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THE FIT AND FLAT COMPONENTS OF BARRELLED SPACES

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The Splitting Theorem says that any given Hamel basis for a (Hausdorff) barrelled space E determines topologically complementary subspaces E_C and E_D, and that E_C is flat, that is, contains no proper dense subspace. By use of this device it was shown that if E is non-flat it must contain a dense subspace of codimension at least $\aleph_0$; here we maximally increase the estimate to $\aleph_1$ under the assumption that the dominating cardinal $\mathfrak{d}$ equals $\aleph_1$ [strictly weaker than the Continuum Hypothesis (CH)]. A related assumption strictly weaker than the Generalised CH allows us to prove that E_D is fit, that is, contains a dense subspace whose codimension in E_D is $\dim(E_D)$, the largest imaginable. Thus the two components are extreme opposites, and E itself is fit if and only if $\dim(E_D) \geqslant \dim(E_C)$, in which case there is a choice of basis for which $E_D = E$. Morover, E is non-flat (if and) only if the codimension of E' is at least $2^{\aleph_1}$ in E^*. These results ensure latitude in the search for certain subspaces of E^* transverse to E', as in the barrelled countable enlargement (BCE) problem, and show that every non-flat GM-space has a BCE.

1. INTRODUCTION

The *codensity character* of a topological vector space E (here always assumed to be Hausdorff) is the supremum of the set of codimensions of all dense subspaces in E. The dimension of E is an obvious upper bound. When E contains a dense subspace with codimension equal to the dimension of E we say E is *fit*. At the other extreme, when the only dense subspace of E is E itself we say E is *flat*. Clearly E is flat if and only if its (continuous) dual E' and its algebraic dual E^* coincide, E has its strongest locally convex topology if and only if E is flat and barrelled, and the only space that is both fit and flat is the space consisting of the single element 0. These and intermediate ideas and examples are found in [5].

In the present note we show that, under an axiomatic assumption strictly weaker than the Generalised Continuum Hypothesis (GCH), every barrelled space can be expressed as the topological direct sum of two subspaces, one of which is fit while the

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other is flat. Consequently, the codensity character of a barrelled space E is always attained as the *maximum* of its defining set (that is, E is *firm* [5]) and is either 0 or at least $\aleph_1$. Under our axiomatic assumption this optimally improves the estimate of $\aleph_0$ given in [3]. Moreover, the simple method we used in [5] is the only way of constructing barrelled spaces that are neither fit nor flat. We also apply the result to obtain important information about the codimension of the dual in the algebraic dual of a barrelled space and to extend the class of spaces for which it can be shown that a barrelled countable enlargement exists [10].

If A is a subset of a topological vector space, $sp(A)$ denotes the linear span of A, with closure $\overline{sp}(A)$. For convenience we state the following theorem from [3], which is crucial for our work.

SPLITTING THEOREM. *Let E be a barrelled space and let $\{x_i, f_i\}_{i\in B}$ be a biorthogonal system in $E \times E^*$ whose first coordinates form a Hamel basis in E. If*

$$E_C = sp(\{x_i : i \in B,\ f_i \in E'\}) \quad \text{and} \quad E_D = sp(\{x_i : i \in B,\ f_i \notin E'\})$$

then E is the topological direct sum of E_C and E_D, and E_C has its strongest locally convex topology.

2. An axiomatic assumption

If A is any set we write $|A|$ for its cardinality, which we consider to be the set of all ordinals of smaller size. For an infinite cardinal μ we employed in [5] a Condition (1) weaker than the GCH, namely that $2^{<\mu} = \mu$. Here we shall make use of the following yet weaker axiomatic condition:

CONDITION (2). *Given any set A of cardinality less than μ there exists a set $\mathcal{F}$ of functions f from A into the set $\mathbb{N}$ of natural numbers such that the cardinality of $\mathcal{F}$ is at most μ and such that for any $g\colon A \to \mathbb{N}$ there exists $f \in \mathcal{F}$ such that $f(a) > g(a)$ for all $a \in A$.*

REMARKS. (i) If $\aleph_0 \leqslant |A| < \mu$ then under GCH the set of *all* functions $g\colon A \to \mathbb{N}$ has cardinality $\aleph_0^{|A|} \leqslant \left(2^{\aleph_0}\right)^{|A|} = 2^{|A|} \leqslant \mu$; if A is finite then this set of functions is countable. Thus GCH implies Condition (2) for all $\mu > \aleph_0$. (The case $\mu = \aleph_0$ does not require any assumptions.)

