_{m,\mathcal{H}},$$

with K a universal constant for fixed m. As in the case of the de Sitter spacetime, the background initial data $\mathring{\mathbf{u}}_\star$ is defined on the whole of $\mathcal{S}$ so that the ***extended data***

$$\mathbf{u}_\star = \mathring{\mathbf{u}}_\star + \mathscr{E}\breve{\mathbf{u}}_\star \tag{16.8}$$

is a well-defined function in $H^m(\mathcal{S}, \mathbb{C}^N)$. The extension of the hyperboloidal data given by (16.8) is non-unique and, in general, will not satisfy the conformal constraint equations on $\mathcal{S} \setminus \mathcal{H}$. As the norm $\parallel \mathscr{E}\breve{\mathbf{u}}_\star \parallel_{m,\mathcal{S}}$ is dominated by the norm $\parallel \breve{\mathbf{u}}_\star \parallel_{m,\mathcal{H}}$, then $\parallel \mathscr{E}\breve{\mathbf{u}}_\star \parallel_{m,\mathcal{S}}$ can be made as small as necessary by making $\parallel \mathbf{u}_\star \parallel_{m,\mathcal{H}}$ suitably small. In complete analogy to the case of the de Sitter spacetime, one says that a hyperboloidal initial data set of the form (16.7a)–(16.7g) is ***ε-small*** (in the norm $\parallel \quad \parallel_{\mathcal{S},m}$) if

$$\parallel \breve{e}_{(\boldsymbol{AB})}{}^{\boldsymbol{i}} \parallel_{\mathcal{S},m} + \parallel \breve{\xi}_{\boldsymbol{ABCD}} \parallel_{\mathcal{S},m} + \parallel \breve{\chi}_{\boldsymbol{ABCD}} \parallel_{\mathcal{S},m}$$
$$+ \parallel \breve{\Theta}_{\boldsymbol{AB}} \parallel_{\mathcal{S},m} + \parallel \breve{\Theta}_{\boldsymbol{ABCD}} \parallel_{\mathcal{S},m} + \parallel \breve{\phi}_{\boldsymbol{ABCD}} \parallel_{\mathcal{S},m} < \varepsilon,$$

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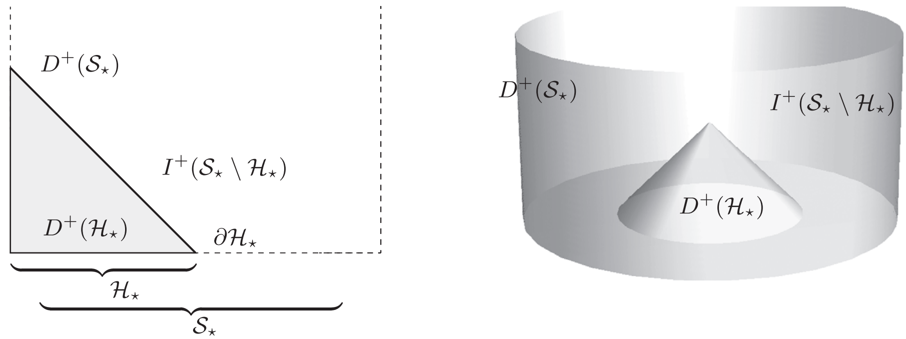

Figure 16.4 Domain of dependence $D^+(\mathcal{H}_\star)$ of data for the conformal evolution equations on a hyperboloid $\mathcal{H}_\star$: on the left, a schematic representation of the setup; on the right, a three-dimensional depiction. To make use of Kato's existence theorem, the data have to be extended to $\mathcal{S}_\star \setminus \mathcal{H}_\star$ where $\mathcal{S}_\star \approx \mathbb{S}^3$. The domain of dependence of the extended data $D^+(\mathcal{S}_\star)$ corresponds, in principle, to the cylinder $[0,\infty)\times\mathbb{S}^3$. The chronological future of the extension on $\mathcal{S}_\star \setminus \mathcal{H}_\star$, denoted by $I^+(\mathcal{S}_\star \setminus \mathcal{H}_\star)$, does not intersect the domain of dependence of the hyperboloidal data $D^+(\mathcal{H}_\star)$, and, thus, it is independent of the particular extension being used.

where it is understood that each of the terms in the above expression comprises a sum over all the independent components of the spinorial field under consideration.

The extended data (16.8) is non-unique. This non-uniqueness does not pose any problem for the considerations of this chapter. While it is true that the development $D^+(\mathcal{S}_\star)$ is clearly dependent on the particular extension of the initial data, one has

$$D^+(\mathcal{H}_\star) \cap I^+(\mathcal{S}_\star \setminus \mathcal{H}_\star) = \emptyset;$$

compare the Remark at the end of Section 14.2. Thus, the particular choice of extension of the data on $\mathcal{H}_\star$ has no effect on $D^+(\mathcal{H}_\star)$; see Figure 16.4 for further details.

