COMPARISON OF COMPLEXES OF MODULES OF GENERALIZED FRACTIONS AND GENERALIZED HUGHES COMPLEXES

by R. Y. SHARP and M. TOUSI

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0. Introduction. Let R be a commutative ring (with non-zero identity) and let M be an R-module.

Suppose that $\mathcal{U}$ is a chain of triangular subsets on R (see [**5**, p. 420]). Then we can construct a complex of modules of generalized fractions $C(\mathcal{U}, M)$. The chain $\mathcal{U}$ determines a family $\mathcal{S}(\mathcal{U})$ of systems of ideals of R (see [**6**, 2.6]), and so the generalized Hughes complex $\mathcal{H}(\mathcal{S}(\mathcal{U}), M)$ for M with respect to $\mathcal{S}(\mathcal{U})$ can be constructed (see [**6**, Section 1]).

One of the main results of [**6**] is Theorem 3.5, which shows that, when R is Noetherian, there is an isomorphism of complexes

$$\Psi = (\psi^i)_{i \geq -2}: C(\mathcal{U}, M) \to \mathcal{H}(\mathcal{S}(\mathcal{U}), M)$$

such that $\psi^{-1}: M \to M$ is the identity mapping Id_M. The proof of that theorem given in [**6**] used the Noetherian property of R in an important way: at the end of [**6**], it was asked whether there is any analogue of that theorem in the case when R is not necessarily Noetherian. The purpose of this paper is to address that question.

We now describe the main results of this paper. We prove that, in general, there is a natural homomorphism of complexes

$$\Theta = (\theta^i)_{i \geq -2}: \mathcal{H}(\mathcal{S}(\mathcal{U}), M) \to C(\mathcal{U}, M)$$

such that $\theta^{-1}: M \to M$ is the identity mapping Id_M. Moreover, we show that, if R is Noetherian, then Θ is an isomorphism of complexes and its inverse is the isomorphism of complexes of [**6**, Theorem 3.5] referred to above. In addition, we show that the class of commutative rings R for which Θ is always an isomorphism of complexes includes the N-rings studied by W. Heinzer and D. Lantz in [**3**]: we say that R is an *N-ring* if and only if, for every ideal $\mathfrak{a}$ of R, there exists a commutative Noetherian extension ring T of R (having the same identity as R) such that $\mathfrak{a}$ is contracted from T, that is, such that $\mathfrak{a} = \mathfrak{a}T \cap R$. It should be noted that an N-ring need not itself be Noetherian (see [**3**, p. 122]).

The final section of this paper provides an example which shows that Θ is not always an isomorphism.

1. Preliminaries. Throughout this paper, R will denote a commutative ring (with non-zero identity) and M will denote an R-module; $\mathcal{C}(R)$ will denote the category of all R-modules and R-homomorphisms. We use $\mathbb{N}_0$ (respectively $\mathbb{N}$) to denote the set of non-negative (respectively positive) integers. For any positive integer n, $D_n(R)$ denotes the set of $n \times n$ lower triangular matrices over R. For $H \in D_n(R)$, the determinant of H is denoted by $|H|$, and we use ^{T} to denote matrix transpose. Given $H \in D_n(R)$ with $n > 1$,

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H^* will denote the $(n-1)\times(n-1)$ submatrix of H obtained by deletion of the nth row and nth column of H.

1.1 Reminder: complexes of modules of generalized fractions. The concept of a chain of triangular subsets on R is explained in [**5**, p. 420] and [**6**, 2.3]. Such a chain $\mathcal{U}=(U_n)_{n\in\mathbb{N}}$ determines a complex of modules of generalized fractions

$$0\to M\xrightarrow{e^0} U_1^{-1}M\xrightarrow{e^1}\ldots\to U_n^{-n}M\xrightarrow{e^n} U_{n+1}^{-n-1}M\to\ldots$$

in which $e^0(m)=m/(1)$ for all $m\in M$ and

$$e^n\left(\frac{m}{(u_1,\ldots,u_n)}\right)=\frac{m}{(u_1,\ldots,u_n,1)}$$

for all $n\in\mathbb{N}$, $m\in M$ and $(u_1,\ldots,u_n)\in U_n$. We shall denote this complex by $C(\mathcal{U},M)$. We shall need to use many of the properties of modules of generalized fractions reviewed in [**6**, Section 2] and, in particular, the descriptions of the cokernels of the e^i $(i\in\mathbb{N}_0)$ which result from [**6**, Lemma 2.7].

