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SPECTRALITY OF A CLASS OF MORAN MEASURES

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Abstract. Let $\{M_n\}_{n=1}^{\infty}$ be a sequence of expanding matrices with $M_n = \text{diag}(p_n, q_n)$, and let $\{\mathcal{D}_n\}_{n=1}^{\infty}$ be a sequence of digit sets with $\mathcal{D}_n = \{(0,0)^t, (a_n, 0)^t, (0, b_n)^t, \pm(a_n, b_n)^t\}$, where p_n, q_n, a_n and b_n are positive integers for all $n \geq 1$. If $\sup_{n\geq 1}\{\frac{a_n}{p_n}, \frac{b_n}{q_n}\} < \infty$, then the infinite convolution

$$\mu_{\{M_n\},\{\mathcal{D}_n\}} = \delta_{M_1^{-1}\mathcal{D}_1} * \delta_{(M_1M_2)^{-1}\mathcal{D}_2} * \cdots$$

is a Borel probability measure (Cantor-Dust-Moran measure). In this paper, we investigate whenever there exists a discrete set Λ such that $\{e^{2\pi i\langle\lambda,x\rangle} : \lambda \in \Lambda\}$ is an orthonormal basis for $L^2(\mu_{\{M_n\},\{\mathcal{D}_n\}})$.

1. Introduction

Let μ be a Borel probability measure with compact support on $\mathbb{R}^n$. We call it a *spectral measure* if there exists a countable subset $\Lambda \subset \mathbb{R}^n$ such that $E_\Lambda = \{e^{2\pi i\langle\lambda,x\rangle} : \lambda \in \Lambda\}$ forms an orthonormal basis for $L^2(\mu)$. The set Λ is called a *spectrum* of μ, we also say that (μ, Λ) is a *spectral pair*. In particular, if μ is the normalized Lebesgue measure supported on a Borel set Ω, then Ω is called a *spectral set*. In the seminal paper [13], Fuglede pioneered the study of spectral sets, and raised the famous conjecture: Ω is a spectral set if and only if Ω is a translational tile. This conjecture has been proved to be false by Tao and others in both directions in dimension three or higher, see [18, 19, 28], but it is still open in dimensions 1 and 2.

The first example of a singular, non-atomic, spectral measure was given by Jorgensen and Pedersen in [17], they showed that the standard middle-fourth Cantor measure on $\mathbb{R}$ is a spectral measure, and Strichartz supplemented their result with a simplified proof [26]. This surprising discovery has drawn great attention in the area of fractal geometry, and the spectrality of self-similar/affine measures has become a important topic. Nowadays, many new spectral measures were found in [3–11, 13, 15–17, 20, 22, 23, 25] and the references therein. Among those, Hu and Lau [16] made a start in studying the spectrality

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of Bernoulli convolutions, they classified the contraction ratios with infinitely many orthogonal exponential functions. It was completed by Dai [3], who proved that the only spectral Bernoulli convolutions are of contraction ratio $1/(2k)$. This is generalized further to the N-Bernoulli convolutions [4, 5]. In the higher dimensional case, Deng and Lau [6], Li [22] and Liu and Luo [25] gave some further spectral results which related to the Sierpinski-type self-affine measures. In another direction, there are considerable studies for the non-spectral measures, please see [21, 24] and the references therein. A more general case is the Moran measure, which is a non-self-similar extension of the Cantor measure through the infinite convolution. Until now, there are few results on the spectrality of Moran measures [1, 2, 12, 14, 27, 29, 30].

Let $\{\rho_n\}_{n=1}^{\infty}$ be a sequence of integers bigger than 1 and let $\{D_n\}_{n=1}^{\infty}$ be a sequence of finite digit sets in $\mathbb{Z}$. Denote $|D| = \max\{|d| : d \in D\}$, it is well known that if $|D_n|/\rho_n < \infty$, then the infinite convolution $\delta_{\rho_1^{-1}D_1} * \delta_{(\rho_1\rho_2)^{-1}D_2} * \cdots := \mu_{\{\rho_n\},\{D_n\}}$ is a Borel probability measure (*Moran measure*), where $*$ is the convolution sign, $\delta_E = \frac{1}{\#E}\sum_{e\in E}\delta_e$, $\#E$ is the cardinality of a set E, δ_e is the Dirac measure at the point e. In [27], Strichartz first studied the spectrality of $\mu_{\{\rho_n\},\{D_n\}}$. After that, An and He [1] considered the case that $D_n = \{0, 1, \ldots, d_n - 1\}$ with $d_n > 1$, they showed that $\mu_{\{\rho_n\},\{D_n\}}$ is a spectral measure if $d_n \mid \rho_n$. Recently, He et al. [14] studied the case that $D_n = \{0, d_n\}$ with $0 < d_n < \rho_n$ for all $n \geq 1$, they proved that if $2 \mid \frac{\rho_n}{\gcd(d_n,\rho_n)}$, then $\mu_{\{\rho_n\},\{D_n\}}$ is a spectral measure. In [30], Wang, Dong and Liu further considered $D_n = \{0, \alpha_n, \beta_n\} = \{0, 1, 2\} \pmod 3$ and gave some sufficient conditions for $\mu_{\{\rho_n\},\{D_n\}}$ to be a spectral measure.

Observe that the above results were obtained under the assumption of $|D_n|/\rho_n \leq 1$ [1, 14], or there exists an increasing subsequence $\{n_k\}_{k=1}^{\infty}$ of $\mathbb{N}$ such that $|D_{n_k}|/\rho_{n_k} \leq 1$ for all $k \geq 1$ [30], can we also say something about the spectrality of $\mu_{\{\rho_n\},\{D_n\}}$ without the above restrictions on the digit sets? Motivated by this question and their results, in this paper, we focus on a class of Moran measures with four-element digit sets on $\mathbb{R}^2$. Let $\{M_n\}_{n=1}^{\infty}$ be a sequence of diagonal matrices as following

$$M_n = \begin{bmatrix} p_n & 0 \\ 0 & q_n \end{bmatrix}, \qquad p_n, q_n \in \mathbb{Z} \cap [2, +\infty), \tag{1.1}$$

and let $\{\mathcal{D}_n\}_{n=1}^{\infty}$ be a sequence of digit sets with

$$\mathcal{D}_n = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} a_n \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ b_n \end{pmatrix}, \pm \begin{pmatrix} a_n \\ b_n \end{pmatrix} \right\}, \qquad a_n, b_n \in \mathbb{Z} \cap [1, +\infty). \tag{1.2}$$

Throughout the paper, we assume that

$$c := \sup_{n\geq 1} \frac{a_n}{p_n} < \infty, \qquad d := \sup_{n\geq 1} \frac{b_n}{q_n} < \infty. \tag{1.3}$$

Denote $\mathcal{M}_n = M_1M_2\cdots M_n = \operatorname{diag}(\mathcal{P}_n, Q_n)$, where $\mathcal{P}_n = \prod_{i=1}^n p_i$, $Q_n = \prod_{i=1}^n q_i$. Under the above assumptions, associated to the sequence $\{M_n, \mathcal{D}_n\}_{n=1}^{\infty}$, there exists a Borel probability measure $\mu_{\{M_n\},\{\mathcal{D}_n\}}$, which is defined by the following infinite convolution

$$\mu_{\{M_n\},\{\mathcal{D}_n\}} = \delta_{\mathcal{M}_1^{-1}\mathcal{D}_1} * \delta_{\mathcal{M}_2^{-1}\mathcal{D}_2} * \cdots , \tag{1.4}$$

and the convergence is in the weak sense. We call the measure $\mu_{\{M_n\},\{\mathcal{D}_n\}}$ a *Cantor-Dust-Moran measure*, which is supported on the following Cantor-Dust-Moran set:

$$T(\{M_n\},\{\mathcal{D}_n\}) := \left\{ \sum_{n=1}^{\infty} \mathcal{M}_n^{-1} d_n : d_n \in \mathcal{D}_n \right\} = \sum_{n=1}^{\infty} \mathcal{M}_n^{-1}\mathcal{D}_n.$$

In this paper, we will study the spectrality of the Cantor-Dust-Moran measures $\mu_{\{M_n\},\{\mathcal{D}_n\}}$. Our first main result is the following theorem.

Theorem 1.1. *Let M_n, $\mathcal{D}_n$ and $\mu_{\{M_n\},\{\mathcal{D}_n\}}$ be defined by* (1.1), (1.2) *and* (1.4) *respectively and satisfy $c, d \le 1$, where c and d are defined by* (1.3). *If* $\gcd(\frac{p_n}{\gcd(a_n,p_n)}, \frac{q_n}{\gcd(b_n,q_n)}) \in 2\mathbb{Z}$ *for all $n \ge 1$, then $\mu_{\{M_n\},\{\mathcal{D}_n\}}$ is a spectral measure.*

To some extent, Theorem 1.1 is an extension of the results in [1, 14, 30], which studied Moran measures on $\mathbb{R}$, and the proof of Theorem 1.1 is inspired by the ideas from them. We remark that the condition $\gcd(\frac{p_n}{\gcd(a_n,p_n)}, \frac{q_n}{\gcd(b_n,q_n)}) \in 2\mathbb{Z}$ for $n \ge 2$ is essential, we will give an example to explain it (see Example 2.3).

Moreover, we further obtain another sufficient condition for $\mu_{\{M_n\},\{\mathcal{D}_n\}}$ to be a spectral measure without the restrictions $c, d \le 1$. Now we state our second main theorem.

Theorem 1.2. *Let M_n, $\mathcal{D}_n$ and $\mu_{\{M_n\},\{\mathcal{D}_n\}}$ be defined by* (1.1), (1.2) *and* (1.4) *respectively and satisfy* $\gcd(\frac{p_n}{\gcd(a_n,p_n)}, \frac{q_n}{\gcd(b_n,q_n)}) \in 2\mathbb{Z}$ *for all $n \ge 1$. If $\lim_{n\to\infty} p_n = \lim_{n\to\infty} q_n = \infty$, then $\mu_{\{M_n\},\{\mathcal{D}_n\}}$ is a spectral measure.*

Some ideas in the proof of Theorem 1.2 are of independent interest. The main difficulty of the proof is to construct candidate spectra for different cases, which are given in the proof of Proposition 3.1. It is worth noting that the proof of Proposition 3.1 is skillful.

Remark 1.3. In the case when $\mathcal{D}_n = \{(0,0)^t, (a_n,0)^t, (0,b_n)^t, (a_n,b_n)^t\}$, the measure $\mu_{\{M_n\},\{\mathcal{D}_n\}}$ is the cross product of two two-dimensional Moran measures. We remark that the conditions $c, d \le 1$ in Theorem 1.1 are similar to those of the results in [1, 14, 30]. Interestingly, Theorem 1.2 implies that $\mu_{\{M_n\},\{\mathcal{D}_n\}}$ can also be a spectral measure without the conditions $c, d \le 1$.