16.3 A priori structure of the conformal boundary

This section discusses the available a priori knowledge of the structure of the conformal boundary of the development of hyperboloidal initial data.

In what follows, assume that $\mathcal{H}_\star$ can be regarded as an open subset of a compact manifold $\mathcal{S}_\star \approx \mathbb{S}^3$. Moreover, assume that $\partial\mathcal{H}_\star \approx \mathbb{S}^2$. On $\mathcal{S}_\star$ one considers a conformal factor Ω such that $\Omega > 0$ in the interior of $\mathcal{H}_\star$ and $\Omega = 0$ on $\partial\mathcal{H}_\star$; that is, the conformal factor Ω can be thought of as a boundary-defining function. Consistent with the hyperbolic reduction procedure for the extended conformal field equations as described in Section 13.4, it is assumed that the domain of dependence $D^+(\mathcal{S}_\star)$ can be covered by a non-singular congruence of

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conformal geodesics with data prescribed on $\mathcal{S}_\star$. In particular, it is required that the conformal geodesics are initially orthogonal to $\mathcal{S}_\star$.

Determining the conformal factor

From Proposition 5.1, it follows that the general form of the conformal factor associated to the congruence of conformal geodesics is given by

$$\Theta = \Theta_\star + \dot{\Theta}_\star \bar{\tau} + \frac{1}{2}\ddot{\Theta}_\star \bar{\tau}^2, \tag{16.9}$$

where the coefficients $\Theta_\star$, $\dot{\Theta}_\star$ and $\ddot{\Theta}_\star$ are functions of the spatial coordinates and are subject to the constraints

$$\dot{\Theta}_\star = \langle \boldsymbol{d}_\star, \dot{\boldsymbol{x}}_\star \rangle, \qquad \Theta_\star \ddot{\Theta}_\star = \frac{1}{2}\boldsymbol{g}^\sharp(\boldsymbol{d}_\star, \boldsymbol{d}_\star). \tag{16.10}$$

It is convenient to set

$$\Theta_\star = \Omega$$

and to make the spatial part of $\boldsymbol{d}_\star$ equal to $\mathbf{d}\Omega$. Accordingly, one finds that

$$d_{\boldsymbol{0}\star} = \dot{\Theta}_\star, \qquad d_{\boldsymbol{i}\star} = D_{\boldsymbol{i}}\Omega.$$

Now, letting

$$\alpha \equiv \Omega^{-1}\dot{\Theta}_\star,$$

one finds from the constraints in (16.10) that

$$2\Omega\ddot{\Theta}_\star = \boldsymbol{h}^\sharp(\mathbf{d}\Omega, \mathbf{d}\Omega) + \alpha^2,$$

where it is recalled that $\boldsymbol{h}^\sharp(\mathbf{d}\Omega, \mathbf{d}\Omega) < 0$ as a consequence of the signature convention. Whenever $\Omega = 0$, it follows from the constraints (16.10) that $\boldsymbol{d}_\star$ must be a null covector as $\mathbf{d}\Theta \neq 0$.

Making use of the above observations, one finds that Equation (16.9) takes the particular form

$$\Theta = \Omega\left(1 + \alpha\bar{\tau} + \left(\frac{1}{4}\alpha^2 - \frac{1}{\omega^2}\right)\bar{\tau}^2\right),$$

where

$$\omega \equiv \frac{2\Omega}{\sqrt{|\boldsymbol{h}^\sharp(\mathbf{d}\Omega, \mathbf{d}\Omega)|}}.$$

A calculation shows that $\Theta = 0$ for

$$\bar{\tau}_\pm \equiv \frac{2\alpha\omega^2 \pm 4\omega}{4 - \alpha^2\omega^2}. \tag{16.11}$$

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Accordingly, it is natural to define the ***future (and, respectively, past) null infinity*** of the development associated to the hyperboloid $\mathcal{H}_\star$ as

$$\mathscr{I}^{\pm} \equiv \Big\{ (\bar{\tau}, x) \in \mathbb{R} \times \mathbb{S}^3 \,|\, \bar{\tau} = \bar{\tau}_{\pm}(x) \Big\}.$$