1.2 Reminder about the construction of generalized Hughes complexes. A system of ideals of R [**1**] is a non-empty set Φ of ideals of R such that, whenever $\mathfrak{a},\mathfrak{b}\in\Phi$, there exists $\mathfrak{c}\in\Phi$ such that $\mathfrak{c}\subseteq\mathfrak{ab}$.

Note that (see [**6**, 1.2]) Φ gives rise to an additive, left exact functor

$$D_\Phi:=\varinjlim_{\mathfrak{b}\in\Phi}\operatorname{Hom}_R(\mathfrak{b},\cdot)$$

from $\mathcal{C}(R)$ to itself.

For each $\mathfrak{b}\in\Phi$ and $x\in M$, we define $\lambda_{\mathfrak{b},x}:\mathfrak{b}\to M$ by $\lambda_{\mathfrak{b},x}(r)=rx$ for all $r\in\mathfrak{b}$. For each R-module G, there is an R-homomorphism

$$\eta_\Phi(G):G\to D_\Phi(G)$$

which is such that, for each $g\in G$, $\eta_\Phi(G)(g)$ is the natural image of $\lambda_{\mathfrak{b},g}$ in $D_\Phi(G)$ (for any $\mathfrak{b}\in\Phi$). Furthermore, as G varies through the category $\mathcal{C}(R)$, the $\eta_\Phi(G)$ constitute a morphism of functors $\eta_\Phi:\mathrm{Id}\to D_\Phi$ from $\mathcal{C}(R)$ to itself. (Of course, Id here denotes the identity functor from $\mathcal{C}(R)$ to itself.)

Let $\mathcal{S}=(\Phi_i)_{i\in\mathbb{N}}$ be a family of systems of ideals of R. The generalized Hughes complex for M with respect to $\mathcal{S}$ has the form

$$0\to M\xrightarrow{h^{-1}}K^0\xrightarrow{h^0}K^1\to\ldots\to K^i\xrightarrow{h^i}K^{i+1}\to\ldots$$

and is denoted by $\mathcal{H}(\mathcal{S},M)$. This complex is a generalization of one constructed by K. R. Hughes in [**4**]. Details of the construction are given in [**6**, 1.3], but its terms and homomorphisms can be essentially described as follows.

Write $K^{-2}=0$, $K^{-1}=M$, and use $h^{-2}:K^{-2}\to K^{-1}$ to denote the zero homomorphism. Then, for all $n\in\mathbb{N}_0$, $K^n:=D_{\Phi_{n+1}}(\operatorname{Coker}h^{n-2})$, while $h^{n-1}:K^{n-1}\to K^n$ is the composition of the natural epimorphism from K^{n-1} to $\operatorname{Coker}h^{n-2}$ and the homomorphism $\eta_{\Phi_{n+1}}(\operatorname{Coker}h^{n-2}):\operatorname{Coker}h^{n-2}\to D_{\Phi_{n+1}}(\operatorname{Coker}h^{n-2})=K^n$.

1.3 Remark. Let $\mathcal{U}=(U_n)_{n\in\mathbb{N}}$ be a chain of triangular subsets on R. By [**6**, Lemma 2.5] (see also M. H. Bijan-Zadeh [**2**]), for each $n\in\mathbb{N}$, the set

$$\Phi(U_n):=\left\{\sum_{i=1}^{n} Ru_i:(u_1,\ldots,u_n)\in U_n\right\}$$

is a system of ideals of R. Thus $\mathcal{S}(\mathcal{U})=(\Phi(U_n))_{n\in\mathbb{N}}$ is a family of systems of ideals of R, and we can form the generalized Hughes complex $\mathcal{H}(\mathcal{S}(\mathcal{U}),M)$. Our purpose in this paper is to compare the complex $\mathcal{H}(\mathcal{S}(\mathcal{U}),M)$ with the complex of modules of generalized fractions $C(\mathcal{U},M)$ described in 1.1.

1.4 Definition. (See [**3**, p. 115].) The ring R is called an *N-ring* if, for every ideal $\mathfrak{a}$ of R, there is a commutative Noetherian ring extension T of R (having the same identity as R) such that $\mathfrak{a}$ is contracted from T, that is, $\mathfrak{a}T\cap R=\mathfrak{a}$.

Of course, if R is Noetherian, then it is an N-ring, but an N-ring need not be Noetherian (see [**3**, p. 122]).

The following theorem of Heinzer and Lantz provides a characterization of N-rings which is very useful for our purpose.