The paper is organized as follows. In Section 2, we introduce some basic definitions and properties of spectral measures. In Sections 3 and 4, we prove Theorems 1.2 and 1.1, respectively.

2. **Preliminaries**

In this section, we give some preliminary definitions and lemmas that we need in proving our main results. Let μ be a probability measure with compact support on $\mathbb{R}^2$. The Fourier transform of μ is defined by $\hat{\mu}(\xi) = \int e^{-2\pi i\langle x,\xi\rangle} d\mu(x)$ as usual. Let M_n, $\mathcal{D}_n$ and $\mu_{\{M_n\},\{\mathcal{D}_n\}}$ be defined by (1.1), (1.2) and (1.4), respectively. It follows from [9] that

$$\hat{\mu}_{\{M_n\},\{\mathcal{D}_n\}}(\xi) = \prod_{n=1}^{\infty} m_{\mathcal{D}_n}(\mathcal{M}_n^{-1}\xi), \tag{2.1}$$

where

$$m_{\mathcal{D}_n}(x) = \frac{1}{4}\sum_{d_n\in\mathcal{D}_n} e^{-2\pi i\langle d_n,x\rangle} = \frac{1}{4}(1 + e^{-2\pi i a_n x_1} + e^{-2\pi i b_n x_2} + e^{\pm 2\pi i(a_n x_1 + b_n x_2)})$$

is the mask polynomial of $\mathcal{D}_n$. It is clear that $m_{\mathcal{D}_n}(x)$ is a $\mathbb{Z}^2$-periodic function since $\mathcal{D}_n \subset \mathbb{Z}^2$. Denote

$$\mathcal{D}_n^+ = \left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} a_n \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ b_n \end{pmatrix}, \begin{pmatrix} a_n \\ b_n \end{pmatrix}\right\} \text{ and } \mathcal{D}_n^- = \left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} a_n \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ b_n \end{pmatrix}, \begin{pmatrix} -a_n \\ -b_n \end{pmatrix}\right\}.$$

Let $\mathcal{Z}(h) = \{x : h(x) = 0\}$ be the zero set of the function h. Then

$$\mathcal{Z}(m_{\mathcal{D}_n}) = \mathcal{Z}(m_{\mathcal{D}_n^+}) \quad \text{or} \quad \mathcal{Z}(m_{\mathcal{D}_n}) = \mathcal{Z}(m_{\mathcal{D}_n^-}), \tag{2.2}$$

where

$$\mathcal{Z}(m_{\mathcal{D}_n^+}) = \left\{\begin{pmatrix} \frac{2k_1+1}{2a_n} \\ \xi_1 \end{pmatrix}, \begin{pmatrix} \xi_2 \\ \frac{2k_2+1}{2b_n} \end{pmatrix} : k_1, k_2 \in \mathbb{Z},\ \xi_1, \xi_2 \in \mathbb{R}\right\},$$

$$\mathcal{Z}(m_{\mathcal{D}_n^-}) = \left\{\begin{pmatrix} \frac{2k_1+1}{2a_n} \\ \frac{k_2}{2b_n} \end{pmatrix}, \begin{pmatrix} \frac{k_1'}{2a_n} \\ \frac{2k_2'+1}{2b_n} \end{pmatrix} : k_1, k_2, k_1', k_2' \in \mathbb{Z}\right\}.$$

It follows from (2.1) that

$$\mathcal{Z}(\hat{\mu}_{\{M_n\},\{\mathcal{D}_n\}}) = \bigcup_{n=1}^{\infty} \mathcal{M}_n(\mathcal{Z}(m_{\mathcal{D}_n})). \tag{2.3}$$

We say that Λ is an *orthogonal set* of μ if E_Λ is an orthonormal family for $L^2(\mu)$. It is easy to show that Λ is an orthogonal set of μ if and only if $\hat{\mu}(\lambda-\lambda') = 0$ for any $\lambda \neq \lambda' \in \Lambda$, which is equivalent to

$$(\Lambda - \Lambda) \setminus \{0\} \subset \mathcal{Z}(\hat{\mu}). \tag{2.4}$$

It is easy to see that a spectrum of a measure is always an orthonormal set of the same measure.

Let $Q_\Lambda(\xi) = \sum_{\lambda\in\Lambda} |\hat{\mu}(\xi + \lambda)|^2$. We recall the fundamental criterion for the spectrality of μ, which is a directed application of Parseval's identity.

Theorem 2.1. *([17]) Let μ be a Borel probability measure with compact support on $\mathbb{R}^n$, and let $\Lambda \subset \mathbb{R}^n$ be a countable subset. Then*

(i) Λ *is an orthonormal set of* μ *if and only if* $Q_\Lambda(\xi) \leq 1$ *for* $\xi \in \mathbb{R}^n$*;*
(ii) Λ *is a spectrum of* μ *if and only if* $Q_\Lambda(\xi) \equiv 1$ *for* $\xi \in \mathbb{R}^n$*;*
(iii) *If* Λ *is an orthonormal set of* μ*, then* $Q_\Lambda(z)$ *is an entire function.*

The following lemma is an effective method to illustrate that a countable set Λ cannot be a spectrum of a measure μ, which will be used in Example 2.3.

Lemma 2.2. *([5]) Let $\mu = \mu_0 * \mu_1$ be the convolution of two probability measures μ_i, $i = 0, 1$, and they are not Dirac measures. Suppose that Λ is an orthogonal set of μ_0 with $0 \in \Lambda$, then Λ is also an orthogonal set of μ, but cannot be a spectrum of μ.*

The following example indicates that the hypothesis $\gcd(\frac{p_n}{\gcd(a_n,p_n)}, \frac{q_n}{\gcd(b_n,q_n)}) \in 2\mathbb{Z}$ for $n \geq 2$ in Theorem 1.1 is essential.

Example 2.3. Let $\mathcal{D}_n = \{(0,0)^t, (1,0)^t, (0,1)^t, \pm(1,1)^t\}$, $M_1 = \mathrm{diag}(4,4)$, $M_2 = \mathrm{diag}(3,3)$ and $M_n = \mathrm{diag}(4,4)$ for $n \geq 3$. Then the measure $\mu_{\{M_n\},\{\mathcal{D}_n\}}$ is not a spectral measure.

Proof. For convenience, we let $\kappa_n := \gcd(\frac{p_n}{\gcd(a_n,p_n)}, \frac{q_n}{\gcd(b_n,q_n)})$ for all $n \geq 1$. It is direct to calculate that $c = d = \frac{1}{3}$ and $\kappa_n = \kappa_1 = 4 \in 2\mathbb{Z}$ for $n \geq 3$, but $\kappa_2 = 3 \notin 2\mathbb{Z}$. Denote $\mu_1 = \delta_{\mathcal{M}_1^{-1}\mathcal{D}_1} * \delta_{\mathcal{M}_3^{-1}\mathcal{D}_3} * \delta_{\mathcal{M}_4^{-1}\mathcal{D}_4} * \cdots$, then $\mu_{\{M_n\},\{\mathcal{D}_n\}} = \delta_{\mathcal{M}_2^{-1}\mathcal{D}_2} * \mu_1$. By (2.1), we have

$$\hat{\mu}_{\{M_n\},\{\mathcal{D}_n\}}(\xi) = m_{\mathcal{D}_1}\left(\frac{\xi}{4}\right) m_{\mathcal{D}_2}\left(\frac{\xi}{12}\right) \prod_{k=2}^{\infty} m_{\mathcal{D}_k}\left(\frac{\xi}{3 \cdot 4^k}\right) = m_{\mathcal{D}_1}\left(\frac{\xi}{4}\right) m_{\mathcal{D}_2}\left(\frac{\xi}{12}\right) \hat{\mu}_{\{M_1\},\{\mathcal{D}_1\}}\left(\frac{\xi}{12}\right).$$

It is easy to check that $\mathcal{Z}(m_{\mathcal{D}_2}(\frac{\xi}{12})) \subset \mathcal{Z}(m_{\mathcal{D}_1}(\frac{\xi}{4}))$, thus $\mathcal{Z}(\hat{\mu}_{\{M_n\},\{\mathcal{D}_n\}}) = \mathcal{Z}(\hat{\mu}_1)$. Hence the assertion follows by Lemma 2.2. □

Throughout this paper, we write

$$\mu_n = \delta_{\mathcal{M}_1^{-1}\mathcal{D}_1} * \cdots * \delta_{\mathcal{M}_n^{-1}\mathcal{D}_n} \quad \text{and} \quad \mu_{>n} = \delta_{\mathcal{M}_{n+1}^{-1}\mathcal{D}_{n+1}} * \delta_{\mathcal{M}_{n+2}^{-1}\mathcal{D}_{n+2}} * \cdots. \tag{2.5}$$

Then $\mu_{\{M_n\},\{\mathcal{D}_n\}} = \mu_n * \mu_{>n}$. Moreover, we let $a_n = 2^{s_n} a'_n$, $b_n = 2^{t_n} b'_n$ for $n \geq 1$, where s_n, $t_n \geq 0$ and a'_n, b'_n are odd. Since $\gcd(\frac{p_n}{\gcd(a_n,p_n)}, \frac{q_n}{\gcd(b_n,q_n)}) \in 2\mathbb{Z}$ for all $n \geq 1$, we can rewrite $p_n = 2^{1+s_n} p'_n$ and $q_n = 2^{1+t_n} q'_n$, where p'_n and q'_n may be even. Denote $\sigma = \sigma_1\sigma_2\cdots$, $\sigma' = \sigma'_1\sigma'_2\cdots \in (2\mathbb{Z}+1)^{\mathbb{N}}$. Now we construct a countable set $\Lambda^{\sigma,\sigma'}$ in terms of $(\{M_n\}, \{s_n\}, \{t_n\}, \{\sigma_n\}, \{\sigma'_n\})$. Let

$$\Lambda_n^{\sigma,\sigma'} = \sum_{i=1}^{n} \mathcal{M}_i \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{\sigma_i}{2^{1+s_i}} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{\sigma'_i}{2^{1+t_i}} \end{pmatrix}, \begin{pmatrix} \frac{\sigma_i}{2^{1+s_i}} \\ \frac{\sigma'_i}{2^{1+t_i}} \end{pmatrix} \right\} \quad \text{and} \quad \Lambda^{\sigma,\sigma'} = \bigcup_{n=1}^{\infty} \Lambda_n^{\sigma,\sigma'}. \tag{2.6}$$

Lemma 2.4. *Let M_n, $\mathcal{D}_n$ and $\mu_{\{M_n\},\{\mathcal{D}_n\}}$ be defined by* (1.1), (1.2) *and* (1.4)*, respectively. With the same notations as above, then the set $\Lambda_n^{\sigma,\sigma'}$ is a spectrum of μ_n and $\Lambda^{\sigma,\sigma'}$ is an orthogonal set of $\mu_{\{M_n\},\{\mathcal{D}_n\}}$ for each σ, $\sigma' \in (2\mathbb{Z}+1)^{\mathbb{N}}$.*

