P 36. Let a model of a closed polyhedron P of genus g be made of wires representing the edges while the faces are not filled in. If each wire has a non-vanishing thickness the total surface of all the wires is a closed orientable surface S. Determine its genus G. Show, in particular, that the wire models of the octahedron and heptahedron are homeomorphic.

H. Helfenstein, University of Ottawa.

Solution by the Proposer. This is a special case of a previous problem: If n compact orientable surfaces $F_j$ of genus $g_1, \ldots, g_n$ are given and $F_j$ is connected to $F_k$ by a finite number of cylindrical non-intersecting tubes then the new surface has genus

$$G = \sum_{j=1}^{n} g_j + T - (n - 1),$$

where T is the total number of inserted tubes. In this special case our $F_j$'s are spheres corresponding to the vertices of P. Hence all $g_j = 0$, $n = v$ = number of vertices of P, and $T = e$ = number of edges of P. Taking into account the Euler formulae for P

$v - e + f = 2 - 2g$ if P is orientable, $v - e + f = 2 - g$ if not,

we obtain

$$G = e - (v - 1) = f + 2g - 1 \quad \text{or} \quad G = f + g - 1,$$

according to the orientability of P.

Thus e.g. the wire models of the octahedron and the heptahedron are topologically equivalent, viz. both are closed orientable surfaces of genus $G = 8 + 0 - 1 = 7 + 1 - 1 = 7$.



https://doi.org/10.1017/S0008439500025947 Published online by Cambridge University Press

P 37. Let $e(n,p)$ denote the exponent of $p$ in the prime power decomposition of $n!$ . Prove that

$$\sum x^{e(n,p)} = \prod_{k=1}^{\infty} \{1 + x^{(p^k-1)/(p-1)} + x^{2(p^k-1)/(p-1)} + \dots + x^{p^k-1}\}.$$

J. Lambek, McGill University

Solution by L. Carlitz, Duke University.
If $n = sp^r + m$, $0 \le s < p$, $0 \le m < p^r$, then

$$e(n,p) = s\left(\frac{p^r - 1}{p - 1}\right) + e(m,p).$$

It follows that

$$\sum_{n=p^{r-1}}^{p^{r+1}-1} x^{e(n,p)} = \sum_{s=0}^{p-1} x^{s(p^r-1)/(p-1)} \sum_{m=1}^{p^r-1} x^{e(m,p)} .$$

Therefore

$$\sum_{n=1}^{p^{r+1}-1} x^{e(n,p)} = \prod_{k=1}^{r} \sum_{s=0}^{p-1} x^{s(p^k-1)/(p-1)} . \sum_{m=1}^{p-1} x^{e(m,p)}$$

$$= (p-1) \prod_{k=1}^{r} \sum_{s=0}^{p-1} x^{s(p^k-1)/(p-1)} ,$$

so that

$$\sum_{n=1}^{\infty} x^{e(n,p)} = (p-1) \prod_{k=1}^{\infty} \sum_{s=0}^{p-1} x^{s(p^k-1)/(p-1)} .$$

Also solved by L. Moser and the proposer.



https://doi.org/10.1017/S0008439500025947 Published online by Cambridge University Press

P 38. Let $\phi(n)$ denote Euler's function and let $\alpha = (\sqrt{5} - 1)/2$. Show that $\prod (1 + \alpha^n)^{\phi(n)/n} = e$.

L. Moser, University of Alberta.

Solution by the proposer. Consider $f(z) = \prod_{n=1}^{\infty} (1 - z^n)^{\phi(n)/n}$.

Then

$$\log f(z) = \sum_{n=1}^{\infty} \frac{\phi(n)}{n} \log (1 - z^n)$$

$$= - \sum_{n=1}^{\infty} \frac{\phi(n)}{n} \sum_{r=1}^{\infty} \frac{z^{nr}}{r} .$$

Let $nr = m$ and get

$$\log f(z) = - \sum_{m=1}^{\infty} \sum_{d|m} \phi(d) \frac{z^m}{m} .$$

A defining property of $\phi$ is $\sum_{d|n} \phi(d)\ n$. Hence

$$\log f(z) = - \sum_{m=1}^{\infty} z^m = \frac{z}{z-1}$$

and $f(z) = e^{z/z-1}$.

Now let

$$\psi (z) = \prod (1 + z^n)^{\phi(n)/n}$$

then $\psi (z) = f(z^2)/f(z)$

$$e^{\frac{z^2}{z^2-1} - \frac{z}{z-1}} = e^{\frac{z}{1-z^2}} .$$



https://doi.org/10.1017/S0008439500025947 Published online by Cambridge University Press

Finally note that $\frac{\alpha}{1-\alpha^2} = 1$ so $f(1) = e$.

Also solved by L. Carlitz and J. D. Dixon.

P 39. Can two regular tetrahedra of edge 1 be placed without overlap inside a cube of edge 1 ?

Ken U. Packit

Solution by the proposer. Two tetrahedra will fit with a fair amount to spare. The point of the proof is that if two tetrahedra are put back to back the largest distance between vertices is $\sqrt{\frac{8}{3}}$ while the longest diagonal of a unit cube is $\sqrt{\frac{9}{3}}$.

P 41. Let $P_1$, $P_2$, $P_3$, $P_4$ be any four points in the plane, no three collinear. On $P_iP_{i+1}$ construct a square with centre $Q_i$ so that the triangles $Q_iP_iP_{i+1}$ all have the same "orientation" ( $i = 1, 2, 3, 4$; $P_5 = P_1$). Show that the segments $Q_1Q_3$ and $Q_2Q_4$ have the same lengths, and the lines containing them are perpendicular.

W. A. J. Luxemburg,
California Institute of Technology

Solution by W. J. Blundon, Memorial University of Newfoundland. In the complex plane let the number $2z_k$ be represented by the point $P_k$. Then with the appropriate "orientation" it is easily verified that the point $Q_k$ represents the complex number $(1 + i)z_k + (1 - i)z_{k-1}$. Thus



https://doi.org/10.1017/S0008439500025947 Published online by Cambridge University Press

$$\overline{Q_4Q_2} = (1 + i)(z_2 - z_4) + (1 - i)(z_3 - z_1)$$

$$= i\,\overline{Q_3Q_1}$$

and the required result follows immediately.

Also solved by A. Makowski, P. Robert, L. Moser, and the proposer.

Note by A. Makowski. This problem was published in the Soviet journal Matematičeskoe Prosveščenie 5 (1960), 254, but its solution has not yet appeared (August 1961).



https://doi.org/10.1017/S0008439500025947 Published online by Cambridge University Press

https://doi.org/10.1017/S0008439500025947 Published online by Cambridge University Press