1.5 Theorem (W. Heinzer and D. Lantz [**3**, Theorem 2.3]). *The ring R is an N-ring if and only if, for every ideal $\mathfrak{b}$ of R, the set $\{(\mathfrak{b}:\mathfrak{c}):\mathfrak{c}$ is an ideal of $R\}$ (partially ordered by inclusion) satisfies the maximal condition.*

2. A morphism of complexes.

The key to our construction of the morphism of complexes mentioned in the introduction is provided by the following lemma.

2.1 Lemma. *Let $n\in\mathbb{N}$ with $n>1$, let U be an expanded triangular subset of R^{n+1} (see [**7**, 3.2]), and let $\bar{U}$ be the restriction of U to R^n [**7**, 3.6]. Let $u=(u_1,\ldots,u_{n+1})\in U$. Let $f\in \operatorname{Hom}_R\left(\sum_{i=1}^{n+1} Ru_i,\ (\bar{U}\times\{1\})^{-n-1}M\right)$. Then there exists $w=(w_1,\ldots,w_{n+1})\in U$ and $H\in D_{n+1}(R)$ such that*

$$\operatorname{Im} f\subseteq\left\{\frac{m}{(w_1,\ldots,w_n,1)}\in(\bar{U}\times\{1\})^{-n-1}M:m\in M\right\}$$

and $Hu^T=w^T$.

Also there is an R-homomorphism

$$\delta_u:\operatorname{Hom}_R\left(\sum_{i=1}^{n+1} Ru_i,(\bar{U}\times\{1\})^{-n-1}M\right)\to U^{-n-1}M$$

which is such that, for f and w as above, so that

$$f(w_{n+1})=\frac{g}{(w_1,\ldots,w_n,1)}$$

for some $g\in M$, we have $\delta_u(f)=g/(w_1,\ldots,w_n,w_{n+1})$.

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Proof. Since $\sum_{i=1}^{n+1} Ru_i$ is a finitely generated ideal of R, and finitely many members of a module of generalized fractions can be put on a common denominator, there exists $t=(t_1,\ldots,t_n)\in \bar{U}$ such that

$$\operatorname{Im} f \subseteq \left\{\frac{m}{(t_1,\ldots,t_n,1)} \in (\bar{U}\times\{1\})^{-n-1}M : m \in M\right\}.$$

Since $\bar{U}$ is the restriction of U to R^n and U is expanded, there exist $w=(w_1,\ldots,w_{n+1})$ in U and $H, K \in D_{n+1}(R)$ such that (with an obvious notation) $Hu^T = w^T = K(t,1)^T$, and, since $K^* t^T = (w_1,\ldots,w_n)^T$, it is clear that w meets the requirements.

To define a map δ_u as described in the statement of the lemma, suppose that $w'=(w'_1,\ldots,w'_{n+1})\in U$ and $H' \in D_{n+1}(R)$ are such that $H'u^T = w'^T$ and

$$\operatorname{Im} f \subseteq \left\{\frac{m}{(w'_1,\ldots,w'_n,1)} \in (\bar{U}\times\{1\})^{-n-1}M : m \in M\right\}.$$

Suppose that

$$f(w'_{n+1}) = \frac{g'}{(w'_1,\ldots,w'_n,1)},$$

where $g' \in M$. We must show that $g'/w' = g/w$ in $U^{-n-1}M$.

There are $P, P' \in D_{n+1}(R)$ and $z=(z_1,\ldots,z_{n+1})\in U$ such that $Pw^T = z^T = P'w'^T$. Let $f(z_{n+1}) = g''/(z_1,\ldots,z_n,1)$, where $g'' \in M$. We show that $g/w = g''/z$ in $U^{-n-1}M$.

Let $P=(p_{ij})$; then $z_{n+1} = \sum_{i=1}^{n+1} p_{n+1\,i} w_i$. Hence

$$z_{n+1}^2 = \sum_{i=1}^{n} a_i w_i + p_{n+1\,n+1}^2 w_{n+1}^2,$$

where $a_1,\ldots,a_n \in \sum_{i=1}^{n+1} Rw_i$. Since

$$\operatorname{Im} f \subseteq \left\{\frac{m}{(w_1,\ldots,w_n,1)} \in (\bar{U}\times\{1\})^{-n-1}M : m \in M\right\},$$

it follows from [**7**, 3.3] that $f\left(\sum_{i=1}^{n} a_i w_i\right) = 0$, and so $f(z_{n+1}^2) = f(p_{n+1\,n+1}^2 w_{n+1}^2)$.