Proof. For convenience, we denote $\mathcal{L}_i = \{(0,0)^t, (\frac{\sigma_i}{2^{1+s_i}}, 0)^t, (0, \frac{\sigma_i'}{2^{1+t_i}})^t, (\frac{\sigma_i}{2^{1+s_i}}, \frac{\sigma_i'}{2^{1+t_i}})^t\}$. Then, for any $\lambda \neq \lambda' \in \Lambda_n^{\sigma,\sigma'}$, we have $\lambda = \sum_{i=1}^n \mathcal{M}_i l_i$ and $\lambda' = \sum_{i=1}^n \mathcal{M}_i l_i'$, where $l_i,\ l_i' \in \mathcal{L}_i$. Let κ be the first index such that $l_\kappa \neq l_\kappa'$, then $\lambda - \lambda' = \mathcal{M}_\kappa[(l_\kappa - l_\kappa') + \sum_{i=\kappa+1}^n \mathcal{M}_i \mathcal{M}_\kappa^{-1}(l_i - l_i')]$. It is easy to check that $l_\kappa,\ l_\kappa'$ and $l_\kappa - l_\kappa'$ belong to $\mathcal{Z}(m_{\mathcal{D}_\kappa})$ (see (2.2)). Note that $p_i = 2^{1+s_i} p_i'$ and $q_i = 2^{1+t_i} q_i'$, we have $M_i(l_i - l_i') \in \mathbb{Z}^2$. Then $\mathcal{M}_i \mathcal{M}_\kappa^{-1}(l_i - l_i') = M_{\kappa+1} \cdots M_i(l_i - l_i') \in \mathbb{Z}^2$ for $i \geq \kappa + 1$. Therefore

$$\lambda - \lambda' \in \mathcal{M}_\kappa(\mathcal{Z}(m_{\mathcal{D}_\kappa}) + \mathbb{Z}^2) = \mathcal{M}_\kappa(\mathcal{Z}(m_{\mathcal{D}_\kappa})) \subset \bigcup_{i=1}^n \mathcal{M}_i(\mathcal{Z}(m_{\mathcal{D}_i})) = \mathcal{Z}(\hat{\mu}_n). \tag{2.7}$$

This and (2.4) imply that $\Lambda_n^{\sigma,\sigma'}$ is an orthogonal set of μ_n. Then $\Lambda_n^{\sigma,\sigma'}$ is a spectrum of μ_n because the dimension of $L^2(\mu_n)$ is 4^n, which is the cardinality of the set $\Lambda_n^{\sigma,\sigma'}$.

Next we prove that $\Lambda^{\sigma,\sigma'}$ is an orthogonal set of $\mu_{\{M_n\},\{\mathcal{D}_n\}}$. For any $\lambda \neq \lambda' \in \Lambda^{\sigma,\sigma'}$, there exists an integer n such that $\lambda,\ \lambda' \in \Lambda_n^{\sigma,\sigma'}$. As $\mu_{\{M_n\},\{\mathcal{D}_n\}} = \mu_n * \mu_{>n}$, it follows from (2.3) that $\mathcal{Z}(\hat{\mu}_n) \subset \mathcal{Z}(\hat{\mu}_{\{M_n\},\{\mathcal{D}_n\}})$. This together with (2.7) yields that $\lambda - \lambda' \in \mathcal{Z}(\hat{\mu}_{\{M_n\},\{\mathcal{D}_n\}})$, by (2.4), the assertion follows. □

3. **Proof of Theorem 1.2**

In this section, we first give a proposition and then use it to complete the proof of Theorem 1.2. Note that $p_n,\ q_n \geq 2$ for $n \geq 1$, then $\mathcal{P}_n,\ \mathcal{Q}_n \geq 2^n$, which will be used many times in the rest of this paper. To simplify notations, we write $\mathcal{P}_{n,m} = \prod_{i=n}^m p_i$ and $\mathcal{Q}_{n,m} = \prod_{i=n}^m q_i$. Denote $\Lambda_n^{\sigma,\sigma'} := (\Lambda_n^1, \Lambda_n^2)^t$, $\Lambda^{\sigma,\sigma'} := (\Lambda^1, \Lambda^2)^t$, which are defined by (2.6). Then $\Lambda_n^1 = \sum_{i=1}^n \mathcal{P}_i \left\{0, \frac{\sigma_i}{2^{1+s_i}}\right\}$, $\Lambda_n^2 = \sum_{i=1}^n \mathcal{Q}_i \left\{0, \frac{\sigma_i'}{2^{1+t_i}}\right\}$. To prove Theorem 1.2, we will construct different sets Λ_n^1 and Λ_n^2 for $\Lambda^{\sigma,\sigma'}$ being spectra.

Proposition 3.1. *Under the conditions of Theorem 1.2, then there exist an increasing subsequence $\{n_k\}_{k=1}^\infty$ of $\mathbb{N}$, some $\sigma,\ \sigma' \in (2\mathbb{Z}+1)^{\mathbb{N}}$ and a constant $\alpha > 0$ such that*

$$|\hat{\mu}_{>n_k}(\xi + \lambda)| \geq \alpha \quad \text{for } k > 1,\ \xi \in [0,1]^2 \text{ and } \lambda \in \Lambda_{n_k}^{\sigma,\sigma'}.$$

Proof. For simplicity, we write

$$J_j(x) := |m_{\mathcal{D}_j}(\mathcal{M}_j^{-1}x)| = \frac{1}{4}|1 + e^{2\pi i \eta_{j,1}(x)} + e^{2\pi i \eta_{j,2}(x)} + e^{\pm 2\pi i(\eta_{j,1}(x) + \eta_{j,2}(x))}|,$$

where $\eta_{j,1}(x) = -\mathcal{P}_j^{-1} a_j x_1,\ \eta_{j,2}(x) = -\mathcal{Q}_j^{-1} b_j x_2$ for $x = (x_1, x_2)^t$. For $\xi = (\xi_1, \xi_2)^t \in [0,1]^2$ and $\lambda = (\lambda_1, \lambda_2)^t \in \Lambda_{n_k}^{\sigma,\sigma'}$, we use $\eta_{j,1},\ \eta_{j,2}$ to denote $\eta_{j,1}(\xi + \lambda)$ and $\eta_{j,2}(\xi + \lambda)$ respectively, that is,

$$\eta_{j,1} = -\mathcal{P}_j^{-1} a_j(\xi_1 + \lambda_1), \quad \eta_{j,2} = -\mathcal{Q}_j^{-1} b_j(\xi_2 + \lambda_2).$$

It follows from (2.5) that $|\hat{\mu}_{>n_k}(\xi + \lambda)| = \prod_{j=n_k+1}^\infty J_j(\xi + \lambda)$. Recall that $a_n = 2^{s_n} a_n'$ and $b_n = 2^{t_n} b_n'$ with $a_n',\ b_n' \notin 2\mathbb{Z}$, for the two sequences $\{s_n\}_{n=1}^\infty$ and $\{t_n\}_{n=1}^\infty$, it is easy to show that there must exist an increasing subsequence$\{n_k\}_{k=1}^{\infty}$ of $\mathbb{N}$ with $n_{k+1} - n_k \geq 2$ and two nonnegative integers $s,\ t$ such that one of the following cases holds:

Case I : $\lim_{k\to\infty} s_{n_k} = \lim_{k\to\infty} t_{n_k} = \infty$;

Case II : $\lim_{k\to\infty} s_{n_k} = \infty$ and $t_{n_k} = t$ for all $k \geq 1$;

Case III : $\lim_{k\to\infty} t_{n_k} = \infty$ and $s_{n_k} = s$ for all $k \geq 1$;

Case IV : $s_{n_k} = s$ and $t_{n_k} = t$ for all $k \geq 1$.

As $\lim_{n\to\infty} p_n = \lim_{n\to\infty} q_n = \infty$, we further suppose that the sequence $\{n_k\}_{k=1}^{\infty}$ satisfies $p_n \geq \max\{10c\pi, 10\}$ and $q_n \geq \max\{10d\pi, 10\}$ for $n \geq n_1$. We will choose different $\sigma,\ \sigma' \in (2\mathbb{Z}+1)^{\mathbb{N}}$ in $\Lambda_n^{\sigma,\sigma'}$ for the above four cases.

By (1.3), it is clear that there exist an increasing subsequence of $\{n_k\}_{k=1}^{\infty}$ (we replace it by the original sequence) and two constants $c',\ d'$ such that

$$\lim_{k\to\infty} \frac{a_{n_k+1}}{p_{n_k+1}} := c' < \infty \quad \text{and} \quad \lim_{k\to\infty} \frac{b_{n_k+1}}{q_{n_k+1}} := d' < \infty. \tag{3.1}$$

We will illustrate our following four steps needed to complete the proof.

Step 1. We first give the choices of σ and σ' for different cases.

In Case I, we can further suppose that the sequence $\{n_k\}_{k=1}^{\infty}$ satisfies $s_{n_k} \geq \max\{2, \lfloor \log_2 4c \rfloor + 1\}$ and $t_{n_k} \geq \max\{2, \lfloor \log_2 4d \rfloor + 1\}$ for all $k \geq 1$, where $\lfloor x \rfloor = \max\{n \in \mathbb{Z} \mid n \leq x\}$. Choose $\sigma_i,\ \sigma_i' = 1$ for all $i \geq 1$. Then, for $\lambda = (\lambda_1, \lambda_2)^t \in \Lambda_{n-1}^{\sigma,\sigma'}$, we have

$$\frac{|\lambda_1|}{\mathcal{P}_n} \leq \sum_{i=1}^{n-1} \frac{1}{\mathcal{P}_{i+1,n}} \frac{1}{2^{1+s_i}} \leq \sum_{i=1}^{n-1} \frac{1}{2\mathcal{P}_{i+1,n}} \leq \frac{1}{p_n} \sum_{i=1}^{\infty} \frac{1}{2^i} \leq \frac{1}{p_n}, \tag{3.2}$$

$$\frac{|\lambda_2|}{\mathcal{Q}_n} \leq \sum_{i=1}^{n-1} \frac{1}{\mathcal{Q}_{i+1,n}} \frac{1}{2^{1+t_i}} \leq \sum_{i=1}^{n-1} \frac{1}{2\mathcal{Q}_{i+1,n}} \leq \frac{1}{q_n} \sum_{i=1}^{\infty} \frac{1}{2^i} \leq \frac{1}{q_n}. \tag{3.3}$$