(ii) In the case $\mu = \aleph_1$, the set A must be countable and so it can be regarded as $\mathbb{N}$ if it is infinite. Condition (2) may then be reformulated as

CONDITION (2′). *There exists a set $\mathcal{F}$ of functions f from $\mathbb{N}$ into $\mathbb{N}$ such that the cardinality of $\mathcal{F}$ is at most $\aleph_1$ and such that for any $g\colon \mathbb{N} \to \mathbb{N}$ there exists $f \in \mathcal{F}$ such that $f(n) > g(n)$ for all $n \in \mathbb{N}$.*

In fact, the minimum cardinality for such an $\mathcal{F}$ is the *dominating cardinal* $\mathfrak{d}$ ([1, Section 3]; see also [7, 9]). It is easily seen that $\aleph_1 \leqslant \mathfrak{d} \leqslant \mathfrak{c}$, and any combination of strict or non-strict inequalities is consistent with the usual ZFC set theory [1, Section 5]; Condition (2′) is equivalent to $\mathfrak{d} = \aleph_1$.

Set-theoretic Lemma. *Let μ be a fixed infinite cardinal. Condition (2) holds for μ if and only if for any fixed set B of cardinality μ there exists a family $\{\{B_{\alpha 1}, B_{\alpha 2}, \ldots\}\}_{\alpha\in\mu}$ of partitions $\{B_{\alpha 1}, B_{\alpha 2}, \ldots\}$ of B into sets $B_{\alpha p}$, each of cardinality μ, such that, given any $g\colon \mu \to \mathbb{N}$ and any $\alpha_0 \in \mu$,*

$$B \setminus \bigcup\{B_{\alpha k}\colon \alpha \leqslant \alpha_0,\, k \leqslant g(\alpha)\} \neq \emptyset.$$

Proof: Suppose Condition (2) holds and let B be a fixed set of cardinality μ. Let $C^1, C^2, \ldots;\ B^\alpha(\alpha \in \mu)$ be a partition of B into μ sets C^p, B^α $(p \in \mathbb{N}, \alpha \in \mu)$ each having μ elements. Now for each $\alpha \in \mu$ there exists a family $\mathcal{F}_\alpha$ of at most μ functions from $A_\alpha = \{\beta \in \mu\colon \beta \leqslant \alpha\}$ into $\mathbb{N}$ which is "dominating" in the sense of Condition (2). Let $\{B^{\alpha f}\}_{f\in\mathcal{F}_\alpha}$ be a partition of B^α into $(\leqslant \mu)$ sets each having μ elements; let

$$B_{\alpha 1} = \Big(\bigcup\{B^\beta\colon \beta < \alpha\}\Big) \cup \Big(\bigcup\{B^{\beta f}\colon \beta \geqslant \alpha,\, f \in \mathcal{F}_\beta,\, f(\alpha) = 1\}\Big) \cup C^1$$

and let

$$B_{\alpha k} = \Big(\bigcup\{B^{\beta f}\colon \beta \geqslant \alpha,\, f \in \mathcal{F}_\beta,\, f(\alpha) = k\}\Big) \cup C^k \quad (k = 2, 3, \ldots).$$

Given any $\alpha \in \mu$, one easily verifies that the sets $B_{\alpha 1}, B_{\alpha 2}, \ldots$ are pairwise disjoint, each has μ elements and their union contains each C^p $(p \in \mathbb{N})$ and each B^β $(\beta \in \mu)$, so that their union is B. Finally, let $g\colon \mu \to \mathbb{N}$ and $\alpha_0 \in \mu$ be fixed and suppose we have α, k, β and f such that

$$\alpha \leqslant \alpha_0,\, 1 \leqslant k \leqslant g(\alpha),\, \beta \geqslant \alpha \text{ and } f \in \mathcal{F}_\beta \text{ with } f(\alpha) = k.$$

Choose $f_0 \in \mathcal{F}_{\alpha_0}$ such that $f_0(\delta) > g(\delta)$ for $\delta \leqslant \alpha_0$. Then if $\beta \neq \alpha_0$ we have $B^{\beta f} \subset B^\beta$, which is disjoint from $B^{\alpha_0} \supset B^{\alpha_0 f_0}$, and if $\beta = \alpha_0$ then $B^{\beta f} = B^{\alpha_0 f}$, which is disjoint from $B^{\alpha_0 f_0}$, since $f_0 \neq f$. (Note that $f(\alpha) = k \leqslant g(\alpha) < f_0(\alpha)$.) Thus in any case $B^{\beta f}$ is disjoint from $B^{\alpha_0 f_0}$ as then is $B_{\alpha k}$, since β and f were arbitrary, subject only to the conditions $\beta \geqslant \alpha$, $f(a) = k$. But $\alpha(\leqslant \alpha_0)$ and $k(\leqslant g(\alpha))$ arbitrary now implies that

$$B \setminus \bigcup\{B_{\alpha k}\colon \alpha \leqslant \alpha_0,\, k \leqslant g(\alpha)\} \supset B^{\alpha_0 f_0} \neq \emptyset.$$