Hence, in $(\bar{U}\times\{1\})^{-n-1}M$,

$$f(z_{n+1}^2) = \frac{z_{n+1} g''}{(z_1,\ldots,z_n,1)} = \frac{p_{n+1\,n+1}^2 w_{n+1} g}{(w_1,\ldots,w_n,1)} = \frac{|P^*|\, p_{n+1\,n+1}^2 w_{n+1} g}{(z_1,\ldots,z_n,1)}.$$

Since $\bar{U}\times\{1\}\subseteq U$, it follows that, in $U^{-n-1}M$,

$$\frac{z_{n+1} g''}{(z_1,\ldots,z_n,1)} = \frac{|P^*|\, p_{n+1\,n+1}^2 w_{n+1} g}{(z_1,\ldots,z_n,1)},$$

that is,

$$\frac{z_{n+1}^3 g''}{(z_1,\ldots,z_n,z_{n+1}^2)} = \frac{z_{n+1}^2\, |P^*|\, p_{n+1\,n+1}^2 w_{n+1} g}{(z_1,\ldots,z_n,z_{n+1}^2)}.$$

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Therefore, by [**8**, 2.1],

$$\frac{z_{n+1}g''}{(z_1,\ldots,z_n,z_{n+1}^2)}=\frac{|P^*|\,p_{n+1\,n+1}^2w_{n+1}g}{(z_1,\ldots,z_n,z_{n+1}^2)}$$

in $U^{-n-1}M$. Let $L=(l_{ij})\in D_{n+1}(R)$ be such that $L^*=P^*$, $l_{n+1\,i}=a_i$ $(1\le i\le n)$ and $l_{n+1\,n+1}=p_{n+1\,n+1}^2$. Then $L(w_1,\ldots,w_n,w_{n+1}^2)^T=(z_1,\ldots,z_n,z_{n+1}^2)^T$ and $|L|=|P^*|\,p_{n+1\,n+1}^2$. Hence, in $U^{-n-1}M$,

$$\frac{g''}{(z_1,\ldots,z_n,z_{n+1})}=\frac{z_{n+1}g''}{(z_1,\ldots,z_n,z_{n+1}^2)}=\frac{|P^*|\,p_{n+1\,n+1}^2w_{n+1}g}{(z_1,\ldots,z_n,z_{n+1}^2)}=\frac{g}{(w_1,\ldots,w_n,w_{n+1})}.$$

Similarly, we can prove that $g'/w'=g''/z$ in $U^{-n-1}M$. Hence $g/w=g'/w'$ in $U^{-n-1}M$. It follows that there is indeed a mapping

$$\delta_u:\mathrm{Hom}_R\left(\sum_{i=1}^{n+1}Ru_i,(\bar U\times\{1\})^{-n-1}M\right)\to U^{-n-1}M,$$

as described in the statement of the lemma; now that the above checking has been completed, it is routine to show that δ_u is an R-homomorphism.

2.2 Proposition. *Let the situation be as in* 2.1. *We denote by* $\Phi(U)$ *the system of ideals of* R *determined by* U (*see* 1.3). *For each* $\mathfrak{b}\in\Phi(U)$, *let*

$$[\quad]:\mathrm{Hom}_R(\mathfrak{b},(\bar U\times\{1\})^{-n-1}M)\to D_{\Phi(U)}((\bar U\times\{1\})^{-n-1}M)$$

be the canonical homomorphism.

There is an R-*monomorphism*

$$\delta:D_{\Phi(U)}((\bar U\times\{1\})^{-n-1}M)\to U^{-n-1}M$$

which is such that, for each $u=(u_1,\ldots,u_{n+1})\in U$ *and each*

$$f\in\mathrm{Hom}_R\left(\sum_{i=1}^{n+1}Ru_i,(\bar U\times\{1\})^{-n-1}M\right),$$

we have $\delta([f])=\delta_u(f)$, *where* δ_u *is the homomorphism defined in Lemma* 2.1.

Proof. Let $u=(u_1,\ldots,u_{n+1})$, $u'=(u'_1,\ldots,u'_{n+1})\in U$ with $\sum_{i=1}^{n+1}Ru'_i\subseteq\sum_{i=1}^{n+1}Ru_i$. We show that the diagram

$$\begin{array}{ccc}\mathrm{Hom}_R\left(\sum_{i=1}^{n+1}Ru_i,(\bar U\times\{1\})^{-n-1}M\right) & & \\ \downarrow & \searrow^{\delta_u} & \\ \mathrm{Hom}_R\left(\sum_{i=1}^{n+1}Ru'_i,(\bar U\times\{1\})^{-n-1}M\right) & \xrightarrow[\delta_{u'}]{} & U^{-n-1}M,\end{array}$$

in which the vertical map is the restriction homomorphism, is commutative.