In Case II, we further suppose that the sequence $\{n_k\}_{k=1}^{\infty}$ satisfies $s_{n_k} \geq \max\{2, \lfloor \log_2 4c \rfloor + 1\}$ for all $k \geq 1$. Let $\sigma_i = 1$ for all $i \geq 1$, then $\frac{|\lambda_1|}{\mathcal{P}_n} \leq \frac{1}{p_n}$ for $\lambda_1 \in \Lambda_{n-1}^1$ by (3.2). For the choice of σ_i', we divide it into two cases. **Case II$_1$** : $d' \in \mathbb{N} + \{\frac{1}{2}\}$. Let $\sigma_i' = 1$ for all $i \geq 1$, then $\frac{|\lambda_2|}{\mathcal{Q}_n} \leq \frac{1}{q_n}$ for $\lambda_2 \in \Lambda_{n-1}^2$ by (3.3). **Case II$_2$** : $d' \notin \mathbb{N} + \{\frac{1}{2}\}$. Let $\sigma_i' = 1$ if $i \notin \{n_k : k \geq 1\}$ and $\sigma_{n_k}' \in \{q_{n_k+1}', q_{n_k+1}' + 1\} \cap (\mathbb{Z} \setminus 2\mathbb{Z})$. As $n_{k+1} - n_k \geq 2$ and $\sigma_{n_k}' \leq q_{n_k+1}$, we have

$$\frac{|\lambda_2|}{\mathcal{Q}_{n_k}} \leq \sum_{i=1}^{n_k-1} \frac{\sigma_i'}{2\mathcal{Q}_{i+1,n_k}} \leq \frac{1}{2q_{n_k}} + \sum_{i=1}^{n_k-2} \frac{1}{2\mathcal{Q}_{i+2,n_k}} \leq \frac{1}{2q_{n_k}}\Big(1 + \sum_{i=0}^{\infty} \frac{1}{2^i}\Big) \leq \frac{3}{2q_{n_k}} \tag{3.4}$$

for $\lambda_2 \in \Lambda_{n_k-1}^2$ and

$$\frac{|\lambda_2|}{\mathcal{Q}_{n_k+1}} \leq \sum_{i=1}^{n_k} \frac{\sigma_i'}{2\mathcal{Q}_{i+1,n_k+1}} \leq \frac{q_{n_k+1}' + 1}{2q_{n_k+1}} + \sum_{i=1}^{n_k-1} \frac{1}{2\mathcal{Q}_{i+2,n_k+1}} \leq \frac{1}{4} + \frac{3}{2q_{n_k+1}} \tag{3.5}$$

for $\lambda_2 \in \Lambda^2_{n_k}$.

In Case III, similar to Case II, we let $\{n_k\}_{k=1}^\infty$ satisfy $t_{n_k} \geq \max\{2, \lfloor \log_2 4d \rfloor + 1\}$ for all $k \geq 1$. Let $\sigma'_i = 1$ for all $i \geq 1$, then $\frac{|\lambda_2|}{\mathcal{Q}_n} \leq \frac{1}{q_n}$ for $\lambda_2 \in \Lambda^2_{n-1}$ by (3.3). For the choice of σ_i, we divide it into two cases. **Case III$_1$** : $c' \in \mathbb{N} + \{\frac{1}{2}\}$. Let $\sigma_i = 1$ for all $i \geq 1$, then $\frac{|\lambda_1|}{\mathcal{P}_n} \leq \frac{1}{p_n}$ for $\lambda_1 \in \Lambda^1_{n-1}$ by (3.2). **Case III$_2$** : $c' \notin \mathbb{N} + \{\frac{1}{2}\}$. Let $\sigma_i = 1$ if $i \notin \{n_k : k \geq 1\}$ and $\sigma_{n_k} \in \{p'_{n_k+1}, p'_{n_k+1} + 1\} \cap (\mathbb{Z} \setminus 2\mathbb{Z})$. Note that $\sigma_{n_k} \leq p_{n_k+1}$, similar to (3.4) and (3.5), we have $\frac{|\lambda_1|}{\mathcal{P}_{n_k}} \leq \frac{3}{2p_{n_k}}$ for $\lambda_1 \in \Lambda^1_{n_k-1}$ and $\frac{|\lambda_1|}{\mathcal{P}_{n_k+1}} \leq \frac{1}{4} + \frac{3}{2p_{n_k+1}}$ for $\lambda_1 \in \Lambda^1_{n_k}$.

In Case IV, the choices of σ_i and σ'_i are the same as Cases III$_\kappa$ and II$_\kappa$ respectively, where $\kappa = 1$ or 2. Here we omit these estimates.

In order to estimate the lower bound of $\prod_{j=n_k+1}^\infty J_j(\xi + \lambda)$, where $\xi = (\xi_1, \xi_2)^t \in [0, 1]^2$ and $\lambda \in \Lambda^{\sigma,\sigma'}_{n_k}$, we will consider two cases: $\lambda \in \Lambda^{\sigma,\sigma'}_{n_k-1}$ and $\lambda \in \Lambda^{\sigma,\sigma'}_{n_k} \setminus \Lambda^{\sigma,\sigma'}_{n_k-1}$.

Step 2. We then estimate the lower bound of $\prod_{j=n_k+1}^\infty J_j(\xi + \lambda)$ for $\lambda = (\lambda_1, \lambda_2)^t \in \Lambda^{\sigma,\sigma'}_{n_k-1}$.

Observe that $\frac{|\lambda_1|}{\mathcal{P}_{n_k}}, \frac{|\lambda_2|}{\mathcal{Q}_{n_k}} \leq \frac{3}{2p_{n_k}}$ for all cases, it follows from $p_n \geq 10c\pi$ and $q_n \geq 10d\pi$ for $n \geq n_1$ that

$$2\pi|\eta_{j,1}| = 2\pi \left| \frac{a_j}{p_j} \frac{\xi_1 + \lambda_1}{\mathcal{P}_{n_k}} \frac{1}{\mathcal{P}_{n_k+1,j-1}} \right| \leq 2\pi c(\frac{1}{p_{n_k}} + \frac{3}{2p_{n_k}}) \frac{1}{2^{j-n_k-1}} \leq \frac{1}{2^{j-n_k}}$$

and similarly, $2\pi|\eta_{j,2}| \leq \frac{1}{2^{j-n_k}}$. Applying $\cos x \geq 1 - \frac{1}{2}x^2$, we obtain

$$\begin{aligned} \prod_{j=n_k+1}^\infty J_j(\xi + \lambda) &\geq \prod_{j=n_k+1}^\infty \frac{1}{4} \left| 1 + \cos(2\pi\eta_{j,1}) + \cos(2\pi\eta_{j,2}) + \cos(\pm 2\pi(\eta_{j,1} + \eta_{j,2})) \right| \\ &\geq \prod_{j=n_k+1}^\infty \frac{1}{4} \left| 1 + 3\cos\frac{1}{2^{j-n_k-1}} \right| \geq \prod_{j=0}^\infty \left(1 - \frac{3}{8}\frac{1}{4^j}\right) := \alpha_1 > 0. \end{aligned} \tag{3.6}$$

Step 3. We estimate the lower bound of $\prod_{j=n_k+2}^\infty J_j(\xi+\lambda)$ for $\lambda = (\lambda_1, \lambda_2)^t \in \Lambda^{\sigma,\sigma'}_{n_k} \setminus \Lambda^{\sigma,\sigma'}_{n_k-1}$.

We first claim that $2\pi|\eta_{j,1}|,\ 2\pi|\eta_{j,2}| \leq \frac{1}{2^{j-n_k-1}}$ for $j \geq n_k + 2$. In Cases I, II and III$_1$, by $\frac{|\lambda_1|}{\mathcal{P}_{n_k+1}} \leq \frac{1}{p_{n_k+1}}$ and $p_n \geq 10c\pi$ for $n \geq n_1$, we have

$$2\pi|\eta_{j,1}| = 2\pi \left| \frac{a_j}{p_j} \frac{\xi_1 + \lambda_1}{\mathcal{P}_{n_k+1}} \frac{1}{\mathcal{P}_{n_k+2,j-1}} \right| \leq 2\pi c(\frac{1}{p_{n_{k+1}}} + \frac{1}{p_{n_{k+1}}}) \frac{1}{2^{j-n_k-2}} \leq \frac{8c\pi}{p_{n_{k+1}}} \frac{1}{2^{j-n_k-1}} \leq \frac{1}{2^{j-n_k-1}}.$$

In Case III$_2$, by $\frac{|\lambda_1|}{\mathcal{P}_{n_k+1}} \leq \frac{3}{2p_{n_k+1}}$ and $p_n \geq \max\{10c\pi, 10\}$ for $n \geq n_1$, we have

$$2\pi|\eta_{j,1}| = 2\pi \left| \frac{a_j}{p_j} \frac{\xi_1 + \lambda_1}{\mathcal{P}_{n_k+1}} \frac{1}{\mathcal{P}_{n_k+2,j-1}} \right| \leq 2\pi c(\frac{1}{4} + \frac{5}{2p_{n_k+1}}) \frac{1}{10c\pi} \frac{1}{2^{j-n_k-3}} \leq \frac{1}{10} \frac{1}{2^{j-n_k-3}} \leq \frac{1}{2^{j-n_k-1}}.$$

Similarly, we can prove that $2\pi|\eta_{j,2}| \leq \frac{1}{2^{j-n_k-1}}$ for all cases. Thus the claim follows. Similar to (3.6), we deduce that

$$\prod_{j=n_k+2}^{\infty} J_j(\xi+\lambda) \geq \prod_{j=n_k+2}^{\infty} \frac{1}{4}\Big|1+3\cos\frac{1}{2^{j-n_k-2}}\Big| \geq \prod_{j=0}^{\infty}\Big(1-\frac{3}{8}\frac{1}{4^j}\Big) > 0. \tag{3.7}$$

Step 4. We estimate the lower bound of $J_{n_k+1}(\xi+\lambda)$ for $\lambda = (\lambda_1,\lambda_2)^t \in \Lambda_{n_k}^{\sigma,\sigma'} \setminus \Lambda_{n_k-1}^{\sigma,\sigma'}$.

The difficult case is that $j = n_k+1$. Since $\lambda \in \Lambda_{n_k}^{\sigma,\sigma'} \setminus \Lambda_{n_k-1}^{\sigma,\sigma'}$, we have

$$-\eta_{n_k+1,1} = \frac{a_{n_k+1}}{p_{n_k+1}}\frac{\xi_1+\lambda_1}{\mathcal{P}_{n_k}} = \frac{a_{n_k+1}}{p_{n_k+1}}\frac{\sigma_{n_k}}{2^{1+s_{n_k}}} + \frac{a_{n_k+1}}{\mathcal{P}_{n_k+1}}(\xi_1+\lambda_1'),$$

$$-\eta_{n_k+1,2} = \frac{b_{n_k+1}}{q_{n_k+1}}\frac{\xi_2+\lambda_2}{\mathcal{Q}_{n_k}} = \frac{b_{n_k+1}}{q_{n_k+1}}\frac{\sigma_{n_k}'}{2^{1+t_{n_k}}} + \frac{b_{n_k+1}}{\mathcal{Q}_{n_k+1}}(\xi_2+\lambda_2')$$

for some $\lambda' = (\lambda_1',\lambda_2')^t \in \Lambda_{n_k-1}^{\sigma,\sigma'}$. It should be noted that $\lambda_1 = \lambda_1'$ and $\lambda_2 = \lambda_2'$ cannot hold at the same time. Without loss of generality, we always suppose that $\lambda_i \neq \lambda_i'$ for all $i = 1,2$. Otherwise, $\lambda_i \in \Lambda_{n_k-1}^i$ for $i = 1$ or 2 and the estimates can be obtained by using the same method as before. We now focus on estimating $J_{n_k+1}(\xi+\lambda)$ for all cases.