Conversely, suppose $B_{\alpha p}(\alpha \in \mu,\, p \in \mathbb{N})$ are given and let A be a set of cardinality less than μ. Condition (2) holds trivially for $A = \emptyset$. If A is nonempty we may assume that for some $\alpha \in \mu$, $A = \{\beta \colon \beta \leqslant \alpha\}$. Then for each $g \colon A \to \mathbb{N}$ we can choose $x_g \in B \setminus \bigcup\{B_{\beta k} \colon \beta \leqslant \alpha,\, k \leqslant g(\beta)\}$ and define $f_{x_g} \colon A \to \mathbb{N}$ by letting $f_{x_g}(\beta)$ be the unique $p \in \mathbb{N}$ such that $x_g \in B_{\beta p}$ $(\beta \leqslant \alpha)$. Clearly, then, $f_{x_g}(\beta) > g(\beta)$ for each g and each $\beta \leqslant \alpha$, and if $x_g = x_h$ for g and h possibly different, we still have $f_{x_g} = f_{x_h}$, so that the cardinality of $\mathcal{F} = \{f_{x_g} \colon g \text{ maps } A \text{ into } \mathbb{N}\}$ cannot exceed μ, the cardinality of B. ∎

3. Fit and flat components

An application of the Set-theoretic Lemma yields our main result, for which we require considerably less than barrelledness. In fact, precisely what we require is the very weakest of all the weak barrelledness conditions currently under study by Saxon and Sánchez Ruiz, strictly weaker than Property (S) or dual locally completeness, for example.

Definition: A locally convex space E has *property* $f|_{L_n}$ if any given linear functional is continuous whenever its restrictions are continuous on each member of some increasing sequence of linear subspaces covering E.

Theorem 1. *Let E be a locally convex space with property $f|_{L_n}$. Suppose $B \cup C$ is a Hamel basis for E with $|B| = \mu \geqslant \aleph_0$, $B \cap C = \emptyset$ (C possibly empty), where the linear coefficient functionals corresponding to elements of B are all discontinuous. Then under Condition (2) for μ there is a dense μ-codimensional subspace E_0 of E of the form $E_0 = sp(B_0 \cup C)$ with $B_0 \subset B$.*

Proof: Let $B_{\alpha p}$ $(\alpha \in \mu,\, p \in \mathbb{N})$ be as in the Set-theoretic Lemma. We inductively define a function $g \colon \mu \to \mathbb{N}$ and choose a family $\{x_\alpha\}_{\alpha \in \mu} \subset B$ such that for each $\alpha \in \mu$:

(i) $x_\alpha \in \bigcup\limits_{k \leqslant g(\alpha)} B_{\alpha k} \setminus \bigcup\{B_{\beta k} \colon \beta < \alpha,\, k \leqslant g(\beta)\}$, and

(ii) $x_\alpha \in \overline{sp}\left(C \cup \bigcup\limits_{k \leqslant g(\alpha)} B_{\alpha k} \setminus \{x_\alpha\}\right)$.

Let $\alpha_0 \in \mu$, and suppose $g(\alpha) \in \mathbb{N}$ and $x_\alpha \in B$ have been chosen for each $\alpha < \alpha_0$ so that (i) and (ii) hold. Choose $\widehat{g} \colon \mu \to \mathbb{N}$ such that $\widehat{g}(\alpha) = g(\alpha)$ for $\alpha < \alpha_0$. The Set-theoretic Lemma guarantees some $x_{\alpha_0} \in B \setminus \bigcup\{B_{\beta k} \colon \beta \leqslant \alpha_0,\, k \leqslant \widehat{g}(\beta)\}$. Let f denote the discontinuous linear functional which is 1 at x_{α_0} and 0 elsewhere in $B \cup C$. Set

$$E_p = sp\left(C \cup \bigcup_{k \leqslant p} B_{\alpha_0 k}\right),$$

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so that $\{E_p\}_{p=1}^{\infty}$ is an increasing sequence of subspaces covering E. The hypothesis on E ensures that $f|_{E_p}$ is discontinuous for some p. Choose $q \geqslant p$ so that $x_{\alpha_0} \in E_q$ and define $g(\alpha_0) = q$. Now (i) and (ii) clearly hold for $\alpha = \alpha_0$, completing the transfinite induction, so that $g\colon \mu \to \mathbb{N}$ and $\{x_\alpha\}_{\alpha\in\mu} \subset B$ exist with (i) and (ii) holding for all $\alpha \in \mu$.