This expression shows how the location of the conformal boundary is predetermined by the initial data Ω and $\boldsymbol{d}_\star$ as long as the underlying congruence of conformal geodesics remains non-singular. As $\Omega \to 0$, one has that either $\bar{\tau}_\pm \to 0$ or $\bar{\tau}_\pm \to -2\dot{\Theta}_\star/\ddot{\Theta}_\star$. It follows that $\mathscr{I}^+$ and $\mathscr{I}^-$ are smooth hypersurfaces whenever $\mathbf{d}\Theta \neq 0$. Moreover, $\partial\mathcal{H}_\star$ is the intersection of $\mathscr{I}^\pm$ with $\mathcal{H}_\star = \{0\} \times \mathcal{H}$ as is to be expected for hyperboloidal data. In analogy to the model case of the hyperboloids in the Minkowski spacetime, *the development of generic hyperboloidal data has a conformal boundary which corresponds to either* $\mathscr{I}^+$ *or* $\mathscr{I}^-$*, but not both*; see Figure 16.5. This information is contained in the sign of the free datum $\dot{\Theta}_\star$. By convention, the conformal factor is positive in the region corresponding to the physical spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$. Accordingly, if $\dot{\Theta}_\star > 0$ on $\partial\mathcal{H}_\star$, then $\tilde{\mathcal{M}}$ lies to the future of the conformal boundary and one speaks of a hyperboloid which intersects past null infinity, and, thus, the conformal boundary is identified with $\mathscr{I}^-$. If, by contrast, $\dot{\Theta}_\star < 0$ on $\partial\mathcal{H}_\star$, then $\tilde{\mathcal{M}}$ lies to the past of the conformal boundary. In this case, the hyperboloid intersects future null infinity and $\mathscr{I}^+$ gives the conformal boundary. *Without loss of generality, in the following, attention will be restricted to hyperboloids intersecting future null infinity so that* $\dot{\Theta} < 0$ *on* $\partial\mathcal{H}_\star$.

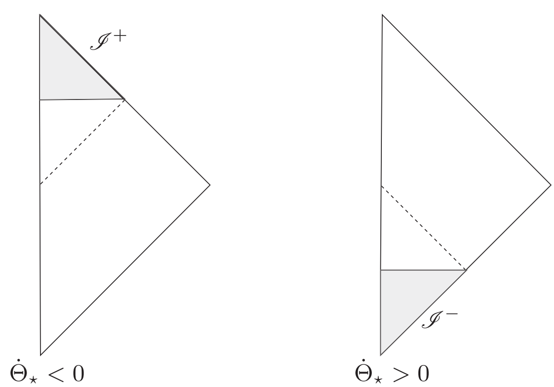

Figure 16.5 The two possible configurations of the conformal boundary for hyperboloidal data as discussed in the main text: on the left, one has the case $\dot{\Theta}_\star < 0$ at $\partial\mathcal{H}_\star$ where the conformal boundary given by the conformal Gaussian gauge system corresponds to $\mathscr{I}^+$; on the right, one has the situation corresponding to $\dot{\Theta}_\star > 0$ at $\partial\mathcal{H}_\star$ so the realised component of the conformal boundary is given by $\mathscr{I}^-$.

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Timelike infinity

To identify the points which can be regarded as representing timelike infinity, one needs to investigate the critical points of Θ on the conformal boundary, that is, the points where $\mathbf{d}\Theta = 0$ and $\bar{\tau} = \bar{\tau}_\pm(x)$. A calculation shows that

$$\mathbf{d}\Theta = \left(1 + \alpha\bar{\tau} + \left(\frac{1}{4}\alpha^2 - \frac{1}{\omega^2}\right)\bar{\tau}^2\right)\mathbf{d}\Omega + \Omega\left(\alpha + 2\bar{\tau}\left(\frac{1}{4}\alpha^2 - \frac{1}{\omega^2}\right)\right)\mathbf{d}\bar{\tau}$$
$$+ \Omega\bar{\tau}\,\mathbf{d}\alpha + \Omega\bar{\tau}^2\left(\frac{1}{2}\alpha\mathbf{d}\alpha + \frac{2}{\omega^3}\mathbf{d}\omega\right).$$

Thus, a necessary condition for having a critical point of Θ on $\mathscr{I}^\pm$ is

$$\alpha + 2\bar{\tau}_\pm\left(\frac{1}{4}\alpha^2 - \frac{1}{\omega^2}\right) = 0.$$

A short computation shows that the above is equivalent to $\boldsymbol{h}(\mathbf{d}\Omega, \mathbf{d}\Omega) = 0$. That is, the critical points of Θ can occur only along conformal geodesics for which $\mathbf{d}\Omega = 0$ on the initial hypersurface $\mathcal{S}_\star$. The standard hyperboloid in the Minkowski spacetime contains precisely one such point. *By continuity, suitably small perturbations of this data will have only one point for which* $\mathbf{d}\Omega = 0$.

Now, for points lying along a conformal geodesic for which $\mathbf{d}\Omega = 0$, Equation (16.11) yields $\bar{\tau}_\pm = -2/\alpha$. Note that $\bar{\tau}_\pm > 0$ if $\alpha < 0$, that is, if $\dot{\Theta}_\star < 0$. To obtain a conformal representation which includes timelike infinity one needs to set $\alpha \neq 0$. *This condition will be assumed in the remainder of this chapter.* Moreover, one defines

$$\bar{\tau}_{i^+} \equiv -2/\alpha.$$

In particular, for the conformal representation of the Minkowski spacetime given by the conformal factor of Equation (16.5) one finds that $\bar{\tau}_{i^+} = 2$.