In Case I, since $s_{n_k} \geq \max\{2, \lfloor\log_2 4c\rfloor+1\}$, $t_{n_k} \geq \max\{2, \lfloor\log_2 4d\rfloor+1\}$, $p_{n_k} \geq 10c\pi > 8c$ and $q_{n_k} \geq 10d\pi > 8d$, it follows from σ_{n_k}, $\sigma_{n_k}' = 1$, (3.2) and (3.3) that

$$|\eta_{n_k+1,1}| = \frac{a_{n_k+1}}{p_{n_k+1}}\frac{1}{2^{1+s_{n_k}}} + \frac{a_{n_k+1}}{p_{n_k+1}}\frac{\xi_1+\lambda_1'}{\mathcal{P}_{n_k}} \leq \frac{c}{2^{1+s_{n_k}}} + \frac{2c}{p_{n_k}} \leq \frac{3}{8}$$

and similarly, $|\eta_{n_k+1,2}| \leq \frac{3}{8}$. Then $(\eta_{n_k+1,1},\eta_{n_k+1,2}) \in [-\frac{3}{8},\frac{3}{8}]^2 := \Theta_0$. Now we consider the continuous function

$$f(x,y) := \frac{1}{4}|1+e^{2\pi ix}+e^{2\pi iy}+e^{\pm 2\pi i(x+y)}|, \quad (x,y)^t \in \mathbb{R}^2$$

and denote $f^+ = \frac{1}{4}|1+e^{2\pi ix}+e^{2\pi iy}+e^{2\pi i(x+y)}|$, $f^- = \frac{1}{4}|1+e^{2\pi ix}+e^{2\pi iy}+e^{-2\pi i(x+y)}|$. It is easy to calculate that $\mathcal{Z}(f) = \mathcal{Z}(f^+)$ or $\mathcal{Z}(f) = \mathcal{Z}(f^-)$, where

$$\mathcal{Z}(f^+) = \left\{\begin{pmatrix}\frac{1}{2}+k_1\\ \xi_1\end{pmatrix}, \begin{pmatrix}\xi_2\\ \frac{1}{2}+k_2\end{pmatrix} : k_1,k_2\in\mathbb{Z},\ \xi_1,\xi_2\in\mathbb{R}\right\},$$

$$\mathcal{Z}(f^-) = \left\{\begin{pmatrix}\frac{1}{2}+k_1\\ \frac{k_2}{2}\end{pmatrix}, \begin{pmatrix}\frac{k_1'}{2}\\ \frac{1}{2}+k_2'\end{pmatrix} : k_1,k_2,k_1',k_2'\in\mathbb{Z}\right\}.$$

It is clear that $\Theta_0 \cap \mathcal{Z}(f) = \emptyset$. Observe that $f(\eta_{j,1},\eta_{j,2}) = J_j(\xi+\lambda)$, we conclude from $(\eta_{n_k+1,1},\eta_{n_k+1,2}) \in \Theta_0$ and the integer-periodicity of f that

$$\min_{\xi\in[0,1]^2,\lambda\in\Lambda_{n_k}^{\sigma,\sigma'}\setminus\Lambda_{n_k-1}^{\sigma,\sigma'}} J_{n_k+1}(\xi+\lambda) \geq \min_{(x,y)\in\Theta_0} f(x,y) := \gamma_0 > 0.$$

Consequently,

$$J_{n_k+1}(\xi+\lambda) \geq \gamma_0 > 0. \tag{3.8}$$

In Case II, it is proved in Case I that $\eta_{n_k+1,1} \in [-\frac{3}{8}, \frac{3}{8}]$, so we only need to estimate $\eta_{n_k+1,2}$. Corresponding to Cases II_1 and II_2, we will give different estimates.

We first consider Case II_1. As $\sigma'_{n_k} = 1$, we have $-\eta_{n_k+1,2} = \frac{1}{2^{1+t}}\frac{b_{n_k+1}}{q_{n_k+1}} + \frac{b_{n_k+1}}{Q_{n_k+1}}(\xi_2 + \lambda'_2)$, where $0 \le t < \infty$. Let $\delta_0 := |(\frac{d'}{2^{1+t}}) - \frac{1}{2}|$, where $(x) = x - \lfloor x \rfloor$, then $\delta_0 > 0$ by $d' \in \mathbb{N} + \{\frac{1}{2}\}$. Since $\lim_{k\to\infty} \frac{b_{n_k+1}}{q_{n_k+1}} = d'$ and $\lim_{k\to\infty} q_{n_k} = \infty$, we further suppose that the sequence $\{n_k\}_{k=1}^{\infty}$ satisfies $|\frac{1}{2^{1+t}}\frac{b_{n_k+1}}{q_{n_k+1}} - \frac{d'}{2^{1+t}}| < \frac{\delta_0}{4}$ and $q_{n_k} \ge \max\{10d\pi, 10, \frac{16d}{\delta_0}\}$ for all $k \ge 1$. Then

$$\frac{1}{2^{1+t}}\frac{b_{n_k+1}}{q_{n_k+1}} \in \Big[(\frac{d'}{2^{1+t}}) - \frac{\delta_0}{4}, (\frac{d'}{2^{1+t}}) + \frac{\delta_0}{4}\Big] + \mathbb{Z} \subset \Big(\Big[-\frac{\delta_0}{4}, \frac{1}{2} - \frac{\delta_0}{4}\Big] \bigcup \Big[\frac{1}{2} + \frac{\delta_0}{4}, 1 + \frac{\delta_0}{4}\Big]\Big) + \mathbb{Z}. \tag{3.9}$$

Furthermore, it follows from $\frac{|\lambda'_2|}{Q_{n_k}} \le \frac{1}{q_{n_k}}$ and $\frac{b_{n_k+1}}{q_{n_k+1}} \le d$ that $\left|\frac{b_{n_k+1}}{q_{n_k+1}}\frac{\lambda'_2}{Q_{n_k}} + \frac{b_{n_k+1}}{q_{n_k+1}}\frac{\xi_2}{Q_{n_k}}\right| \le \frac{2d}{q_{n_k}} \le \frac{\delta_0}{8}$. This implies from (3.9) that

$$\eta_{n_k+1,2} \in \Big(\Big[-\frac{3\delta_0}{8}, \frac{1}{2} - \frac{\delta_0}{8}\Big] \bigcup \Big[\frac{1}{2} + \frac{\delta_0}{8}, 1 + \frac{3\delta_0}{8}\Big]\Big) + \mathbb{Z} := E_0.$$

Let $\Theta_1 := [-\frac{3}{8}, \frac{3}{8}] \times E_0$, so $(\eta_{n_k+1,1}, \eta_{n_k+1,2}) \in \Theta_1$. Clearly, $\Theta_1 \cap \mathcal{Z}(f) = \emptyset$. Similar to (3.8), we have $J_{n_k+1}(\xi + \lambda) \ge \gamma_1 > 0$, where $\gamma_1 := \min_{(x,y)\in\Theta_1} f(x,y)$.

We then consider Case II_2. As $\sigma'_{n_k} \in \{q'_{n_k+1}, q'_{n_k+1} + 1\} \cap (\mathbb{Z} \setminus 2\mathbb{Z})$, we have $\frac{b_{n_k+1}}{q_{n_k+1}}\frac{\sigma'_{n_k}}{2^{1+t}} \in \{\frac{b'_{n_k+1}}{2^{2+t}}, \frac{b'_{n_k+1}}{2^{2+t}} + \frac{1}{2^{1+t}}\frac{b_{n_k+1}}{q_{n_k+1}}\}$, where $0 \le t < \infty$. Let

$$\delta_1 := \min_{1 \le j \le 2^{2+t}-1, j \notin 2\mathbb{Z}} \left\{(\frac{j}{2^{2+t}} + (\frac{d'}{2^{1+t}}) - \frac{1}{2}), 1 - (\frac{j}{2^{2+t}} + (\frac{d'}{2^{1+t}}) - \frac{1}{2}), \frac{1}{2^{2+t}}\right\},$$

where $(x) = x - \lfloor x \rfloor$, then $\delta_1 > 0$ by $d' \notin \mathbb{N} + \{\frac{1}{2}\}$. Since $\lim_{k\to\infty} \frac{b_{n_k+1}}{q_{n_k+1}} = d'$ and $\lim_{k\to\infty} q_{n_k} = \infty$, suppose that the sequence $\{n_k\}_{k=1}^{\infty}$ satisfies $|\frac{1}{2^{1+t}}\frac{b_{n_k+1}}{q_{n_k+1}} - \frac{d'}{2^{1+t}}| < \frac{\delta_1}{4}$ and $q_{n_k} \ge \max\{10d\pi, 10, \frac{20d}{\delta_1}\}$ for all $k \ge 1$. Then we conclude from $b'_{n_k+1} \in \mathbb{Z} \setminus 2\mathbb{Z}$ that

$$\frac{b'_{n_k+1}}{2^{2+t}} + \frac{1}{2^{1+t}}\frac{b_{n_k+1}}{q_{n_k+1}} \in \Big[\frac{j}{2^{2+t}} + (\frac{d'}{2^{1+t}}) - \frac{\delta_1}{4}, \frac{j}{2^{2+t}} + (\frac{d'}{2^{1+t}}) + \frac{\delta_1}{4}\Big] + \mathbb{Z}.$$

By the definition of δ_1, we have

$$\frac{j}{2^{2+t}} + (\frac{d'}{2^{1+t}}) \in \Big(\Big[0, \frac{1}{2} - \delta_1\Big] \bigcup \Big[\frac{1}{2} + \delta_1, 1\Big]\Big) + \mathbb{Z} \subset \Big(\Big[0, \frac{1}{2} - \frac{\delta_1}{2}\Big] \bigcup \Big[\frac{1}{2} + \frac{\delta_1}{2}, 1\Big]\Big) + \mathbb{Z}.$$

Hence

$$\frac{b'_{n_k+1}}{2^{2+t}} + \frac{1}{2^{1+t}}\frac{b_{n_k+1}}{q_{n_k+1}} \in \Big(\Big[-\frac{\delta_1}{4}, \frac{1}{2} - \frac{\delta_1}{4}\Big] \bigcup \Big[\frac{1}{2} + \frac{\delta_1}{4}, 1 + \frac{\delta_1}{4}\Big]\Big) + \mathbb{Z} := E_{\delta_1}.$$