Let $f\in\mathrm{Hom}_R\left(\sum_{i=1}^{n+1}Ru_i,\ (\bar U\times\{1\})^{-n-1}M\right)$. Then there exists $w=(w_1,\ldots,w_{n+1})\in U$ and H, $H'\in D_{n+1}(R)$ such that

$$\mathrm{Im}\,f\subseteq\left\{\frac{m}{(w_1,\ldots,w_n,1)}\in(\bar U\times\{1\})^{-n-1}M:m\in M\right\}$$

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and $Hu^T = w^T = H'u'^T$. Let $f(w_{n+1}) = g/(w_1, \ldots, w_n, 1)$. Then

$$f|_{\sum_{i=1}^{n+1} Ru_i'}(w_{n+1}) = \frac{g}{(w_1, \ldots, w_n, 1)}.$$

It follows from the definition that

$$\delta_u(f) = g/w = \delta_{u'}(f|_{\sum_{i=1}^{n+1} Ru_i'}).$$

Hence there is an R-homomorphism δ as described in the statement of the proposition. We show that δ is injective.

Let $u = (u_1, \ldots, u_{n+1}) \in U$ and $f \in \operatorname{Hom}_R\left(\sum_{i=1}^{n+1} Ru_i,\ (\bar{U} \times \{1\})^{-n-1}M\right)$ be such that $\delta([f]) = \delta_u(f) = 0$. There exist $H \in D_{n+1}(R)$ and $w = (w_1, \ldots, w_{n+1}) \in U$ such that $Hu^T = w^T$ and

$$\operatorname{Im} f \subseteq \left\{\frac{m}{(w_1, \ldots, w_n, 1)} \in (\bar{U} \times \{1\})^{-n-1}M : m \in M\right\}.$$

Let $f(w_{n+1}) = g/(w_1, \ldots, w_n, 1)$. Then $g/w = 0$ in $U^{-n-1}M$. Therefore there exist $Q \in D_{n+1}(R)$ and $z = (z_1, \ldots, z_{n+1}) \in U$ such that $Qw^T = z^T$ and $|Q|\, g \in \sum_{i=1}^{n} z_i M$. Let $Q = (q_{ij})$. Then $z_{n+1}^2 = \sum_{i=1}^{n} b_i w_i + q_{n+1\,n+1}^2 w_{n+1}^2$, where $b_1, \ldots, b_n \in \sum_{i=1}^{n+1} Rw_i$. It follows from [**7**, 3.3], and the fact that

$$\operatorname{Im} f \subseteq \left\{\frac{m}{(w_1, \ldots, w_n, 1)} \in (\bar{U} \times \{1\})^{-n-1}M : m \in M\right\},$$

that $f(z_i^2) = 0$ $(1 \le i \le n)$ and $f(z_{n+1}^2) = q_{n+1\,n+1}^2 f(w_{n+1}^2)$. Since

$$Q^*(w_1, \ldots, w_n)^T = (z_1, \ldots, z_n)^T$$

and $|Q| = |Q^*|\, q_{n+1\,n+1}$, it follows from [**7**, 3.3] that

$$\frac{q_{n+1\,n+1} g}{(w_1, \ldots, w_n, 1)} = \frac{q_{n+1\,n+1}\, |Q^*|\, g}{(z_1, \ldots, z_n, 1)} = \frac{|Q|\, g}{(z_1, \ldots, z_n, 1)} = 0$$

in $(\bar{U} \times \{1\})^{-n-1}M$. Hence

$$\frac{q_{n+1\,n+1}^2 w_{n+1} g}{(w_1, \ldots, w_n, 1)} = 0$$

in $(\bar{U} \times \{1\})^{-n-1}M$. Hence $q_{n+1\,n+1}^2 f(w_{n+1}^2) = 0$, that is, $f(z_{n+1}^2) = 0$. Since $f(z_i^2) = 0$ $(1 \le i \le n)$ and $f(z_{n+1}^2) = 0$, the restriction of f to $\sum_{i=1}^{n+1} Rz_i^2$ is zero, and so $[f] = 0$. Therefore δ is injective.

