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# A CHARACTERIZATION OF LINE SPACES

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ABSTRACT. The line spaces of J. Cantwell are characterized among the axiomatic convexity spaces defined by Kay and Womble. This characterization is coupled with a recent result of Doignon to give an intrinsic solution of the linearization problem.

§1. **Introduction.** A convexity space is a pair $(X, \mathscr{C})$ where $X$ is a non-empty set and $\mathscr{C}$ is a family of subsets closed under arbitrary intersection and includes $\phi$ and $X$. In [7] Kay and Womble introduce such spaces and raise the linearization problem: derive necessary and sufficient conditions for a convexity space to be a vector space over an ordered field for which the members of $\mathscr{C}$ are the convex sets.

The purpose of this note is to present a solution to this problem by characterizing those convexity spaces that are line spaces [3] and using Doignon's recent result [6] that a line space, generally, is a linearly open convex subset of an affine space. This solution differs from those given in [8] and [9] each of which impose conditions *extrinsic* to the convexity structure. Another intrinsic solution has been obtained recently by David Kay using an approach different than the one presented here.

The results presented in this paper are a part of the second named author's Master's thesis. Also the authors wish to thank Professor Peter Mah and the referee for several helpful suggestions.

§2. **Definitions.** Let $(X, \mathscr{C})$ be a convexity space. For any $A \subseteq X$, the convex hull of $A$ is defined as $\mathscr{C}(A) = \cap\{C : C \in \mathscr{C}, A \subseteq C\}$. The operation of forming the convex hull is a (non-topological) closure operator $\mathscr{C}$ satisfying, for $A, B \subseteq X$: (i) $A \subseteq \mathscr{C}(A)$, (ii) $\mathscr{C}(A) \subseteq \mathscr{C}(B)$ when $A \subseteq B$, (iii) $\mathscr{C}(\mathscr{C}(A)) = \mathscr{C}(A)$. Also one has that $A \in \mathscr{C}$ if and only if $\mathscr{C}(A) = A$.

We will denote singletons $\{a\}$ by $a$ and the convex hull of finite sets $\{a, b, c, \ldots\}$ by $\mathscr{C}(a, b, c, \ldots)$. For $a, b \in X$ we will denote $\mathscr{C}(a, b)$ by $ab$ and call it the *segment* with endpoints $a$ and $b$. The corresponding *open segment* is $(ab) = ab \backslash \{a, b\}$. Note that if $a = b$, $(ab)$ is not necessarily empty.

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It is easy to verify that for $x \in X$ and $A \subseteq X$, $\cup\{xb : b \in \mathscr{C}(A)\} \subseteq \mathscr{C}(\cup\{xa : a \in A\}) = \mathscr{C}(x \cup A)$. The reverse inclusion does not always hold. If it does, $(X, \mathscr{C})$ is said to be *join-hull commutative* (JHC).

$(X, \mathscr{C})$ is said to be *domain-finite* (DF) if, for each $A \subseteq X$, $\mathscr{C}(A) = \cup\{\mathscr{C}(F) : F \subseteq A, |F| < \infty\}$. ($|F|$ denotes the cardinality of F.)

2.1 Remark. When $(X, \mathscr{C})$ is both DF and JHC, (i) if $A, B \in \mathscr{C}$ and $x \in \mathscr{C}(A \cup B)$, then $x \in ab$ for some $a \in A$, $b \in B$; (ii) $A \in \mathscr{C}$ if and only if $ab \subseteq A$ whenever $a, b \in A$.

Let $a/b = \{x : x \neq a, a \in xb\}$. The *line* determined by $a, b \in X$, $a \neq b$, is the set $l(a, b) = ab \cup (a/b) \cup (b/a)$. If $F \subseteq X$ and $l(a, b) \subseteq F$ whenever $a, b \in F$, then $F$ is called a *flat*. Let $\mathscr{A}$ be the family of all flats in $(X, \mathscr{C})$. Then $(X, \mathscr{A})$ is a convexity space and $\mathscr{A}(A)$ is called the *affine hull* of A. The *dimension* of $X$, $\dim X$, is defined inductively: $\dim X = 0$, if $X$ is a singleton; $\dim X = n$, if $X = \mathscr{A}(a_0, a_1, \ldots, a_n)$ and $\dim X \neq n-1$; $\dim X = \infty$, if $X \neq \mathscr{A}(F)$ for every finite subset $F \subseteq X$.