It is easy to see that $\frac{b'_{n_k+1}}{2^{2+t}} \in E_{\delta_1}$. Consequently, $\frac{b_{n_k+1}}{q_{n_k+1}}\frac{\sigma'_{n_k}}{2^{1+t}} \in E_{\delta_1}$. As $\frac{|\lambda'_2|}{Q_{n_k}} \le \frac{3}{2q_{n_k}}$ and $\frac{b_{n_k+1}}{q_{n_k+1}} \le d$, we further have $\left|\frac{b_{n_k+1}}{q_{n_k+1}}\frac{\lambda'_2}{Q_{n_k}} + \frac{b_{n_k+1}}{q_{n_k+1}}\frac{\xi_2}{Q_{n_k}}\right| \le \frac{5d}{2q_{n_k}} \le \frac{\delta_1}{8}$. It follows that

$$\eta_{n_k+1,2} \in \Big(\Big[-\frac{3\delta_1}{8}, \frac{1}{2} - \frac{\delta_1}{8}\Big]\bigcup\Big[\frac{1}{2} + \frac{\delta_1}{8}, 1 + \frac{3\delta_1}{8}\Big]\Big) + \mathbb{Z} := E_1.$$

Let $\Theta_2 := [-\frac{3}{8}, \frac{3}{8}] \times E_1$, then $(\eta_{n_k+1,1}, \eta_{n_k+1,2}) \in \Theta_2$. Similar to (3.8), by $\Theta_2 \cap \mathcal{Z}(f) = \emptyset$, we have $J_{n_k+1}(\xi + \lambda) \ge \gamma_2 > 0$, where $\gamma_2 := \min_{(x,y)\in\Theta_2} f(x, y)$.

In Case III, it is proved in Case I that $\eta_{n_k+1,2} \in [-\frac{3}{8}, \frac{3}{8}]$. The estimate of $\eta_{n_k+1,1}$ is similar to that of $\eta_{n_k+1,2}$ in Case II. Using the similar argument as in Case II, we can prove that there exists a constant $\gamma_3 > 0$ such that $J_{n_k+1}(\xi + \lambda) \ge \gamma_3$.

In Case IV, the estimates of $\eta_{n_k+1,1}$ and $\eta_{n_k+1,2}$ are similar to that of $\eta_{n_k+1,2}$ in Case II. Hence, using the similar argument as before, we can find a constant $\gamma_4 > 0$ such that $J_{n_k+1}(\xi + \lambda) \ge \gamma_4$.

Therefore, we have $J_{n_k+1}(\xi + \lambda) \ge \min\{\gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_4\} := \gamma > 0$ for all cases. By (3.7), we obtain $\prod_{j=n_k+1}^{\infty} J_j(\xi + \lambda) \ge \gamma \prod_{j=0}^{\infty}\left(1 - \frac{3}{8}\frac{1}{4^j}\right) := \alpha_2 > 0$ for $\lambda \in \Lambda_{n_k}^{\sigma,\sigma'} \setminus \Lambda_{n_{k-1}}^{\sigma,\sigma'}$. This together with (3.6) shows that $|\hat{\mu}_{>n_k}(\xi + \lambda)| = \prod_{j=n_k+1}^{\infty} J_j(\xi + \lambda) \ge \alpha$ for $\lambda \in \Lambda_{n_k}^{\sigma,\sigma'}$, where $\alpha := \min\{\alpha_1, \alpha_2\}$. Hence, the proof is completed. □

Define

$$Q_n(\xi) = \sum_{\lambda\in\Lambda_n^{\sigma,\sigma'}} |\hat{\mu}_{\{M_n\},\{\mathcal{D}_n\}}(\xi + \lambda)|^2 \quad \text{and} \quad Q_{\Lambda^{\sigma,\sigma'}}(\xi) = \sum_{\lambda\in\Lambda^{\sigma,\sigma'}} |\hat{\mu}_{\{M_n\},\{\mathcal{D}_n\}}(\xi + \lambda)|^2.$$

We now present the proof of Theorem 1.2.

Proof of Theorem 1.2. Let $\Lambda^{\sigma,\sigma'} = \bigcup_{k=1}^{\infty} \Lambda_{n_k}^{\sigma,\sigma'}$, where $\Lambda_{n_k}^{\sigma,\sigma'}$ is given in Proposition 3.1. We will prove that $\Lambda^{\sigma,\sigma'}$ is a spectrum of $\mu_{\{M_n\},\{\mathcal{D}_n\}}$. It follows from $\hat{\mu}_{\{M_n\},\{\mathcal{D}_n\}} = \hat{\mu}_{n_k}\hat{\mu}_{>n_k}$ and Proposition 3.1 that $|\hat{\mu}_{\{M_n\},\{\mathcal{D}_n\}}(\xi + \lambda)|^2 \ge \alpha^2|\hat{\mu}_{n_k}(\xi + \lambda)|^2$. By Theorem 2.1(ii) and Lemma 2.4, we have $\sum_{\lambda\in\Lambda_{n_k}^{\sigma,\sigma'}} |\hat{\mu}_{n_k}(\xi + \lambda)|^2 = 1$. Then, for $k > 1$, we get

$$\begin{aligned} Q_{n_k}(\xi) &= Q_{n_{k-1}}(\xi) + \sum_{\lambda\in\Lambda_{n_k}^{\sigma,\sigma'}\setminus\Lambda_{n_{k-1}}^{\sigma,\sigma'}} |\hat{\mu}_{\{M_n\},\{\mathcal{D}_n\}}(\xi + \lambda)|^2 \\ &\ge Q_{n_{k-1}}(\xi) + \alpha^2\Big(1 - \sum_{\lambda\in\Lambda_{n_{k-1}}^{\sigma,\sigma'}} |\hat{\mu}_{n_k}(\xi + \lambda)|^2\Big). \end{aligned} \tag{3.10}$$

If $Q_{\Lambda^{\sigma,\sigma'}}(\xi) \not\equiv 1$, then there exists $\xi_0 \in [0, 1]^2$ such that $Q_{\Lambda^{\sigma,\sigma'}}(\xi_0) < 1$. Let ϑ_0 satisfy $\max\{Q_{\Lambda^{\sigma,\sigma'}}(\xi_0), e^{-2}\} < \vartheta_0 < 1$. Assume that $\{n_k\}_{k=1}^{\infty}$ satisfies the conditions of Proposition 3.1 and $n_{k+1} - n_k \ge 2 - \log_4 \ln \vartheta_0^{-1/2}$ for $k \ge 1$ (otherwise we choose a subsequence of $\{n_k\}_{k=1}^{\infty}$ to replace it). Now we prove that

$$|\hat{\mu}_{>n_k}(\xi_0 + \lambda)| \ge \sqrt{\vartheta_0} \quad \text{for } k > 1 \text{ and } \lambda \in \Lambda_{n_{k-1}}^{\sigma,\sigma'}.$$

It can be seen from the proof of Proposition 3.1 that $2\pi|\eta_{j,1}(\xi_0+\lambda)|,\ 2\pi|\eta_{j,2}(\xi_0+\lambda)| \le \frac{1}{2^{j-n_{k-1}-1}}$ for $j > n_k$ and $\lambda \in \Lambda_{n_{k-1}}^{\sigma,\sigma'}$. Hence, using $\cos x \ge 1-\frac{1}{2}x^2$ and $\ln(1-x) \ge -5x$ for $0 \le x \le 4/5$, we have

$$\begin{aligned}
|\hat{\mu}_{>n_k}(\xi_0+\lambda)| &= \prod_{j=n_k+1}^{\infty} J_j(\xi_0+\lambda) \ge \prod_{j=n_k+1}^{\infty} \frac{1}{4}\Big(1+3\cos\frac{1}{2^{j-n_{k-1}-2}}\Big) \\
&= \prod_{j=0}^{\infty} \frac{1}{4}\Big(1+3\cos\frac{1}{2^{j+n_k-n_{k-1}-1}}\Big) \ge \prod_{j=0}^{\infty}\Big(1-\frac{3}{8}\frac{1}{4^{j+n_k-n_{k-1}-1}}\Big) \\
&\ge \exp\Big(-\frac{15}{8}\sum_{j=0}^{\infty}\frac{1}{4^{j+n_k-n_{k-1}-1}}\Big) \ge \exp(-4^{2-n_k+n_{k-1}}) \ge \sqrt{\vartheta_0}.
\end{aligned}$$

Thus $|\hat{\mu}_{\{M_n\},\{\mathcal{D}_n\}}(\xi_0+\lambda)|^2 \ge \vartheta_0|\hat{\mu}_{n_k}(\xi_0+\lambda)|^2$, and hence (3.10) becomes

$$Q_{n_k}(\xi_0) \ge Q_{n_{k-1}}(\xi_0)+\alpha^2(1-\vartheta_0^{-1}\sum_{\lambda\in\Lambda_{n_{k-1}}^{\sigma,\sigma'}}|\hat{\mu}_{\{M_n\},\{\mathcal{D}_n\}}(\xi_0+\lambda)|^2) \ge Q_{n_{k-1}}(\xi_0)+\alpha^2(1-\vartheta_0^{-1}Q_{\Lambda^{\sigma,\sigma'}}(\xi_0)).$$

Therefore

$$1 \ge Q_{\Lambda^{\sigma,\sigma'}}(\xi_0) \ge Q_{n_k}(\xi_0) \ge Q_{n_1}(\xi_0)+(k-1)\alpha^2(1-\vartheta_0^{-1}Q_{\Lambda^{\sigma,\sigma'}}(\xi_0))$$

for $k > 1$, which is impossible when k is large enough. Hence, $Q_{\Lambda^{\sigma,\sigma'}}(\xi) \equiv 1$ and the assertion follows by Theorem 2.1. □

4. **Proof of Theorem 1.1**

In this section, we will prove Theorem 1.1 by decomposing it into two cases. **Case A** : There exists an increasing subsequence $\{n_k\}_{k=1}^{\infty}$ of $\mathbb{N}$ such that $\frac{a_{n_k+1}}{p_{n_k+1}},\ \frac{b_{n_k+1}}{q_{n_k+1}} \le r$ for some $r<1$. **Case B** : $\lim_{n\to\infty}\frac{a_n}{p_n}=1$ or $\lim_{n\to\infty}\frac{b_n}{q_n}=1$. We begin with two propositions for the above two cases respectively. Recall that $\Lambda_n^{\sigma,\sigma'}$ and $\Lambda^{\sigma,\sigma'}$ are defined by (2.6).