If $\alpha_1 < \alpha_2$, then $x_{\alpha_1} \in \bigcup_{k\leqslant g(\alpha_1)} B_{\alpha_1 k} \subset \bigcup\{B_{\beta k}\colon \beta < \alpha_2, k \leqslant g(\beta)\}$, and x_{α_2} is excluded from the latter set by (i), so $x_{\alpha_1} \neq x_{\alpha_2}$; that is, the x_α's $(\alpha \in \mu)$ are distinct. This also shows that

$$C \cup B \setminus \{x_\alpha\}_{\alpha\in\mu} \supset C \cup \bigcup_{k\leqslant g(\alpha_1)} B_{\alpha_1 k} \setminus \{x_\alpha\}_{\alpha\leqslant\alpha_1} = X;$$

thus if we suppose that $F = \overline{sp}(C \cup B \setminus \{x_\alpha\}_{\alpha\in\mu})$ contains x_α for all $\alpha < \alpha_1$, we find that

$$F \supset \overline{sp}(X \cup \{x_\alpha\}_{\alpha<\alpha_1}),$$

which contains x_{α_1} by (ii). Since μ is well-ordered, we have $x_\alpha \in F$ for all $\alpha \in \mu$, so that $F = sp(C \cup B) = E$, and $E_0 = sp(C \cup B \setminus \{x_\alpha\}_{\alpha\in\mu})$ is a dense μ-codimensional subspace of the desired form. ∎

Now let $\{x_i\colon i \in B\}$ be a Hamel basis in a barrelled space E and let $E = E_C \oplus E_D$ be the corresponding decomposition given by the Splitting Theorem. Certainly E_C, having its strongest locally convex topology, is flat. The restriction to E_D of the coefficient functional corresponding to any member of D is still discontinuous, and D is either empty or infinite. In the first case $E_D = \{0\}$ is, trivially, fit. And for D infinite, applying Theorem 1 with $B = D$, $C = \emptyset$ ensures E_D is fit. Thus in any case we have

Theorem 2. [Assume Condition (2) generally.] *Every barrelled space E splits with respect to a given basis into a fit component E_D and a flat component E_C.*

Corollary 1. *Under the above assumptions the codensity character of E is* $\dim(E_D)$ *and is firm, that is, is attained.*

Proof: Let G be a dense subspace of E_D with codimension in E_D equal to $\dim(E_D)$. Then $E_C + G$ is a dense subspace of E with codimension equal to $\dim(E_D)$.

Now if F is any dense subspace of E, its image under the projection of E onto E_C along E_D is dense in E_C and thus is all of the flat E_C. Therefore $F + E_D = E$, and $\operatorname{codim}_E(F) \leqslant \dim(E_D)$. ∎

Theorem 2 also shows that barrelled spaces that are neither fit nor flat can be constructed only as in [5]; indeed, since a non-flat barrelled space must have dimension at least $\aleph_1$ we have

COROLLARY 2. *A barrelled space E is neither fit nor flat if and only if $E = E_1 \oplus E_2$ (topologically), where E_1 is flat, E_2 is fit and* $\dim(E_1) > \dim(E_2) \geqslant \aleph_1$.

The next result says the codensity character of a non-flat barrelled space is at least $\aleph_1$, which extends Corollary 1 of Theorem 3 in [3] and was announced without proof in [6]. This extension is optimal under our assumption that $\aleph_1 = \mathfrak{d}$, since the metrisable (and thus fit) barrelled space $\psi_{\mathfrak{b}}$ of [7] must then have codensity character $\aleph_1$.

THEOREM 3. [Assume $\mathfrak{d} = \aleph_1$.] *Every non-flat barrelled space E has a dense subspace of codimension at least* $\aleph_1$.

PROOF: Take B any subset of a given basis for E with $B \subset E_D$ and $|B| = \aleph_1$; apply Theorem 1. ∎

COROLLARY 1. [Assume $\mathfrak{d} = \aleph_1$.] *If E is a non-flat barrelled space, then*

$$\operatorname{codim}_{E^*}(E') \geqslant 2^{\aleph_1}.$$

PROOF: By Theorem 3 E has an $\aleph_1$-codimensional dense subspace. The result then follows immediately from Theorem 4 of [3]. ∎

REMARK. Under the stronger assumption that the Continuum Hypothesis (CH) holds we get analogously that the codimension of E' in E^* must be at least $2^{\mathfrak{c}}$; consequently, there is no barrelled space whose dual has codimension $\mathfrak{c}$ in the algebraic dual.