$(X, \mathscr{C})$ is *regular* (REG) if its segments are (i) non-discrete: $(ab) \neq \phi$ when $a \neq b$; (ii) decomposable: if $x \in ab$, then $ax \cap xb = x$ and $ax \cup xb = ab$; and, (iii) extendable: $a/b$ is non-empty when $a \neq b$.

Finally, we say that $(X, \mathscr{C})$ is *straight* (STR) if the union of two segments having more than one point in common is a segment.

2.2 Remarks. (i) If $(X, \mathscr{C})$ is REG then the following properties obtain: (1) $\mathscr{C}(a) = a$ for all $a \in X$; (2) if $a \in bc$ and $b \in ac$, $a \neq b \neq c$, then $a = b$; and, (3) for distinct points $a, b, c$, if $a \in bc$, then $b \notin ac$ and $c \notin ab$. (ii) The segments in a regular space can be given a natural linear ordering as decomposability essentially yields a betweenness relation. (iii) In a straight, regular space $(X, \mathscr{C})$ lines are uniquely determined by two points. In particular, for $a, b \in X$, $l(a, b) = \mathscr{A}(a, b)$. Further, as for segments above, lines have a natural linear ordering. (iv) The paradigm of a convexity space with any or all of the above properties is a real vector space. However, there are many other models of a convexity space and, in fact, each of the properties can be shown to be independent as is seen in the final section.

§3. **Line spaces.** In 1974 Cantwell [3] introduced line spaces (see definition below). Subsequently Doignon [6] has shown that line spaces of dimension three or greater or of dimension two and desarguesian are linearly open convex subsets of a real affine space. Recently Cantwell and Kay [4] have also obtained essentially the same result for dimension $\geq 3$ using different techniques.

In this section we will characterize those convexity spaces that are line spaces. Then Doignon's result will yield the desired linearization theorem.

A pair $(X, \mathscr{L})$, $X$ a non-empty set whose members are called *points* and $\mathscr{L}$ a family of subsets of $X$ whose members are called *lines*, is called a *line space*

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(Cantwell [3]) if the following conditions are satisfied: (i) every line is uniquely determined by two points; (ii) every line $l \in \mathcal{L}$ is a linearly ordered set with ordering $<_l$ and is order isomorphic to the reals; and, (iii) (Pasch's axiom) for any $a, b, c \in X$, $x \in [a, b]$, and $y \in [c, x]$, then there is $z \in [a, c]$ so that $y \in [b, z]$ where $[a, b] = \{x \in l = l(a, b) : a \leq_l x \leq_l b\}$ and $l(a, b)$ is the line uniquely determined by $a$ and $b$. (If $a = b$, $l(a, b) = a$.)

If $(X, \mathcal{L})$ is a line space and $C \subseteq X$, we say $C$ is a *convex set*, if $[a, b] \subseteq C$ whenever $a, b \in C$. Letting $\mathcal{C}_{\mathcal{L}}$ denote the family of all convex sets in $(X, \mathcal{L})$, it can be shown that $(X, \mathcal{C}_{\mathcal{L}})$ is a convexity space that is DF, JHC, REG, STR and complete (see definition below). The main result of this section will be to show the converse of this statement.

3.1 Lemma. *Let* $(X, \mathcal{C})$ *be* DF, JHC, REG, *and* STR *and let* $a, b, c \in X$. (i) (*Pasch's axiom*). *If* $y \in ac$ *and* $z \in by$ *then there is* $x \in ab$ *such that* $z \in cx$. (ii) (*Peano's axiom*). *If* $x \in ab$ *and* $y \in ac$, *then* $by \bigcap cx \neq \phi$. *Further, if* $a, b, c$ *are non-collinear then* $by \cap cx$ *is a singleton.*

**Proof.** (i) Pasch's axiom is an immediate consequence of JHC. (ii) Since the result is straightforward, if $a, b, c$ are collinear, it suffices to consider the case where $a, b, c$ are non-collinear and $x \in (ab)$, $y \in (ac)$.

