Introduction

In this introduction, we describe in detail the content of the first volume of the book, simultaneously highlighting its sources and the interconnections between various fragments of the book.

In Part I, "Ergodic Theory and Geometric Measures," we first bring up in Chapter 1, with proofs, some basic and fundamental concepts and theorems from abstract and geometric measure theory. These include, in particular, the three classical covering theorems: $4r$, Besicovitch, and Vitali type. We also include a short section on probability theory: conditional expectations and martingale theorems. We devote a large amount of space to treating Hausdorff and packing measures. In particular, we formulate and prove Frostman Converse Lemmas, which form an indispensable tool for proving that a Hausdorff or packing measure is finite, positive, or infinite. Some of them are frequently called, in particular in the fractal geometry literature, the mass redistribution principle, but these lemmas involve no mass redistribution. We then deal with Hausdorff, packing, and box counting, as well as the dimensions of sets and measures, and provide tools to calculate and estimate them.

In the next four chapters of Part I, we deal with classical ergodic theory, both finite (probability) and, which we would like to emphasize, also infinite.

In Chapter 2, we deal with both finite and infinite invariant measures. We start with quasi-invariant measures and early on, in the second section of this chapter, we introduce the powerful concept of the first return map. This concept, along with that of nice sets (see Chapter 12), will form our most fundamental tool in Part IV in Volume 2 of our book, which is devoted to presenting a refined ergodic theory of elliptic functions. We introduce in Chapter 2 the notions of ergodicity and conservativity (always satisfied for finite invariant measures) and prove the Poincaré Recurrence Theorem, the Birkhoff Ergodic Theorem, and the Hopf Ergodic Theorem, the last pertaining

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to infinite measures. We also provide a powerful, though perhaps somewhat neglected by the ergodic community, tool for proving the existence of invariant σ-finite measures that are absolutely continuous with respect to given quasi-invariant measures. This stems from the work of Marco Martens [**Mar**].

We then prove, in Chapter 3, the Bogolyubov–Krylov Theorem about the existence of Borel probability invariant measures for continuous dynamical systems acting on compact metrizable topological spaces. We also establish in this chapter the basic properties of invariant and ergodic measures and provide a large variety of examples of such measures.

Chapter 4 is devoted to the stochastic laws for measurable endomorphisms, preserving a probability measure that are finer than the mere Birkhoff Ergodic Theorem. Under appropriate hypotheses, we prove the Law of the Iterated Logarithm. We then describe another powerful method of ergodic theory, namely L. S. Young towers (recently also frequently called Kakutani towers), which she developed in [**LSY2**] and [**LSY3**]; see also [**Go1**] for further progress. With appropriate assumptions imposed on the first return time, the Young construction yields the exponential decay of correlations, the Central Limit Theorem, and the Law of the Iterated Logarithm, which follows too.

In Chapter 5, we deal with refined stochastic laws for dynamical systems preserving an infinite measure. This is primarily the Darling–Kac Theorem. We make use of some recent progress on this theorem and related issues, mainly due to Zweimm̈uller, Thaler, Theresiu, Melbourne, Gouëzel, Bruin, Aaronson, and others, but we do not go into the most recent subtleties and developments of this branch of infinite ergodic theory. We do not need them for our applications to elliptic functions.

In Chapter 6, we provide a classical account of Kolmogorov–Sinai metric entropy for measure-preserving dynamical systems. We prove the Shannon–McMillan–Breiman Theorem and, based on the Abramov formula, define the concept of the Krengel entropy of a conservative system preserving a (possibly infinite) invariant measure.

In Chapter 7, the last chapter of Part I, we collect and prove the basic concepts and theorems of classical thermodynamic formalism. This includes topological pressure, the variational principle, and equilibrium states. We have written this chapter very rigorously, taking care of some inaccuracies which have persisted in some expositions of thermodynamic formalism since the 1970s and 1980s; we have not singled them out explicitly. We do not deal with Gibbs states in this chapter.

We provide many examples in Part I of the book: mainly of invariant measures, ergodic invariant measures, and counterexamples of infinite ergodic theory. We do emphasize once more that we treat the latter with care and detail.

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Part II, "Complex Analysis, Conformal Measures, and Graph Directed Markov Systems," devotes the first chapter, Chapter 8, to some selected topics of geometric function theory. Its character is entirely classical, meaning that no dynamics is involved. We deal here at length with Riemann surfaces, normal families, and Montel's Theorem, extremal lengths, and moduli of topological annuli. However, we think that the central theme of this chapter consists of various versions of the Koebe Distortion Theorems. These theorems, which were proved in the early years of the nineteenth century by Koebe and Bieberbach, form a beautiful, elegant, and powerful tool for complex analysis. We prove them carefully and provide their many versions of analytic and geometric character. These theorems also form an absolutely indispensable tool for nonexpanding holomorphic dynamics and their applications occur very frequently throughout the book; most notably when dealing with holomorphic inverse branches, conformal measures, and Hausdorff and packing measures. The version of the Riemann–Hurwitz Formula, appropriate in the context of transcendental meromorphic functions, which we treat at length in the last section of Chapter 8, is a very helpful tool to prove the existence of holomorphic inverse branches and an elegant and probably the best tool to control the topological structure of connected components of inverse images of open connected sets under meromorphic maps, especially to make sure that such connected components are simply connected. Another reason why we devoted a lot of time to the Riemann–Hurwitz Formula is that in the standard monographs on Riemann surfaces this formula is usually formulated only for compact surfaces and its proofs are somewhat sketchy; we do need, as a matter of fact almost exclusively, the noncompact case. Our approach to the Riemann–Hurwitz Formula stems from that of Alan Beardon in [**Bea**], which is designed to deal with rational functions of the Riemann sphere. We modify it to fit to our context of transcendental meromorphic functions.