To conclude the discussion of timelike infinity, it is necessary to analyse the Hessian of the conformal factor Θ. *In what follows it is assumed that one has obtained a solution to the conformal field equations and that the associated unphysical metric* $\boldsymbol{g}$ *has been determined.*

From the general discussion of the conformal field equations in Chapter 8 it follows that Θ satisfies the equations

$$\nabla_{\boldsymbol{a}}\Theta = \Sigma_{\boldsymbol{a}}, \tag{16.12a}$$
$$\nabla_{\boldsymbol{a}}\Sigma_{\boldsymbol{b}} = s\eta_{\boldsymbol{ab}} - \Theta L_{\boldsymbol{ab}}, \tag{16.12b}$$
$$\nabla_{\boldsymbol{a}}s = -L_{\boldsymbol{ac}}\Sigma^{\boldsymbol{c}}, \tag{16.12c}$$

where s denotes the Friedrich scalar, $\boldsymbol{\nabla}$ is the Levi-Civita connection of the unphysical metric $\boldsymbol{g} \equiv \Theta^2\tilde{\boldsymbol{g}}$ and $L_{\boldsymbol{ab}}$ are the components of the Schouten tensor of $\boldsymbol{\nabla}$ with respect to the Weyl propagated frame $\{\boldsymbol{e_a}\}$. If s and $L_{\boldsymbol{ab}}$ are *regular* at the points for which $\bar{\tau} = \bar{\tau}_{i^+}$, one finds that

$$\boldsymbol{Hess}\,\Theta|_{i^+} = s|_{i^+}\boldsymbol{g}|_{i^+}.$$

If, in addition, $s|_{i^+} \neq 0$ – which, as will be seen, is the case for perturbations of the Minkowski spacetime – one concludes that the Hessian of the conformal factor Θ is non-degenerate at i^+, and, consequently, the point i^+ can be rightfully regarded as the timelike infinity of the development of the hyperboloidal initial data prescribed on $\mathcal{H}_\star$.

The Cauchy horizon of the hyperboloidal data and the conformal boundary

The discussion in the previous two subsections can be further refined to show that the conformal boundary $\mathscr{I}^+$ coincides with the ***Cauchy horizon*** $H^+(\mathcal{H}_\star)$ of the initial data prescribed on $\mathcal{H}_\star$. General results of ***Lorentzian causal theory*** as described in Chapter 14 imply that the Cauchy horizon $H^+(\mathcal{H}_\star)$ is generated by null geodesic segments with endpoints on $\partial\mathcal{H}_\star$; see Proposition 14.4. Since $\partial\mathcal{H}_\star$ is assumed to be a smooth two-dimensional manifold, it follows that $H^+(\mathcal{H}_\star)$ is, in a neighbourhood of $\partial\mathcal{H}_\star$, a $\boldsymbol{g}$-null hypersurface.

Setting $\Sigma_{\boldsymbol{a}} \equiv \nabla_{\boldsymbol{a}}\Theta$ it follows from the initial data on $\mathcal{H}_\star$ that

$$\Omega = 0, \qquad \text{and} \qquad \Sigma_{\boldsymbol{a}}\Sigma^{\boldsymbol{a}} = \eta^{\boldsymbol{ab}} d_{\boldsymbol{a}} d_{\boldsymbol{b}} = 0, \qquad \text{on} \quad \partial\mathcal{H}_\star,$$

where the various fields are expressed in terms of their components with respect to the Weyl propagated frame $\{\boldsymbol{e_a}\}$. Accordingly, the null directions tangent to $H^+(\mathcal{H}_\star)$ – the so-called ***null generators of null infinity*** – are given on $\partial\mathcal{H}_\star$ by $\Sigma_{\boldsymbol{a}}$. On $\partial\mathcal{H}_\star$ one can define $\boldsymbol{g}$-null vectors $\boldsymbol{l}$ and $\boldsymbol{n}$ by requiring

$$l_{\boldsymbol{a}} = \Sigma_{\boldsymbol{a}}, \qquad \boldsymbol{n} \perp \partial\mathcal{H}_\star, \qquad \boldsymbol{g}(\boldsymbol{l}, \boldsymbol{n}) = 1, \qquad \text{on} \quad \partial\mathcal{H}_\star.$$