By REG, there exists $d \in X$ such that $a \in (xd)$. By Pasch's axiom, there is $e \in cx$ such that $y \in de$ and $c \neq e$, otherwise $y = c = e$ which contradicts $y \in (ac)$. Now $x \in ab$ and $a \in bd$ so $e \in cx \subseteq \mathcal{C}(b, c, d)$. By JHC, there is $f \in bc$ such that $e \in df$. Again by Pasch's axiom there is $z \in by$ such that $e \in cz$. $z \in by \subseteq \mathcal{C}(a, b, c)$ so, by JHC, there is $w \in ab$ such that $z \in cw$. Since $e \in cz \subseteq cw$ and $e \in cx$, by STR, $c$, $w$, and $x$ are collinear. Thus $w = x$ and $z \in cx$.

Finally if $z_1, z_2 \in by \cap cx$, then $by \cup cx$ is a segment. Thus $b, x, c, y$, and hence $a, b, c$, are collinear.

Before proceeding to show that lines in such convexity spaces are order isomorphic to the reals, we need the following definition.

A convexity space $(X, \mathcal{C})$ is said to be *complete* (CMP) provided, for each $C \in \mathcal{C}$, if $a \in C$ and $b \in X \backslash C$, then there is $d \in ab$ such that $(ad) \subseteq C$ and $(db) \subseteq X \backslash C$.

3.2 Remarks. Let $(X, \mathcal{C})$ be DF, JHC, REG, STR, and CMP. (i) Each open segment in $(X, \mathcal{C})$ is conditionally complete, that is, every non-empty bounded subset has a greatest lower bound and a least upper bound. (ii) If $(X, \mathcal{C})$ is also of dimension at least two, that is, there are three non-collinear points in $X$, then using the Pasch and Peano axioms one can show that any two open segments in $(X, \mathcal{C})$ are order isomorphic. Further, by using Theorem 12.61 in Coxeter's book [5], one can show that each open segment in $X$ is order isomorphic to a line, and conversely.

3.3 Proposition. *If* $(X, \mathcal{C})$ *is* DF, JHC, REG, STR, *and* CMP *with dimension at least* 2, *then each line is order isomorphic to the real numbers* $\mathbb{R}$.

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**Proof.** By 3.2(ii), it is sufficient to show that an open segment is order isomorphic to $\mathbb{R}$. Further, it follows from Theorem 24, Chapter VIII of [1] that to show an open segment is order isomorphic to $\mathbb{R}$ it suffices to show that it contains a countable dense subset, is conditionally complete and has no endpoints. By 3.2(i), it is sufficient to produce a countable dense subset. This is done using the following nice construction due to Doignon [6].

Let $a, b, c \in X$ be distinct non-collinear points. Define a sequence by setting $x_1 = a$ and choosing $x_{m+1} \in (x_m b)$. By CMP, there is $d \in ab$ such that $\{x_m\} \subseteq ad$, for each $y \in (ad)$ there is some $x_m \in (yd)$ and $d \neq x_m$ for any $m$.

Consider the open segment $(da) \subseteq l(a, b)$ and order the line so that $d < a$. Choose $e \in (dc)$. For $u, v \in (da)$, construct $u+v \in (da)$ as follows: let $y = (ea) \cap (cu)$, $z = y/d \cap (ca)$, $w = (ya) \cap (zv)$ and set $u+v = w/c \cap (da)$. By definition $u+v > u$ and $u+v > v$; moreover $+$ is strictly increasing in each of its arguments. In particular, if $t, u, v \in (da)$ and $u < v$, then $u+t < v+t$ and $t+u < t+v$. Also, it is easily shown that, if $u < w$, then there is a unique $v \in (da)$ such that $u+v = w$.

For $u \in (da)$ and $n$ a positive integer, define $n \cdot u$ and $(n+1)u = n \cdot u + u$. If $u < v$, there is some $n$ such that $nu > v$; for, if not, then $nu \leq v$ for all $n$. Let $\bar{u} = \sup\{n \cdot u : n \in N\}$. Now $\bar{u} > u$, so there is $w \in (da)$ such that $u + w = \bar{u}$. Since $w < \bar{u}$, there is $n \in N$ such that $n \cdot u > w$ and $(n+1)u > w + u = \bar{u}$ which is a contradiction.