2.3 THEOREM. *Let the situation be as in* 2.2. *If R is an N-ring* (*see* 1.4) (*and so*, *in particular*, *if R is Noetherian*), *then the R-monomorphism δ of* 2.2 *is an isomorphism.*

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Proof. It is enough to show that δ is surjective. Let $m/(u_1, \ldots, u_{n+1}) \in U^{-n-1}M$, where $m \in M$, $(u_1, \ldots, u_{n+1}) \in U$. It follows from 1.5 that there exists $t \in \mathbb{N}$ such that $\left(\sum_{i=1}^{n} Ru_i : u_{n+1}^t\right) = \left(\sum_{i=1}^{n} Ru_i : u_{n+1}^{t+1}\right)$. Therefore there exists an R-homomorphism

$$f: \sum_{i=1}^{n} Ru_i + Ru_{n+1}^{t+1} \to (\bar{U} \times \{1\})^{-n-1}M$$

for which

$$f\left(\sum_{i=1}^{n} a_i u_i + a_{n+1} u_{n+1}^{t+1}\right) = \frac{a_{n+1} u_{n+1}^t m}{(u_1, \ldots, u_n, 1)}$$

for all $a_1, \ldots, a_{n+1} \in R$. (To see this, reason as in the proof of [**6**, Lemma 3.1].) By 2.1, we have

$$\delta_{(u_1, \ldots, u_n, u_{n+1}^{t+1})}(f) = \frac{u_{n+1}^t m}{(u_1, \ldots, u_n, u_{n+1}^{t+1})} = \frac{m}{(u_1, \ldots, u_n, u_{n+1})}.$$

Therefore $m/(u_1, \ldots, u_{n+1}) \in \operatorname{Im} \delta$.

A similar result is available for triangular subsets of R. Its proof is similar to, but simpler than, the above proofs of 2.2 and 2.3, and so we merely state the result here and leave the proof to the reader.

2.4 Proposition. *Let U be an expanded triangular subset of R. We denote by $\Phi(U)$ the system of ideals of R determined by U. For each $\mathfrak{b} \in \Phi(U)$, let $[\ \ ]: \operatorname{Hom}_R(\mathfrak{b}, M) \to D_{\Phi(U)}(M)$ be the canonical homomorphism.*

There is a monomorphism $\delta: D_{\Phi(U)}(M) \to U^{-1}M$ which is such that $\delta([f]) = f(u_1)/(u_1)$ for each $f \in \operatorname{Hom}_R(Ru_1, M)$ where $(u_1) \in U$. Moreover, if R is an N-ring (and, in particular, if R is Noetherian), δ is an isomorphism.

2.5 Theorem. *Let $\mathcal{U} = (U_n)_{n \in \mathbb{N}}$ be a chain of triangular sets on R. Denote the complex $C(\mathcal{U}, M)$ of modules of generalized fractions by*

$$0 \to M \xrightarrow{f^{-1}} F^0 \xrightarrow{f^0} F^1 \to \ldots \to F^n \xrightarrow{f^n} F^{n+1} \to \ldots$$

(so that $F^n = U_{n+1}^{-n-1}M$ and $f^{n-1} = e^n$ for all $n \in \mathbb{N}_0$), and set $F^{-1} = M$.

Let $\mathcal{S}(\mathcal{U}) = (\Phi(U_n))_{n \in \mathbb{N}}$ be the family of systems of ideals of R determined by $\mathcal{U}$. Denote the generalized Hughes complex $\mathcal{H}(\mathcal{S}(\mathcal{U}), M)$ for M with respect to $\mathcal{S}(\mathcal{U})$ by

$$0 \to M \xrightarrow{h^{-1}} K^0 \xrightarrow{h^0} K^1 \to \ldots \to K^n \xrightarrow{h^n} K^{n+1} \to \ldots$$

and set $K^{-1} = M$.

Then there is a homomorphism of complexes

$$\Theta = (\theta^i)_{i \geq -2}: \mathcal{H}(\mathcal{S}(\mathcal{U}), M) \to C(\mathcal{U}, M)$$

such that $\theta^{-1}: F^{-1} \to K^{-1}$ is the identity mapping on M. Moreover, Θ is an isomorphism if R is an N-ring (and, in particular, when R is Noetherian).

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Proof. The homomorphism $\Theta = (\theta^i)_{i \geq -2}$ is constructed by a straightforward inductive process, and most of the details are left to the reader.