Moreover, on suitable open sets $\mathcal{O} \subset \partial\mathcal{H}_\star$ one can supplement $\boldsymbol{l}$ and $\boldsymbol{n}$ with complex vectors $\boldsymbol{m}$ and $\bar{\boldsymbol{m}}$ tangent to $\partial\mathcal{H}_\star$ with $\boldsymbol{g}(\boldsymbol{m}, \bar{\boldsymbol{m}}) = -1$. The resulting Newman-Penrose frame $\{\boldsymbol{l}, \boldsymbol{n}, \boldsymbol{m}, \bar{\boldsymbol{m}}\}$ can be propagated along the null generators of $H^+(\mathcal{H}_\star)$ which terminate on $\mathcal{O} \subset \partial\mathcal{H}_\star$ by parallel transport in the direction of $\boldsymbol{l}$; that is, one has

$$l^{\boldsymbol{a}}\nabla_{\boldsymbol{a}} l^{\boldsymbol{b}} = 0, \qquad l^{\boldsymbol{a}}\nabla_{\boldsymbol{a}} n^{\boldsymbol{b}} = 0, \qquad l^{\boldsymbol{a}}\nabla_{\boldsymbol{a}} m^{\boldsymbol{b}} = 0.$$

Assuming now that the conformal field equations are satisfied on $H^+(\mathcal{H}_\star)$, it follows from transvecting Equations (16.12a) and (16.12b) with $l^{\boldsymbol{a}}$ and $m^{\boldsymbol{a}}$ that

$$\begin{aligned}
&l^{\boldsymbol{a}}\nabla_{\boldsymbol{a}}\Theta = l^{\boldsymbol{a}}\Sigma_{\boldsymbol{a}}, \\
&l^{\boldsymbol{a}}\nabla_{\boldsymbol{a}}\big(l^{\boldsymbol{b}}\Sigma_{\boldsymbol{b}}\big) = -\Theta\left(L_{\boldsymbol{ab}}\right), \\
&l^{\boldsymbol{a}}\nabla_{\boldsymbol{a}}\big(m^{\boldsymbol{b}}\Sigma_{\boldsymbol{b}}\big) = -\Theta\big(L_{\boldsymbol{ab}} l^{\boldsymbol{a}} m^{\boldsymbol{b}}\big).
\end{aligned}$$

These equations can be regarded as *ordinary differential equations* for the scalars Θ, $l^{\boldsymbol{b}}\Sigma_{\boldsymbol{b}}$ and $m^{\boldsymbol{b}}\Sigma_{\boldsymbol{b}}$ along the generators of $\mathscr{I}^+$. By construction, these fields vanish on $\mathcal{O} \subset \partial\mathcal{H}_\star$. Therefore, following a generator on $H^+(\mathcal{H}_\star)$ off $\partial\mathcal{H}_\star$ one finds that $\Theta = 0$, $l^{\boldsymbol{b}}\Sigma_{\boldsymbol{b}} = 0$, $m^{\boldsymbol{b}}\Sigma_{\boldsymbol{b}} = 0$ until, possibly, one arrives at a caustic point. Consequently, there is at least a portion of $H^+(\mathcal{H}_\star)$ where the

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conformal factor vanishes. It follows from the above that on $\mathcal{O}$ the field $\Sigma_{\boldsymbol{a}}$ must be proportional to $l_{\boldsymbol{a}}$ – more precisely, one can write

$$\Sigma^{\boldsymbol{a}} = \left(n^{\boldsymbol{b}}\Sigma_{\boldsymbol{b}}\right) l^{\boldsymbol{a}} \qquad \text{on} \quad \mathcal{O} \subset \partial\mathcal{H}_\star. \tag{16.13}$$

The portion of the Cauchy horizon where Θ vanishes can be identified with a portion of $\mathscr{I}^+$ as given by Equation (16.11). On this part of $H^+(\mathcal{H}_\star)$, the conformal field Equations (16.12b) and (16.12c) imply that

$$\begin{aligned} &l^{\boldsymbol{a}}\nabla_{\boldsymbol{a}}\left(n^{\boldsymbol{b}}\Sigma_{\boldsymbol{b}}\right) = s, \\ &l^{\boldsymbol{a}}\nabla_{\boldsymbol{a}}s = -\left(n^{\boldsymbol{c}}\Sigma_{\boldsymbol{c}}\right)L_{\boldsymbol{ab}}l^{\boldsymbol{a}}l^{\boldsymbol{b}}. \end{aligned}$$

Since $n^{\boldsymbol{a}}\Sigma_{\boldsymbol{a}} = 1$ on $\partial\mathcal{H}_\star$ it follows from the homogeneity of the above equations that s and $n^{\boldsymbol{b}}\Sigma_{\boldsymbol{b}}$ cannot vanish simultaneously. Moreover, contracting $\nabla_{\boldsymbol{a}}\Theta = \Sigma_{\boldsymbol{a}}$ with $m^{\boldsymbol{a}}$, $\bar{m}^{\boldsymbol{a}}$ one obtains

$$s = -\left(n^{\boldsymbol{a}}\Sigma_{\boldsymbol{a}}\right)\rho, \tag{16.14}$$

where $\rho \equiv m^{\boldsymbol{a}}\bar{m}^{\boldsymbol{b}}\nabla_{\boldsymbol{b}}l_{\boldsymbol{a}}$ is the Newman-Penrose spin coefficient associated to the expansion of the congruence of null generators. Thus, ρ is a measure of its convergence; see, for example, Stewart (1991), section 2.7. It follows from Equation (16.14) that $\rho \to \infty$ if $\mathbf{d}\Theta = 0$ at some point $p \in H^+(\mathcal{H}_\star)$; see Figure 16.6.