Let $Q = \{m \cdot x_n : m, n \in N\}$ which is a countable subset of $(da)$. Let $f, g \in (da)$, $f < g$. There is $h \in (da)$ such that $f + h = g$ and, for some $n$, $x_n < h$. If $x_n > f$, then $f < x_n < g$. Otherwise, choosing the largest $m$ such that $m \cdot x_n \leq f$, one obtains $f < (m+1)x_n < g$. Thus $Q$ is dense in $(da)$ and the proposition is proved.

The results of this section can be summarized in the following.

3.4 Theorem. *Let $(X, \mathscr{C})$ be a* DF, JHC, REG. STR, *and* CMP *convexity space of dimension at least two and let $\mathscr{L}_{\mathscr{C}}$ be the collection of lines in $(X, \mathscr{C})$. Then $(X, \mathscr{L}_{\mathscr{C}})$ is a line space. Conversely, if $(X, \mathscr{L})$ is a line space and $\mathscr{C}_{\mathscr{L}}$ is the collection of convex subsets of X, then $(X, \mathscr{C}_{\mathscr{L}})$ is a* DF, JHC, REG, STR, *and* CMP *convexity space.*

The linearization result is a corollary of Theorem 3.4 and Doignon's result mentioned above or the Cantwell–Kay result, if $\dim X \geq 3$. For $\dim X \geq 3$, both [4] and [6] achieve the same result, but [4] is self-contained while [6] depends on a 1938 theorem of Sperner.

3.5 Corollary. *Let $(X, \mathscr{C})$ be a convexity space of dimension 2 and desarguesian or of dimension $>2$, then $(X, \mathscr{C})$ is isomorphic to a linearly open convex subset of a real affine space if and only if $(X, \mathscr{C})$ is* DF, JHC, REG, STR *and* CMP.

§4. **Examples.** The topologists 'longline' with intervals as members of $\mathscr{C}$ is a 1-dimensional convexity space satisfying DF, JHC, REG, STR, and CMP but it is not a line space (cf. Theorem 3.4). The Moulton plane, which is a 2-dimensional non-desarguesian line space, cannot be embedded in an affine space (cf. Corollary 3.5).

Finally, several examples, each designated by the *one property which fails* to obtain, are given to exhibit the independence of DF, JHC, REG, STR, and CMP. In each of the examples $\dim X \geq 2$ and if $\dim X = 2$, it is desarguesian.

DF: Let $X = \mathbb{R}^2$ and $\mathscr{C}$ be the compact convex sets in $\mathbb{R}^2$ together with $\mathbb{R}^2$.

JHC: Let $X = \mathbb{R}^3$ and $\mathscr{C}$ be the convex sets in $\mathbb{R}^3$ of dimension less than or equal to 2 together with $\mathbb{R}^3$.

REG(i): (Segments fail to be non-discrete.) Let $X = \mathbb{R}^2 \backslash D$ where $D$ is an open disc in $\mathbb{R}^2$ and let $\mathscr{C}$ consist of the sets of the form $C \cap X$ where $C$ is convex in $\mathbb{R}^2$.

REG(ii): (Segments fail to be decomposable.) Let $X = P^3$, the classical projective 3-space. Points in $X$ are lines, in $\mathbb{R}^4 \backslash \{0\}$, which pass through the origin. For $a, b \in X$, $ab = a$ when $a = b$ and $ab$ is the unique projective line determined by $a$ and $b$ whenever $a \neq b$. Then $C \in \mathscr{C}$ if and only if $ab \subset C$ whenever $a, b \in C$.

REG(iii): (Segments fail to be extendable.) Let $X = D$ where $D$ is a closed disc in $\mathbb{R}^2$ and $\mathscr{C}$ consists of the sets of the form $C \cap D$ for $C$ convex in $\mathbb{R}^2$.

STR: Let $X = U \cup S$ where $U$ is the open unit disc in $\mathbb{R}^2$ and $S = \{(x, 0) : x \geq 1\}$. For $a, b \in X$ define $ab$ as follows: $ab = [a, b]$, if $a, b \in U$ or $a, b \in S$ and $ab = [a, p] \cup [p, b]$ where $p = (0, 1)$, if $a \in U$ and $b \in S$. Then $C \in \mathscr{C}$ if and only if $ab \subset C$.

CMP: Let $X = \{(x, y) \in \mathbb{R}^2 : x, y \text{ are rational}\}$ and $\mathscr{C}$ consists of sets of the form $C \cap X$ where $C$ is convex in $\mathbb{R}^2$.

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