Proposition 4.1. *Under the conditions of Theorem 1.1, suppose that Case A holds and let l be a positive integer such that $\frac{2^{l-1}+1}{2^l-1} < \min\{\frac{1}{2r},\frac{2}{\pi}\}$. Then there exist $\sigma,\ \sigma' \in \{-1,1\}^{\mathbb{N}}$ such that the following statements hold:*

(i) *There exists $\alpha_{r,l} > 0$ such that $|\hat{\mu}_{>n_k}(\xi+\lambda)| \ge \alpha_{r,l}$ for $k>1$, $\xi\in[0,1]^2$ and $\lambda\in\Lambda_{n_k}^{\sigma,\sigma'}$;*
(ii) *If $n_k-n_{k-1} \ge 3$, then $|\hat{\mu}_{>n_k}(\xi+\lambda)| \ge \exp(-4^{4-n_k+n_{k-1}})$ for $k>1$, $\xi\in[0,1]^2$ and $\lambda\in\Lambda_{n_{k-1}}^{\sigma,\sigma'}$.*

Proof. (i) Without loss of generality, we assume that $\{n_k\}_{k=1}^{\infty}$ satisfies $n_{k+1}-n_k \ge l$ for each $k\ge1$ and $n_1 \ge l$. Choose σ and σ' such that $\sigma_i,\ \sigma_i' = 1$ if $i\in\{n_k : k\ge1\}$ and $\sigma_i,\ \sigma_i' = -1$ for otherwise. First, we claim that

$$\frac{|\lambda_1|}{\mathcal{P}_{n_k}} \le \frac{2^{l-1}}{2^l-1},\quad \frac{|\lambda_2|}{\mathcal{Q}_{n_k}} \le \frac{2^{l-1}}{2^l-1}\quad \text{for } \lambda=(\lambda_1,\lambda_2)^t\in\Lambda_{n_k}^{\sigma,\sigma'} \text{ and } k>1. \tag{4.1}$$

In fact, for $\lambda \in \Lambda_{n_k-1}^{\sigma,\sigma'}$, we have $\lambda_1 \in \sum_{i=1}^{n_k-1} \mathcal{P}_i\{0, \frac{\sigma_i}{2^{1+s_i}}\}$ and $\lambda_2 \in \sum_{i=1}^{n_k-1} \mathcal{Q}_i\{0, \frac{\sigma_i'}{2^{1+t_i}}\}$. Then

$$\frac{|\lambda_1|}{\mathcal{P}_{n_k}} \leq \sum_{i=1}^{n_k-1} \frac{\mathcal{P}_i}{\mathcal{P}_{n_k}} \frac{1}{2^{1+s_i}} = \frac{1}{p_{n_k}}\Big(\frac{1}{2^{1+s_{n_k-1}}} + \sum_{i=1}^{n_k-2} \frac{1}{\mathcal{P}_{i+1,n_k-1}} \frac{1}{2^{1+s_i}}\Big) \leq \frac{1}{2^{1+s_{n_k}}} \sum_{i=1}^{\infty} \frac{1}{2^i} = \frac{1}{2^{1+s_{n_k}}} \leq \frac{2^{l-1}}{2^l - 1}$$

and similarly, $\frac{|\lambda_2|}{\mathcal{Q}_{n_k}} \leq \frac{2^{l-1}}{2^l-1}$. For $\lambda \in \Lambda_{n_k}^{\sigma,\sigma'} \setminus \Lambda_{n_k-1}^{\sigma,\sigma'}$, we have $\lambda_1 \in \sum_{i=1}^{n_k} \mathcal{P}_i\{0, \frac{\sigma_i}{2^{1+s_i}}\}$ and $\lambda_2 \in \sum_{i=1}^{n_k} \mathcal{Q}_i\{0, \frac{\sigma_i'}{2^{1+t_i}}\}$. Then there exist $l_i \in \{0, \frac{\sigma_i}{2^{1+s_i}}\}$ and $l_i' \in \{0, \frac{\sigma_i'}{2^{1+t_i}}\}$ such that $\lambda_1 = \sum_{i=1}^{n_k} \mathcal{P}_i l_i$ and $\lambda_2 = \sum_{i=1}^{n_k} \mathcal{Q}_i l_i'$, where $l_{n_k} = 0$ and $l_{n_k}' = 0$ cannot hold at the same time. Without loss of generality, we always suppose that $l_{n_k} = \frac{\sigma_{n_k}}{2^{1+s_{n_k}}}$ and $l_{n_k}' = \frac{\sigma_{n_k}'}{2^{1+t_{n_k}}}$. Otherwise, $\lambda_i \in \Lambda_{n_k-1}^i$ for $i = 1$ or 2 and the estimates can be obtained by using the same method as before. Hence, by $\sigma_{n_k} = \sigma_{n_k}' = 1$ and $n_k - n_i = \sum_{j=i}^{k-1}(n_{j+1} - n_j) \geq (k-i)l$, we have

$$\frac{\lambda_1}{\mathcal{P}_{n_k}} \geq \frac{1}{2^{1+s_{n_k}}} - \frac{1}{p_{n_k}}\Big(\frac{1}{2^{1+s_{n_k-1}}} + \sum_{i=1}^{n_k-2} \frac{1}{\mathcal{P}_{i+1,n_k-1}} \frac{1}{2^{1+s_i}}\Big) \geq \frac{1}{2^{1+s_{n_k}}}(1 - \sum_{i=1}^{n_k-1} \frac{1}{2^{n_k-i}}) \geq 0$$

and similarly, $\frac{\lambda_2}{\mathcal{Q}_{n_k}} \geq 0$. Moreover,

$$\frac{\lambda_1}{\mathcal{P}_{n_k}} \leq \frac{1}{2^{1+s_{n_k}}} + \sum_{i=1}^{k-1} \frac{1}{\mathcal{P}_{n_i+1,n_k}} \frac{1}{2^{1+s_{n_i}}} \leq \frac{1}{2}(1 + \sum_{i=1}^{k-1} \frac{1}{2^{n_k-n_i}}) \leq \frac{1}{2} \sum_{i=0}^{\infty} \frac{1}{2^{li}} = \frac{2^{l-1}}{2^l - 1}$$

and similarly, $\frac{\lambda_2}{\mathcal{Q}_{n_k}} \leq \frac{2^{l-1}}{2^l-1}$. Thus the claim follows.

Therefore, for $\xi = (\xi_1, \xi_2)^t \in [0,1]^2$, since $n_1 \geq l$ and $\mathcal{P}_{n_k}, \mathcal{Q}_{n_k} \geq 2^{n_k}$, by (4.1), we have

$$\left|\frac{\xi_1 + \lambda_1}{\mathcal{P}_{n_k}}\right| \leq \frac{1}{\mathcal{P}_{n_k}} + \frac{2^{l-1}}{2^l - 1} \leq \frac{2^{l-1}+1}{2^l - 1}, \quad \left|\frac{\xi_2 + \lambda_2}{\mathcal{Q}_{n_k}}\right| \leq \frac{1}{\mathcal{Q}_{n_k}} + \frac{2^{l-1}}{2^l - 1} \leq \frac{2^{l-1}+1}{2^l - 1}. \tag{4.2}$$

When $j = n_k + 1, \ldots, n_k + 3$, since $\frac{2^{l-1}+1}{2^l-1} < \min\{\frac{1}{2r}, \frac{2}{\pi}\}$, it follows from $a_j \leq p_j,\ b_j \leq q_j$, (4.2) and $\frac{a_{n_k+1}}{p_{n_k+1}}, \frac{b_{n_k+1}}{q_{n_k+1}} \leq r$ that

$$|\eta_{j,1}| = \left|\frac{a_j}{p_j} \frac{\xi_1 + \lambda_1}{\mathcal{P}_{n_k}} \frac{1}{\mathcal{P}_{n_k+1,j-1}}\right| \leq \frac{2^{l-1}+1}{2^l - 1}\left|\frac{a_{n_k+1}}{p_{n_k+1}}\right| \leq \frac{2^{l-1}+1}{2^l - 1} r < \frac{1}{2} \tag{4.3}$$

and similarly, $|\eta_{j,2}| \leq \frac{2^{l-1}+1}{2^l-1} r < \frac{1}{2}$. Let $\Theta_3 := [-\frac{2^{l-1}+1}{2^l-1} r, \frac{2^{l-1}+1}{2^l-1} r]^2$, then $(\eta_{j,1}, \eta_{j,2}) \in \Theta_3$. Similar to (3.8), by $\Theta_3 \cap \mathcal{Z}(f) = \emptyset$, we obtain

$$\prod_{j=n_k+1}^{n_k+3} J_j(\xi + \lambda) \geq \beta_{r,l}^3 > 0, \quad \text{where } \beta_{r,l} := \min_{(x,y)\in\Theta_3} f(x,y). \tag{4.4}$$

When $j \geq n_k + 4$, as $a_j \leq p_j,\ b_j \leq q_j$, it follows from (4.2) and $\frac{2^{l-1}+1}{2^l-1} < \min\{\frac{1}{2r}, \frac{2}{\pi}\}$ that

$$2\pi|\eta_{j,1}| = 2\pi\left|\frac{a_j}{p_j} \frac{\xi_1 + \lambda_1}{\mathcal{P}_{n_k}} \frac{1}{\mathcal{P}_{n_k+1,j-1}}\right| \leq \frac{2^{l-1}+1}{2^l - 1} \frac{\pi}{2^{j-n_k-2}} \leq \frac{1}{2^{j-n_k-3}} \tag{4.5}$$

and similarly, $2\pi|\eta_{j,2}| \leq \frac{1}{2^{j-n_k-3}}$. Then $\prod_{j=n_k+4}^{\infty} J_j(\xi + \lambda) \geq \prod_{j=0}^{\infty}\left(1 - \frac{3}{8}\frac{1}{4^j}\right) > 0$ (similar to (3.6)). Combining with (4.4), we get that $|\hat{\mu}_{>n_k}(\xi + \lambda)| \geq \beta_{r,l}^3 \prod_{j=0}^{\infty}\left(1 - \frac{3}{8}\frac{1}{4^j}\right) := \alpha_{r,l} > 0$.