Use 2.4 to define θ^0. Suppose, inductively, that $n \geq 1$ and we have constructed R-homomorphisms $\theta^{-2}, \theta^{-1}, \ldots, \theta^{n-1}$ so that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & \cdots & \longrightarrow & K^{n-2} & \xrightarrow{h^{n-2}} & K^{n-1} \\ \downarrow & & \downarrow{\scriptstyle \theta^{-1}} & & & & \downarrow{\scriptstyle \theta^{n-2}} & & \downarrow{\scriptstyle \theta^{n-1}} \\ 0 & \longrightarrow & M & \longrightarrow & \cdots & \longrightarrow & F^{n-2} & \xrightarrow{f^{n-2}} & F^{n-1} \end{array}$$

commutes, and suppose we have shown that $\theta^{-1}, \theta^0, \ldots, \theta^{n-1}$ are all isomorphisms when R is an N-ring. The above diagram induces a homomorphism $\overline{\theta^{n-1}}: \operatorname{Coker} h^{n-2} \rightarrow \operatorname{Coker} f^{n-2}$, and the latter cokernel is isomorphic to $(U_n \times \{1\})^{-n-1}M$ by [**6**, 2.7]. Application of the functor $D_{\Phi(U_{n+1})}$ and use of 2.2 provide us with R-homomorphisms

$$D_{\Phi(U_{n+1})}(\overline{\theta^{n-1}}): K^n = D_{\Phi(U_{n+1})}(\operatorname{Coker} h^{n-2}) \rightarrow D_{\Phi(U_{n+1})}(\operatorname{Coker} f^{n-2}),$$

$$D_{\Phi(U_{n+1})}(\operatorname{Coker} f^{n-2}) \xrightarrow{\cong} D_{\Phi(U_{n+1})}((U_n \times \{1\})^{-n-1}M)$$

and

$$\delta: D_{\Phi(U_{n+1})}((U_n \times \{1\})^{-n-1}M) \rightarrow U_{n+1}^{-n-1}M = F^n,$$

and it is routine to check that θ^n, the composition of these, has all the properties required to complete the inductive step.

2.6 Remark. It is easy to check that, when R is Noetherian, the isomorphism of complexes of 2.5 is the inverse of the isomorphism provided by [**6**, Theorem 3.5].

3. A counterexample. A multiplicatively closed subset of R is a triangular subset of R. We give an example of a commutative ring R and a multiplicatively closed subset S of R for which the natural map

$$\delta: \varinjlim_{sR \in \Phi(S)} \operatorname{Hom}_R(sR, R) = D_{\Phi(S)}(R) \rightarrow S^{-1}R$$

of 2.4 is not surjective. Since S can be incorporated into the chain of triangular subsets $\mathcal{U} = (U_n)_{n \in \mathbb{N}}$ on R, where $U_1 = S$ and $U_n = S \times \{1\} \times \ldots \times \{1\} \subseteq R^n$ for all $n \in \mathbb{N}$ with $n > 1$, this example is enough to show that the morphism of complexes of 2.5 is not always an isomorphism.

Consider $R = k[X_1, X_2, \ldots, X_n, \ldots]/\mathfrak{c}$ where k is a field and

$$\mathfrak{c} := (X_1X_2, X_1^2X_3, \ldots, X_1^{n-1}X_n, \ldots).$$

Let x_i denote the natural image of X_i in R. We show that $(0:_R x_1^{n-1}) \subset (0:_R x_1^n)$, for each $n \in \mathbb{N}$.

Since $x_1^n x_{n+1} = 0$, we have $x_{n+1} \in (0:_R x_1^n)$. It is enough to show that $x_{n+1} \notin (0:_R x_1^{n-1})$. Suppose that $x_{n+1} \in (0:_R x_1^{n-1})$, so that $X_1^{n-1}X_{n+1} \in \mathfrak{c}$. Hence there are $t \in \mathbb{N}$ and $f_1(X_1, \ldots, X_t), \ldots, f_t(X_1, \ldots, X_t) \in k[X_1, \ldots, X_t]$ such that $t > n+1$ and

$$X_1^{n-1}X_{n+1} = \sum_{i=1}^{t} X_1^i X_{i+1} f_i(X_1, \ldots, X_t)$$

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in $k[X_1, \ldots, X_{t+1}]$. Evaluate at $X_2 = \ldots = X_n = X_{n+2} = \ldots = X_{t+1} = 0$ in $k[X_1, \ldots, X_{t+1}]$. We obtain that

$$X_1^{n-1}X_{n+1} = X_1^n X_{n+1} f_n(X_1, 0, \ldots, 0, X_{n+1}, 0, \ldots, 0),$$

and this contradiction shows that $x_{n+1} \notin (0:_R x_1^{n-1})$.