In Chapter 9, we encounter holomorphic dynamics for the first time in the book. Its settings are somehow technical and it has, on the one hand, a very preparatory character serving the needs of constructing and controlling the Sullivan conformal measures in various subsequent parts of the book; for example, in Chapter 10. On the other hand, this chapter is important and interesting on its own. Indeed, its hypotheses are very general and flexible, and under such weak assumptions it establishes in the context of holomorphic dynamics such important results as Pesin's Theory, Ruelle's Inequality, and Volume Lemmas.

In Chapter 10, we encounter for the first time the beautiful, elegant, and powerful concept of conformal measures, which is due to Patterson (see [**Pat1**] and also [**Pat2**]) in the context of Fuchsian groups and due to Sullivan in the context of all Kleinian groups and rational functions of the Riemann sphere (see [**Su2**]–[**Su7**]). We deal in Chapter 10 with conformal measures being in the settings of the previous chapter, namely Chapter 9. The Sullivan conformal measures and their invariant versions will form the central theme of Volume 2, starting from Chapter 17 onward. In fact, Chapter 10 is the first and essential step for the construction of the Sullivan conformal measures for general elliptic functions that is completed in the last two parts of the second volume of the book. It deals with holomorphic maps f defined on some open neighborhood of a compact f-invariant subset X of a parabolic Riemann surface. We provide a fairly complete account of the Sullivan conformal measures in such a setting. We also introduce in Chapter 10 several dynamically significant concepts and sets, such as radial or conical points and several fractal dimensions defined in dynamical terms. We relate them to exponents of conformal measures,

However, motivated by [**DU1**] and choosing the most natural, at least in our opinion, framework, we do not restrict ourselves to conformal dynamical systems only but present in the first section of Chapter 10 a fairly complete account of the theory of general conformal measures.

Chapter 11 deals with conformal graph directed Markov systems, its special case of iterated function systems, and thermodynamic formalism of countable alphabet subshifts of finite type, frequently also called topological Markov chains. This theory started in the papers [**MU1**] and [**MU4**] and the book [**MU2**]. It was in [**MU1**] and [**MU2**] that the concept of conformal measures due to Patterson and Sullivan was adapted to the realm of conformal graph directed Markov systems and iterated function systems. We present some elements of this theory in Chapter 11, primarily those related to conformal measures and a version of Bowen's Formula for the Hausdorff dimension of limit sets of such systems. In particular, we get an almost cost-free, effective, lower estimate for the Hausdorff dimension of such limit sets. More about conformal graph directed Markov systems can be found in many papers and books, such as [**MU3**]–[**MU5**], [**MPU**], [**MSzU**], [**U5**], [**U6**], [**CTU**], [**CLU1**], [**CLU2**], and [**CU**].

Afterwards, in Parts IV and VI (the second volume of the book), we apply the techniques developed here to get quite a good, explicit estimate from below of Hausdorff dimensions of Julia sets of all elliptic functions and to explore stochastic properties of invariant versions of conformal measures for parabolic and subexpanding elliptic functions. Getting these stochastic properties is possible for us by combining several powerful methods which we have already mentioned. Namely, having proved the existence of nice and pre-nice sets (see Chapter 12 for their definition and construction), it turns out that the holomorphic inverse branches of the first return map that these nice sets generate form a conformal iterated function system. So, the whole theory of conformal graph directed Markov systems applies and we also enhance it by the Young tower techniques developed in [**LSY2**] and [**LSY3**]; see also [**Go1**].

As already discussed in the previous paragraph, in the last chapter of this volume, Chapter 12, "Nice Sets for Analytic Maps," we introduce and thoroughly study the objects related to the powerful concept of nice and pre-nice sets, which will be our indispensable tool in the last part of the second volume of the book, Part VI, "Compactly nonrecurrent Elliptic Functions: Fractal Geometry, Stochastic Properties, and Rigidity," leading, along with the theory of conformal graph directed Markov systems, the first return map method, and the techniques of Young towers, to such stochastic laws as the exponential decay of correlations, the Central Limit Theorem, and the Law of the Iterated Logarithm, which follows for large classes of elliptic functions constituted by subexpanding and parabolic ones. However, the main objective of the discussed chapter is to prove the existence of nice and pre-nice sets.