ON PARTITIONS OF n INTO k SUMMANDS

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1. Introduction

In his recent paper on partitions (1), Jakub Intrator proved that the number $p(n, k)$ of partitions of n into exactly k summands, $1 < k \leqq n$, is given by a polynomial of degree exactly $k-1$ in n, the first $[(k+1)/2]$ coefficients of which (starting with the coefficient of the highest degree term), are independent of n and the rest depend on the residue of n modulo the least common multiple of the integers 1, 2, 3, ..., k. He even showed (ignoring the case $k = 3$) that the $[(k+3)/2]$-th coefficient in the polynomial depends only on the parity of n and is not the same for n even and n odd.

The object of this note is to prove the more precise

Theorem. *The coefficient of n^j in the polynomial for $p(n, k)$ depends on the residue of n modulo the least common multiple of the integers* 1, 2, 3, ..., $[k/(j+1)]$, $0 \leqq j \leqq k-1$.

All polynomials in this paper will be deemed to have coefficients rational but not necessarily integral. Polynomials with integral coefficients will be denoted by capital letters.

2. Preliminaries

Our proof of the theorem will depend on the

Lemma. *If $F_k(x)$ denotes*

$$(1-x)(1-x^2)(1-x^3)\dots(1-x^k), \tag{1}$$

then $1/F_k(x)$ can be written in the form

$$\sum_{r=1}^{k} \sum_{h=1}^{[k/r]} f_{r,h}(x)/(1-x^r)^h, \tag{2}$$

where

$$f_{r,h}(x) = a_0(r, h) + a_1(r, h)x + \dots + a_{r-1}(r, h)x^{r-1} \tag{3}$$

is a polynomial in x of degree at most $r-1$.

Proof. One way of getting the desired expression is the following. Let

$$G_r(x) = 1 + g_1x + g_2x^2 + \dots + g_tx^t,$$

where $t = \phi(r)$ and ϕ is Euler's totient function, be the irreducible polynomial which divides $1-x^r$ but not $1-x^s$ for any positive $s < r$. Then

$$F_k(x) = \prod_{r=1}^{k} \{G_r(x)\}^{[k/r]}. \tag{4}$$

Also

$$1-x^r = \prod_{d \mid r} G_d(x) = G_r(x)H_r(x), \text{ say.}$$

Since the G's are mutually prime in pairs, polynomials $b_r(x)$ of degree less than $\phi(r)[k/r]$ can be determined uniquely, such that

$$1/F_k(x) = \sum_{r=1}^{k} b_r(x)/\{G_r(x)\}^{[k/r]}. \tag{5}$$

As an immediate consequence of (5) we have

$$1/F_k(x) = \sum_{r=1}^{k} c_r(x)/(1-x^r)^{[k/r]}, \tag{6}$$

where

$$c_r(x) = b_r(x)\{H_r(x)\}^{[k/r]}.$$

The expression on the right of (6) can readily be written in the form stated in the lemma.

Our method gives an explicit expression for $1/F_k(x)$.

Since the denominators in (2) are not prime in pairs any more, a little manipulation will generally give simpler expressions for $1/F_k(x)$. For example, our method gives

$$72/F_3(x) = 17/(1-x)+18/(1-x)^2+12/(1-x)^3+9(1-x)/(1-x^2) + 8(2-x-x^2)/(1-x^3).$$

A simpler expression is provided by

$$12/F_3(x) = 3/(1-x)^2+2/(1-x)^3+3/(1-x^2)+4/(1-x^3).$$

3. Proof of the theorem

We observe that $p(n, k)$ is the coefficient of x^{n-k} in the expansion of $1/F_k(x)$ as a formal power series. For any r, let

$$n-k = q_r r+t_r, \quad 0 \leqq t_r \leqq (r-1).$$

Then, the coefficient of x^{n-k} in $f_{r,h}(x)/(1-x^r)^h$ is given by

$$\binom{h+q_r-1}{h-1} a_{t_r}(r, h). \tag{7}$$

Replacing q_r by $(n-k-t_r)/r$ in (7), it is readily seen that (7) is a polynomial in n of degree $h-1$ at the most, with coefficients which depend on t_r, the residue of $n-k$ modulo r, and, therefore, on the residue of n modulo r.

Keeping h fixed and letting r vary from 1 to $[k/h]$, we get a term of $p(n, k)$, which is a polynomial in n of degree at most $h-1$ and with coefficients depending on the residues of n modulo each of the integers 1, 2, 3, …, $[k/h]$ and hence on the residue of n modulo the least common multiple of these integers. Since n^j can occur in (7) only if $h \geqq (j+1)$, the theorem follows.

4. Remarks

(1) The value of

$$(1-x^r)^{[k/r]}/(1-x)(1-x^2)(1-x^3)\dots(1-x^k) \tag{8}$$

at w_r, a primitive r-th root of unity, is different from zero. Hence, for $h = [k/r]$,

$$f_{r,h}(w_r) = a_0(r, h)+a_1(r, h)w_r+\dots+a_{r-1}(r, h)w_r^{r-1}$$

is different from zero too. The $a(r, [k/r])$'s are, therefore, not all equal. In particular, for $r = 1$, the value of (8) at $x = 1$ is $1/k!$. Therefore $a_0(1, k) = 1/k!$. The polynomial for $p(n, k)$ is consequently of degree exactly $(k-1)$. Again, for $r = 2$, the value of (8) at $x = -1$, is

$$(h!\,2^{k-h})^{-1}, \text{ where } h = [k/2].$$

Hence

$$a_0(2, [k/2])-a_1(2, [k/2]) = \{[k/2]!\,2^{[(k+1)/2]}\}^{-1}.$$

This result is in agreement with Intrator's.

(2) As has already been hinted at in Section **1**, Intrator's statement that the $[(k+3)/2]$-th term in the polynomial for $p(n, k)$ depends only on the parity of n, does not hold for $k = 3$. We have, in fact,

$$p(n, 3) = [(n^2+3)/12] = (n^2+v)/12,$$

where $v = 0, -1, -4, 3, -4, -1$ according as $n \equiv 0, 1, 2, 3, 4, 5 \pmod 6$; so that v does not depend merely on the parity of n.

(3) By a slight variation in the argument it can be shown that the theorem holds for $P(n, k)$, the number of partitions of n into at most k parts, and also for $q(n, k)$, the number of partitions of n into k distinct parts.

(4) For values of $k \leqq 12$, the polynomials for $P(n, k)$ are given in (**2**).

REFERENCES

(**1**) Jakub Intrator, Partitions I, *Czechoslovak Math. J.* **18** (93) (1968), 16-24; MR **37** (1969) #181.

(**2**) Hansraj Gupta, Partitions in terms of combinatory functions, *Res. Bull. Panjab Univ.* **94** (1956) 153-159; MR **19** (1958), 252.

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