We note in passing that the strictly ascending chain

$$(0:_R x_1) \subset (0:_R x_1^2) \subset \ldots \subset (0:_R x_1^n) \subset (0:_R x_1^{n+1}) \subset \ldots$$

shows that R is not an N-ring.

Take $S = \{x_1^i : i \in \mathbb{N}_0\}$. We show that $1/x_1 \notin \operatorname{Im} \delta$. Suppose that $1/x_1 \in \operatorname{Im} \delta$. Then there are $l \in \mathbb{N}$ and $f \in \operatorname{Hom}_R(x_1^l R, R)$ such that $1/x_1 = f(x_1^l)/x_1^l$ in $S^{-1}R$. Note that $(0:_R x_1^l) \subseteq (0:_R f(x_1^l))$.

We can assume that

$$f(x_1^l) = \sum_{(i_1, \ldots, i_u) \in \Lambda} a_{i_1, \ldots, i_u} x_1^{i_1} \ldots x_u^{i_u}$$

for some $u \in \mathbb{N}$ with $u \geqslant 2$, finite subset Λ of $\mathbb{N}_0^u$, and $a_{i_1, \ldots, i_u} \in k((i_1, \ldots, i_u) \in \Lambda)$. If, for any $i = (i_1, \ldots, i_u) \in \Lambda$, one of the components of i other than the first, say i_j where $2 \leqslant j \leqslant u$, is positive, then $x_1^{j-1} a_{i_1, \ldots, i_u} x_1^{i_1} \ldots x_u^{i_u} = 0$ in R, and hence

$$\frac{a_{i_1, \ldots, i_u} x_1^{i_1} \ldots x_u^{i_u}}{x_1^l} = 0$$

in $S^{-1}R$. Hence, in $S^{-1}R$,

$$\frac{1}{x_1} = \frac{f(x_1^l)}{x_1^l} = \sum_{(i_1, 0, \ldots, 0) \in \Lambda} \frac{a_{i_1, 0, \ldots, 0} x_1^{i_1}}{x_1^l}.$$

For each $(i_1, 0, \ldots, 0) \in \Lambda$, write b_{i_1} for $a_{i_1, 0, \ldots, 0}$. Then there exists $x_1^q \in S$ such that $x_1^{q+l} = \sum_{(i_1, 0, \ldots, 0) \in \Lambda} b_{i_1} x_1^{i_1+q+1}$ in R. It follows from the definition of c that

$$X_1^{q+l} = \sum_{(i_1, 0, \ldots, 0) \in \Lambda} b_{i_1} X_1^{i_1+q+1}$$

in $k[X_1]$. Hence we can assume that the only member of Λ of the form $(i_1, 0, \ldots, 0)$ is $(l-1, 0, \ldots, 0)$, and that $b_{l-1} = 1$. Thus

$$f(x_1^l) = x_1^{l-1} + \sum_{(i_1, i_2, \ldots, i_u) \in \Lambda'} a_{i_1, \ldots, i_u} x_1^{i_1} \ldots x_u^{i_u},$$

where Λ' is a finite subset of $\mathbb{N}_0^u$ and $\Lambda' \cap (\mathbb{N}_0 \times \{0\} \times \ldots \times \{0\}) = \emptyset$. Now

$$x_1^{u-1}\Bigg(\sum_{(i_1, \ldots, i_u) \in \Lambda'} a_{i_1, \ldots, i_u} x_1^{i_1} \ldots x_u^{i_u}\Bigg) = 0.$$

Hence $x_1^{u-1} f(x_1^l) = x_1^{u+l-2}$ and $x_{u+l} x_1^{u-1} f(x_1^l) = x_{u+l} x_1^{u+l-2} \neq 0$, since $x_{u+l} \notin (0:_R x_1^{u+l-2})$. However $x_{u+l} x_1^{u-1}(x_1^l) = x_{u+l} x_1^{u+l-1} = 0$. We have thus shown that

$$x_{u+l} x_1^{u-1} \in (0:_R x_1^l) \backslash (0:_R f(x_1^l)),$$

and this contradiction show that $1/x_1 \notin \operatorname{Im} \delta$.

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Department of Pure Mathematics
University of Sheffield
Hicks Building
Sheffield S3 7RH

Department of Mathematics
Faculty of Science
Tehran University
Enghelab Avenue
Tehran
Iran