The discussion of the previous subsection shows that the development of hyperboloidal data suitably close to Minkowski data will contain an isolated point i^+ on the conformal boundary for which $\mathbf{d}\Theta = 0$. As $\mathscr{I}^+$ and $H^+(\mathcal{H}_\star)$ coincide wherever there are no caustics, it follows that the null geodesics on $H^+(\mathcal{H}_\star)$ must converge to i^+. Accordingly, $H^+(\mathcal{H}_\star)$ is the past light cone of i^+, and the causal past $J^-(i^+)$ and the future domain of dependence $D^+(\mathcal{H}_\star)$ coincide.

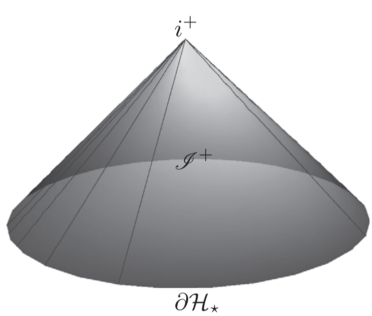

Figure 16.6 Null generators of $\mathscr{I}^+$ meeting at i^+, as discussed in the main text. The causal past of the caustic point i^+ corresponds to the future domain of dependence of hyperboloidal data; that is, $J^-(i^+) = D^+(\mathcal{H}^+_\star)$.

16.4 The proof of the main existence and stability result

Once the background Minkowski spacetime has been analysed in terms of a conformal Gaussian system adapted to the standard hyperboloid, a proof of semiglobal existence and stability is obtained by a procedure almost identical to the one used for the de Sitter spacetime in Section 15.4.

As in the case of the analysis of the de Sitter spacetime, it is convenient to consider an ansatz for the solutions to the conformal evolution equations of the form

$$e_{\boldsymbol{AB}}{}^0 = \breve{e}_{\boldsymbol{AB}}{}^0, \quad e_{\boldsymbol{AB}}{}^\alpha = \bar{e}_{\boldsymbol{AB}}{}^\alpha + \breve{e}_{\boldsymbol{AB}}{}^\alpha,$$
$$\xi_{\boldsymbol{ABCD}} = \bar{\xi}_{\boldsymbol{ABCD}} + \breve{\xi}_{\boldsymbol{ABCD}}, \quad \chi_{\boldsymbol{ABCD}} = \bar{\chi}_{\boldsymbol{ABCD}} + \breve{\chi}_{\boldsymbol{ABCD}}, \quad f_{\boldsymbol{AB}} = \breve{f}_{\boldsymbol{AB}},$$
$$\Theta_{\boldsymbol{ABCD}} = \bar{\Theta}_{\boldsymbol{ABCD}} + \breve{\Theta}_{\boldsymbol{ABCD}}, \quad \phi_{\boldsymbol{ABCD}} = \breve{\phi}_{\boldsymbol{ABCD}},$$

where

$$\bar{e}_{\boldsymbol{AB}}{}^\mu, \qquad \bar{\xi}_{\boldsymbol{ABCD}}, \qquad \bar{\chi}_{\boldsymbol{ABCD}}, \qquad \bar{\Theta}_{\boldsymbol{ABCD}}$$

are the values of the exact conformal Minkowski spacetime as discussed in Section 16.1. For conciseness, the above ansatz will be written schematically as $\mathbf{u} = \bar{\mathbf{u}} + \breve{\mathbf{u}}$. Taking into account that the background fields are also a solution to the conformal evolution equations and writing the (explicitly known) conformal gauge fields Θ and $d_{\boldsymbol{a}}$ in the form

$$\Theta = \breve{\Theta} + \breve{\Theta}, \qquad d_{\boldsymbol{a}} = \breve{d}_{\boldsymbol{a}} + \breve{d}_{\boldsymbol{a}},$$