(ii) For $\lambda = (\lambda_1, \lambda_2)^t \in \Lambda_{n_{k-1}}^{\sigma,\sigma'}$ and $\xi = (\xi_1, \xi_2)^t \in [0,1]^2$, similar to (4.2) and (4.5), we can obtain that $\left|\frac{\xi_1+\lambda_1}{\mathcal{P}_{n_{k-1}}}\right|$, $\left|\frac{\xi_2+\lambda_2}{Q_{n_{k-1}}}\right| \le \frac{2^{l-1}+1}{2^l-1}$ and $2\pi|\eta_{j,1}|$, $2\pi|\eta_{j,2}| \le \frac{1}{2^{j-n_{k-1}-3}}$. Therefore, according to $\cos x \ge 1 - \frac{1}{2}x^2$ and $\ln(1-x) \ge -5x$ for $0 \le x \le 4/5$, we have

$$
\begin{aligned}
|\hat{\mu}_{>n_k}(\xi+\lambda)| &= \prod_{j=n_k+1}^{\infty} J_j(\xi+\lambda) \ge \prod_{j=n_k+1}^{\infty} \frac{1}{4}\Big(1 + 3\cos\frac{1}{2^{j-n_{k-1}-4}}\Big) \\
&= \prod_{j=0}^{\infty} \frac{1}{4}\Big(1 + 3\cos\frac{1}{2^{j+n_k-n_{k-1}-3}}\Big) \ge \prod_{j=0}^{\infty}\Big(1 - \frac{3}{8}\frac{1}{4^{j+n_k-n_{k-1}-3}}\Big) \\
&\ge \exp\Big(-\frac{15}{8}\sum_{j=0}^{\infty}\frac{1}{4^{j+n_k-n_{k-1}-3}}\Big) \ge \exp(-4^{4-n_k+n_{k-1}}).
\end{aligned}
$$

Hence, we complete the proof of Proposition 4.1. □

Proposition 4.2. *Under the conditions of Theorem 1.1, suppose that Case B holds. Then there exist an increasing subsequence $\{n_k\}_{k=1}^{\infty}$ of $\mathbb{N}$, some σ, $\sigma' \in (2\mathbb{Z}+1)^{\mathbb{N}}$ and a constant $\alpha > 0$ such that*

$$
|\hat{\mu}_{>n_k}(\xi+\lambda)| \ge \alpha \quad \text{for } k > 1,\ \xi \in [0,1]^2 \text{ and } \lambda \in \Lambda_{n_k}^{\sigma,\sigma'}.
$$

Proof. Without loss of generality, we only consider the case that $\lim_{n\to\infty} \frac{a_n}{p_n} = 1$. Then $\lim_{n\to\infty} p_n = \infty$. There are two possible situations: $\lim_{n\to\infty} \frac{b_n}{q_n} = 1$ or $\lim_{n\to\infty} \frac{b_n}{q_n} \ne 1$. If $\lim_{n\to\infty} \frac{b_n}{q_n} = 1$, then $\lim_{n\to\infty} q_n = \infty$. This implies that the conditions of Theorem 1.2 hold, and hence the assertion follows by Proposition 3.1.

We now prove the case that $\lim_{n\to\infty} \frac{b_n}{q_n} \ne 1$. In this case, there exist a subsequence $\{n_k\}_{k=1}^{\infty}$ of $\mathbb{N}$, a constant $r_0 \in (0,1)$ and a positive integer l_0 such that $\frac{b_{n_k+1}}{q_{n_k+1}} \le r_0$ and $n_1, n_{k+1} - n_k \ge l_0$ for $k \ge 1$, where l_0 satisfies $\frac{2^{l_0-1}+1}{2^{l_0}-1} < \min\{\frac{1}{2r_0}, \frac{2}{\pi}\}$. That is, the conditions of Proposition 4.1 hold. Choose $\sigma'_i = 1$ if $i \in \{n_k : k \ge 1\}$ and $\sigma'_i = -1$ for otherwise in Λ_n^2. Then, for $\lambda_2 \in \Lambda_{n_k}^2$ and $\xi_2 \in [0,1]$, we can similarly prove that $\eta_{j,2} = -Q_j^{-1} b_j(\xi_2 + \lambda_2) \in E_2 := [-\frac{2^{l_0-1}+1}{2^{l_0}-1} r_0, \frac{2^{l_0-1}+1}{2^{l_0}-1} r_0] \subset (-\frac{1}{2}, \frac{1}{2})$ for $n_k + 1 \le j \le n_k + 3$ and $2\pi|\eta_{j,2}| \le \frac{1}{2^{j-n_k-3}}$ for $j \ge n_k + 4$. For the choice of σ in Λ_n^1, note that $c' = 1$ (see (3.1)), we use the same choice of Case I or Case III$_2$ of Proposition 3.1, which depends on s_{n_k} (in this process, maybe we need a subsequence of $\{n_k\}_{k=1}^{\infty}$ to replace it). Therefore, applying a similar argument as that in the proof of Proposition 3.1, we can show that there exists a closed set E_3 with $\frac{\kappa}{2} \notin E_3$ ($\kappa \notin 2\mathbb{Z}$) such that $\eta_{n_k+1,1} \in E_3$ and $2\pi|\eta_{j,1}| \le \frac{1}{2^{j-n_k-1}}$ for $j \ge n_k + 2$.

Hence, when $j \ge n_k + 4$, by $2\pi|\eta_{j,2}| \le \frac{1}{2^{j-n_k-3}}$ and $2\pi|\eta_{j,1}| \le \frac{1}{2^{j-n_k-1}} \le \frac{1}{2^{j-n_k-3}}$, similar to (3.6), we obtain that

$$
\prod_{j=n_k+4}^{\infty} J_j(\xi+\lambda) \ge \prod_{j=0}^{\infty}\Big(1 - \frac{3}{8}\frac{1}{4^j}\Big) > 0. \tag{4.6}
$$

When $j = n_k + 1, \dots, n_k + 3$, let $E_4 = [-\frac{1}{2\pi}, \frac{1}{2\pi}]$, then $(\eta_{n_k+1,1}, \eta_{n_k+1,2}) \in E_3 \times E_2$ and $(\eta_{n_k+i,1}, \eta_{n_k+i,2}) \in E_4 \times E_2$ for $i = 2, 3$. Similar to (3.8), it follows from $(E_3 \times E_2) \cap \mathcal{Z}(f) = \emptyset$ and $(E_4 \times E_2) \cap \mathcal{Z}(f) = \emptyset$ that $\prod_{j=n_k+1}^{n_k+3} J_j(\xi + \lambda) \geq \gamma_5^3 > 0$, where $\gamma_5 := \min_{(x,y)\in(E_3\times E_2)\cup(E_4\times E_2)} f(x,y)$. This and (4.6) imply $|\hat{\mu}_{>n_k}(\xi + \lambda)| \geq \gamma_5^3 \prod_{j=0}^{\infty}\left(1 - \frac{3}{8}\frac{1}{4^j}\right) := \alpha > 0$. Hence the proof is completed. □

Now we are ready to prove Theorem 1.1. Recall that

$$Q_n(\xi) = \sum_{\lambda\in\Lambda_n^{\sigma,\sigma'}} |\hat{\mu}_{\{M_n\},\{\mathcal{D}_n\}}(\xi + \lambda)|^2 \quad \text{and} \quad Q_{\Lambda^{\sigma,\sigma'}}(\xi) = \sum_{\lambda\in\Lambda^{\sigma,\sigma'}} |\hat{\mu}_{\{M_n\},\{\mathcal{D}_n\}}(\xi + \lambda)|^2.$$

Proof of Theorem 1.1. If Case A holds, without loss of generality, we assume that the sequence $\{n_k\}_{k=1}^{\infty}$ satisfies the conditions of Proposition 4.1. Since $\hat{\mu}_{\{M_n\},\{\mathcal{D}_n\}} = \hat{\mu}_{n_k}\hat{\mu}_{>n_k}$, it follows from Proposition 4.1(i) that $|\hat{\mu}_{\{M_n\},\{\mathcal{D}_n\}}(\xi + \lambda)|^2 \geq \alpha_{r,l}^2 |\hat{\mu}_{n_k}(\xi + \lambda)|^2$. By Theorem 2.1(ii) and Lemma 2.4, we know that $\sum_{\lambda\in\Lambda_{n_k}^{\sigma,\sigma'}} |\hat{\mu}_{n_k}(\xi + \lambda)|^2 = 1$. Hence, for $k > 1$, we have

$$\begin{aligned} Q_{n_k}(\xi) &= Q_{n_{k-1}}(\xi) + \sum_{\lambda\in\Lambda_{n_k}^{\sigma,\sigma'}\setminus\Lambda_{n_{k-1}}^{\sigma,\sigma'}} |\hat{\mu}_{\{M_n\},\{\mathcal{D}_n\}}(\xi + \lambda)|^2 \\ &\geq Q_{n_{k-1}}(\xi) + \alpha_{r,l}^2\Big(1 - \sum_{\lambda\in\Lambda_{n_{k-1}}^{\sigma,\sigma'}} |\hat{\mu}_{n_k}(\xi + \lambda)|^2\Big). \end{aligned} \tag{4.7}$$

If $Q_{\Lambda^{\sigma,\sigma'}}(\xi) \not\equiv 1$, then there exists $\xi_0 \in [0, 1]^2$ such that $Q_{\Lambda^{\sigma,\sigma'}}(\xi_0) < 1$. Let ϑ_0 satisfy $\max\{Q_{\Lambda^{\sigma,\sigma'}}(\xi_0), e^{-2}\} < \vartheta_0 < 1$. Without loss of generality, we can assume that $n_k - n_{k-1} \geq \max\{l, 4 - \log_4 \ln \vartheta_0^{-1/2}\}$ for $k > 1$ (otherwise we choose a subsequence of $\{n_k\}_{k=1}^{\infty}$ to replace it). By Proposition 4.1(ii), we have $|\hat{\mu}_{>n_k}(\xi_0 + \lambda)| \geq \exp(-4^{4-n_k+n_{k-1}}) \geq \sqrt{\vartheta_0} > 0$ for $\lambda \in \Lambda_{n_{k-1}}^{\sigma,\sigma'}$, and hence $|\hat{\mu}_{\{M_n\},\{\mathcal{D}_n\}}(\xi_0 + \lambda)|^2 \geq \vartheta_0 |\hat{\mu}_{n_k}(\xi_0 + \lambda)|^2$. Consequently,

$$\sum_{\lambda\in\Lambda_{n_{k-1}}^{\sigma,\sigma'}} |\hat{\mu}_{n_k}(\xi_0 + \lambda)|^2 \leq \frac{1}{\vartheta_0} \sum_{\lambda\in\Lambda_{n_{k-1}}^{\sigma,\sigma'}} |\hat{\mu}_{\{M_n\},\{\mathcal{D}_n\}}(\xi_0 + \lambda)|^2 \leq \frac{1}{\vartheta_0} Q_{\Lambda^{\sigma,\sigma'}}(\xi_0) < 1.$$

This implies from (4.7) that $Q_{n_k}(\xi_0) \geq Q_{n_{k-1}}(\xi_0) + \alpha_{r,l}^2(1 - \vartheta_0^{-1} Q_{\Lambda^{\sigma,\sigma'}}(\xi_0))$. By recursion,

$$1 \geq Q_{\Lambda^{\sigma,\sigma'}}(\xi_0) \geq Q_{n_k}(\xi_0) \geq Q_{n_1}(\xi_0) + (k-1)\alpha_{r,l}^2(1 - \vartheta_0^{-1} Q_{\Lambda^{\sigma,\sigma'}}(\xi_0))$$

for $k > 1$, which is impossible when k is large enough. This implies $Q_{\Lambda^{\sigma,\sigma'}}(\xi) \equiv 1$, and the result follows by Theorem 2.1.

For the proof of Case B, it is similar to that of Theorem 1.2, we only need to use Proposition 4.2 instead of Proposition 3.1. □

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