Now we recall two definitions from [5]. Let $\{x_i, f_i\}_{i\in B}$ be a biorthogonal system in $E \times E^*$ whose first coordinates form a Hamel basis in E. Then $\{x_i, f_i\}_{i\in B}$ is called a *discontinuous* basis if each f_i is discontinuous, and a *fully discontinuous* basis if $sp(\{f_i : i \in B\}) \cap E' = \{0\}$. It is shown in Theorem 6 of [5], without any non-ZFC assumptions, that any fit topological vector space has a fully discontinuous basis. Combining this with Theorem 1 we have immediately:

THEOREM 4. *When E has property $f|_{L_n}$ the following are equivalent:*

(a) *E is fit;*
(b) *E has a fully discontinuous basis;*
(c) *E has a discontinuous basis.*

NOTE. Generally, (a) $\Rightarrow$ (b) $\Rightarrow$ (c) and neither arrow is reversible [5].

For a barrelled space E with a given basis, the fit and flat components E_D and E_C are uniquely defined. By Corollary 1 to Theorem 2 the dimension of E_D is always the codensity character of E, which is independent of the basis. However, the topological nature of E_D may vary widely with the choice of basis, as may the algebraic nature of E_C. To see this, let $E = M \oplus N$ be the topological direct sum of two barrelled spaces M and N of the same infinite dimension, with M metrisable and N flat, as considered in [5]. Let N_1 and N_2 represent arbitrary algebraic complements in N; they must be topologically complementary as well, since N has its strongest locally convex topology, and E is the topological direct sum of $M + N_1$ and N_2. Since M is fit (indeed, M is Baire-like and thus quasi-Baire) and $\dim(M) \geqslant \dim(N_1)$, we have $M + N_1$ is also fit. By Theorem 4 (or by [5, Theorem 6]) there is a (fully) discontinuous basis B_1 for $M + N_1$. If B_2 is any basis for N_2, then $B = B_1 \cup B_2$ is a basis for E with respect to which E_D is $M + N_1 = M \oplus N_1$ and E_C is N_2. Now the choice of dimensions for N_1 and N_2 ranges maximally from 0 to $\dim(E)$, thus also for E_C, and E_D is metrisable if and only if $\dim(N_1)$ is finite. In fact, all distinct infinite values for $\dim(N_1)$ yield non-isomorphic values for E_D, for if a subspace $N^{(1)}$ of $M \oplus N_1$ has infinite dimension exceeding $\dim(N_1)$ then $N^{(1)}$ contains an infinite-dimensional (metrisable) subspace of M and cannot be flat.

4. An application to the BCE problem

The following problem has been extensively studied in recent years (for a survey see [**10**]):

Problem. Let E be a non-flat barrelled space. Does there exist a subspace M of E^* such that $\dim(M) = \aleph_0$, $E' \cap M = \{0\}$ and E is barrelled under the Mackey topology $\tau(E, E' + M)$?

When the answer is "yes" we say E has a *barrelled countable enlargement* (BCE).

According to Eberhardt and Roelcke [**2**] a *GM-space* is a locally convex space E such that any linear mapping $t\colon E \to F$, where F is any metrisable locally convex space and t has closed graph, is continuous. Since F may be any Banach space it is immediate that GM-spaces are barrelled.

Let E be a non-flat GM-space. If we assume $\mathfrak{d} = \aleph_1$ then Theorem 3 guarantees that E has a dense $\aleph_1$-codimensional subspace. Since every dense subspace of a GM-space is barrelled [**2**, 1.9], Theorem 5 of [**8**] now tells us that E has a BCE. (The cardinal $\mathfrak{b}$ which appears in Theorem 5 of [**8**] satisfies $\aleph_1 \leqslant \mathfrak{b} \leqslant \mathfrak{d}$ [**1**, 3.1], so that under $\mathfrak{d} = \aleph_1$ all three coincide.)

Theorem 5. [Assume $\mathfrak{d} = \aleph_1$] *Every non-flat GM-space has a BCE.*

Remark. In Section 4 of [**4**], Theorem 5 is proved, without non-ZFC assumptions, for $\aleph_0$-products, a subclass of the class of GM-spaces [**2**, Section 3]. Since $\aleph_0$-products are complete, their proper dense subspaces are non-flat GM-spaces which are not $\aleph_0$-products.

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