one finds evolution equations for the perturbation fields of the form

$$\partial_{\bar{\tau}} \breve{\boldsymbol{v}} = \mathbf{K}\breve{\boldsymbol{v}} + \mathbf{Q}(\bar{\boldsymbol{\Gamma}} + \breve{\boldsymbol{\Gamma}})\breve{\boldsymbol{v}} + \mathbf{Q}(\breve{\boldsymbol{\Gamma}})\bar{\boldsymbol{v}} + \mathbf{L}(x)\breve{\boldsymbol{\phi}}, \tag{16.15a}$$
$$\big(\mathbf{I} + \mathbf{A}^0(\bar{e} + \breve{e})\big)\partial_{\bar{\tau}}\breve{\boldsymbol{\phi}} + \mathbf{A}^\alpha(\bar{e} + \breve{e})\partial_\alpha\breve{\boldsymbol{\phi}} = \mathbf{B}(\bar{\boldsymbol{\Gamma}} + \breve{\boldsymbol{\Gamma}})\breve{\boldsymbol{\phi}}, \tag{16.15b}$$

in the conventions of Proposition 13.3. The natural domains for solutions to the above equations are sets of the form

$$\mathcal{M}_{\bar{\tau}_\bullet} \equiv [0, \bar{\tau}_\bullet] \times \mathbb{S}^3$$

for some $\bar{\tau}_\bullet > 0$.

Using the evolution Equations (16.15a) and (16.15b) one obtains the following technical version of the main theorem of this chapter:

Theorem 16.1 (***semiglobal existence and stability for perturbations of hyperboloidal data***) *Let $\mathbf{u}_\star = \bar{\mathbf{u}}_\star + \breve{\mathbf{u}}_\star$ be hyperboloidal initial data for the conformal Einstein field equations given on a hyperboloidal manifold $\mathcal{H}$. Given $m \geq 4$ and $\bar{\tau}_\bullet > 2$ there exists $\varepsilon > 0$ such that:*

(i) For $\| \breve{\mathbf{u}}_\star \|_m < \varepsilon$ *there exists a solution* $\mathbf{u} = \bar{\mathbf{u}} + \breve{\mathbf{u}}$ *to the conformal propagation equations with a minimal existence interval* $[0, \bar{\tau}_\bullet]$ *such that*

$$\mathbf{u} \in C^{m-2}([0, \bar{\tau}_\bullet] \times \mathbb{S}^3),$$

and the associated congruence of conformal geodesics contains no conjugate points in $[0, \bar{\tau}_\bullet]$.

(ii) For every $\breve{\mathbf{u}}_\star$ *with* $\| \breve{\mathbf{u}}_\star \|_m < \varepsilon$ *there is a unique point* p_+ *in the interior of* $\mathcal{H}$ *such that* $\mathbf{d}\Omega = 0$ *with* $\tau_{i^+} \equiv \bar{\tau}_+(p_+) \in [0, \bar{\tau}_\bullet]$.

The solution $\mathbf{u} = \bar{\mathbf{u}} + \breve{\mathbf{u}}$ *is unique on* $D^+(\mathcal{H}_\star)$ *and implies, wherever* $\Theta \neq 0$, *a* C^{m-2} *solution to the vacuum Einstein field equations with a vanishing cosmological constant for which the set* $\mathscr{I}^+$*, as defined by*

$$\mathscr{I}^+ \equiv \{(\tau, p) \in \mathbb{R} \times \mathbb{S}^3 \mid \tau = \tau_\pm(p_+)\},$$

represents null infinity, while the point $i^+ \equiv (\bar{\tau}_{i^+}, x^\alpha(p_+))$ *represents timelike infinity. Moreover, one has*

$$D^+(\mathcal{H}_\star) = J^-(i^+).$$

Proof The assertion in (i) follows from the general existence result from symmetric hyperbolic systems in Theorem 12.4 along lines similar to the ones used in the proofs of Propositions 15.1 and 15.3. The key observation in this respect is that as $(\mathbf{I} + \mathbf{A}^0(\bar{\boldsymbol{e}}))|_\star$ is positive definite and bounded away from zero, then $\big(\mathbf{I} + \mathbf{A}^0(\bar{\boldsymbol{e}} + \breve{\boldsymbol{e}})\big)|_\star$ can also be made positive definite and bounded away from zero by choosing $\varepsilon > 0$ small enough. This observation and the general structure of the evolution Equations (16.15a) and (16.15b) ensure the existence of C^{m-2} solutions $\breve{\mathbf{u}}$ with $\| \breve{\mathbf{u}}_\star \|_m < \varepsilon$ on $[0, \bar{\tau}_\bullet] \times \mathbb{S}^3$ with $\bar{\tau}_\bullet > 2$. The regularity of the congruence of conformal geodesics defining the gauge is obtained by supplementing the conformal evolution equations with evolution equations for the conformal deviation fields, Equations (13.67a) and (13.67b), and recalling that the deviation fields for the expanding Einstein cylinder are given by Equation (15.14).

The proof of point (ii) follows from the discussion in Section 16.3 and by observing that the spatial conformal factor $\bar{\Omega}$ for the exact (background) hyperboloidal data has an isolated critical point (in fact, a maximum) at $\psi = 0$. Accordingly, by continuity, any suitably small perturbations of this data will also have a unique isolated critical point of its spatial conformal factor. Again, choosing $\varepsilon > 0$ sufficiently small, one can ensure that $\bar{\tau}_+ < \bar{\tau}_\bullet$.

The final remarks in Theorem 16.1 follow from a *propagation of the constraints argument* using the properties of the subsidiary evolution system as given by Proposition 13.4 and the assumption that the initial data satisfy the conformal constraint equations on $\mathcal{H}_\star$. The solution to the conformal field equations obtained by the above argument implies a solution to the vacuum Einstein field equations whenever $\Theta \neq 0$ as a consequence of Proposition 8.3. Finally, the

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statements about the interpretation of $\mathscr{I}^+$ as the conformal boundary and the structure of i^+ follow from the analysis in Section 16.3. □

Remarks

(i) For conciseness, Theorem 16.1 is restricted to perturbations of the data implied by the Minkowski spacetime on the *standard hyperboloid*. An inspection of the argument, however, shows that this simplifying assumption is non-essential and that an analogous result can be obtained, at the expense of some further technical details, for perturbations of Minkowski data on *arbitrary hyperboloids*. In other words, the location of the initial hyperboloid within null infinity is irrelevant. A more subtle consequence of this observation is that it is, in principle, hard to quantify how far away a given hyperboloidal initial data set lies from spatial infinity or even whether there is any (asymptotically Euclidean) Cauchy initial data for the Einstein field equations whose development contains the hyperboloidal data.

(ii) Theorem 16.1 can be combined with the *method of exterior gluing* discussed in Section 11.8.2 to show the existence of a large class of asymptotically simple spacetimes with a complete conformal boundary, that is, whose null generators are inextendible geodesics starting at i^0 and ending at i^+ and, respectively, i^-. These ideas are discussed in more detail in Section 20.5.

(iii) The future domain of dependence $D^+(\mathcal{H}_\star)$ as given by Theorem 16.1 provides an infinite portion of spacetime where the framework of ***asymptopia***, as discussed in Chapter 10, can be applied; see also, for example, chapter 3 of Stewart (1991). In particular, if the hyperboloidal initial data are constructed using the methods of Theorem 11.2, one can obtain a development which has any desired degree of smoothness and, accordingly, satisfies the ***peeling behaviour***; see the discussion in Section 10.2.

16.5 Extensions and further reading

The first semiglobal existence and stability result for hyperboloidal vacuum data of Minkowski-like spacetimes was obtained in the seminal work by Friedrich (1986b). This analysis used the standard vacuum conformal field equations and gauge source functions. The approach adopted in this chapter, employing the extended conformal field equations and a gauge based on the properties of conformal geodesics, is adapted from the discussion given in Lübbe and Valiente Kroon (2009). Similar semiglobal existence and stability results have been obtained in Anderson and Chruściel (2005) for arbitrary even-dimensional spacetimes using the conformal equations given by the Graham-Fefferman obstruction tensor.

The main result of this chapter can be extended to the case of the Einstein-Maxwell and Einstein-Yang-Mills equations. This was done in Friedrich (1991)

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where the standard conformal field equations and a hyperbolic reduction procedure based on gauge source functions were used. An alternative proof of the semi-global existence and stability result for the Einstein-Maxwell equations has been obtained in Lübbe and Valiente Kroon (2012) using an approach similar in spirit to the one used in this chapter, that is, employing the extended conformal field equations and a conformal gauge based on the properties of conformal curves. Conformal curves were preferred in this analysis as they provide an explicit expression for the conformal factor. In the presence of matter, a standard conformal Gaussian system does not provide an explicit expression for the conformal factor. There is, however, no reason why a semi-global result of the type discussed in this chapter cannot be obtained using a gauge based on conformal geodesics. Another way of generalising the main result of this chapter is to consider the Einstein-conformally invariant scalar field system; see Hübner (1995).

The methods in this chapter can be adapted to analyse semiglobal existence and stability of asymptotically simple spacetimes with vanishing cosmological constant which are neither the Minkowski spacetime nor perturbations thereof – so-called ***purely radiative spacetimes***. These vacuum spacetimes consist of gravitational radiation (hence the name) which is not necessarily weak, but still tame enough to not form a black hole; see, for example, Friedrich (1986c) and the discussion in Chapter 19. Stability of these types of spacetimes from the perspective of a hyperboloidal initial value problem has been analysed, for the vacuum case, in Lübbe and Valiente Kroon (2010) and, for the Einstein-Maxwell case, in Lübbe and Valiente Kroon (2012).

The main theorem of this chapter has been beautifully verified in numerical simulations in Hübner (2001a). In particular, the numerical results show how the null generators of the conformal boundary converge, to machine precision, at timelike infinity. These numerical simulations are further discussed in Section 21.3.