Proceedings of the Edinburgh Mathematical Society (1983) **26**, 343–351 ©

ISOMORPHISMS BETWEEN RADICAL WEIGHTED CONVOLUTION ALGEBRAS

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(Received 20th July 1982)

In [**4**] we have shown that any two semi-simple weighted convolution algebras $L^1(\omega_1)$ and $L^1(\omega_2)$ are isomorphic. In this paper, given any two radical weighted convolution algebras $L^1(\omega_1)$ and $L^1(\omega_2)$ we find necessary and sufficient conditions, in terms of ω_1 and ω_2, for $L^1(\omega_1)$ and $L^1(\omega_2)$ to be isomorphic.

We call a continuous and positive function ω on the non-negative real numbers R^+ a weight function if $\omega(s+t) \leqq \omega(s)\omega(t)$ for every $s, t \in R^+$, and if $\omega(0)=1$. The weighted convolution algebra $L^1(\omega)$ is the (complex) Banach algebra of all equivalence classes of Lebesgue measurable functions f such that $\|f\| = \int_0^\infty |f(t)|\omega(t)\,dt < \infty$, under pointwise addition, scalar multiplication of functions, and convolution product:

$$(f*g)(x) = \int_0^x f(x-t)g(t)\,dt \qquad (f, g \in L^1(\omega), \quad \text{a.e.} \quad x \in R^+).$$

The elementary properties of the algebras $L^1(\omega)$ are given in [**3**]. We use the theory developed in [**1**], [**2**] and [**4**], and adopt the notation of [**4**].

We shall repeatedly use Titchmarsh's convolution theorem, which asserts that, if $\mu \neq 0$ and $\nu \neq 0$ are any two locally finite measures, then $\mu * \nu \neq 0$, or in its equivalent form $\alpha(\mu*\nu) = \alpha(\mu) + \alpha(\nu)$, where for every $\mu \neq 0$, $\alpha(\mu)$ is the infimum of the support of μ (see [**1**] for a proof).

If θ is an algebra isomorphism from $L^1(\omega_1)$ onto $L^1(\omega_2)$ then it is continuous [**5**; Remark 3(a)].

In this paper all of the algebras $L^1(\omega)$ are radical, or equivalently $\lim_{t\to\infty} \omega(t)^{1/t} = 0$.

In the following proposition, $M(\omega)$ is as defined in [**4**].

Proposition 1. *Suppose θ is an isomorphism from $L^1(\omega_1)$ onto $L^1(\omega_2)$. Then the formula $\bar{\theta}(\mu)(f) = \theta(\mu * \theta^{-1}(f))$ $(\mu \in M(\omega_1),$ $f \in L^1(\omega_2))$ defines a continuous isomorphism $\bar{\theta}: M(\omega_1) \to M(\omega_2)$ which extends θ.*

Proof. For every $\mu \in M(\omega_1)$, let $T_\mu: L^1(\omega_2) \to L^1(\omega_2)$ be defined by $T_\mu(f) = \theta(\mu * \theta^{-1}(f))$ $(f \in L^1(\omega_2))$. Then T_μ is obviously linear and we have

$$T_\mu(f*g) = \theta(\mu * \theta^{-1}(f*g)) = \theta(\mu * \theta^{-1}(f) * \theta^{-1}(g))$$
$$= \theta(\mu * \theta^{-1}(f)) * g = T_\mu(f) * g.$$

Thus, T_μ is a multiplier on $L^1(\omega_2)$. By an identification of the multiplier algebra of $L^1(\omega_2)$ with $M(\omega_2)$ [4; Theorem 1.4], there exists a measure, say $\bar{\theta}(\mu)$, in $M(\omega_2)$ such that $T_\mu(f) = \bar{\theta}(\mu) * f$ $(f \in L^1(\omega_2))$. We prove that the map $\mu \to \bar{\theta}(\mu)$ is an extension of θ to an isomorphism from $M(\omega_1)$ onto $M(\omega_2)$. This map is obviously linear. Let $\mu, \nu \in M(\omega_1)$ and $f \in (L^1(\omega_2)\backslash\{0\})$. Then

$$\bar{\theta}(\mu * \nu) * f = \theta(\mu * \nu * \theta^{-1}(f)) = \theta(\mu * \theta^{-1}\theta(\nu * \theta^{-1}(f)))$$

$$= \theta(\mu * \theta^{-1}(\bar{\theta}(\nu) * f) = \bar{\theta}(\mu) * \bar{\theta}(\nu) * f,$$

which together with Titchmarsh's convolution theorem implies $\bar{\theta}(\mu * \nu) = \bar{\theta}(\mu) * \bar{\theta}(\nu)$.

Let $\bar{\theta}(\mu) = 0$. Then for every $f \in (L^1(\omega_2)\backslash\{0\})$

$$\theta(\mu * \theta^{-1}(f)) = \bar{\theta}(\mu) * f = 0,$$

whence $\mu * \theta^{-1}(f) = 0$, since θ is an isomorphism. Hence by Titchmarsh's convolution theorem $\mu = 0$. Thus θ is injective.

To show that $\bar{\theta}$ is onto, let $\mu \in M(\omega_2)$, then $f \to \theta^{-1}(\mu * \theta(f))$ is a multiplier on $L^1(\omega_1)$, whence there exists $\nu \in M(\omega_1)$ such that $\theta^{-1}(\mu * \theta(f)) = \nu * f$ $(f \in L^1(\omega_1))$. If we apply $\bar{\theta}$ to both sides of this equality we obtain $\mu * \theta(f) = \bar{\theta}(\nu) * \theta(f)$ $(f \in L^1(\omega_1))$. Another application of the Titchmarsh's convolution theorem implies $\mu = \bar{\theta}(\nu)$. It is easily verified that $\bar{\theta}$ extends θ.

We also note that $(\bar{\theta})^{-1} = \overline{(\theta^{-1})}$.

Lemma 1. *Suppose θ is an isomorphism from $L^1(\omega_1)$ onto $L^1(\omega_2)$ and $\bar{\theta}$ is its extension as described in Proposition 1. Then there exists a constant $A_\theta > 0$, such that*

$$\alpha(\bar{\theta}(\delta_x)) = A_\theta x \qquad (x \in R^+),$$

where δ_x is the unit mass concentrated at x.

Proof. We consider the function $\beta: R^+ \to R^+$ defined by $\beta(x) = \alpha[\bar{\theta}(\delta_x)]$. For every $x, y \in R^+$, by Titchmarsh's convolution theorem we have,

$$\beta(x+y) = \alpha[\bar{\theta}(\delta_{x+y})] = \alpha[\bar{\theta}(\delta_x) * \bar{\theta}(\delta_y)]$$

$$= \alpha[\bar{\theta}(\delta_x)] + \alpha[\bar{\theta}(\delta_y)] = \beta(x) + \beta(y). \tag{1}$$

Next we prove that β is continuous from the right at every $x \in R^+$. It suffices to do this for $x = 0$. Let $x_n > 0$ and $x_n \to 0$. Then $\delta_{x_n} \xrightarrow{bso} \delta_0$. (For the definition of the topology *bso* and the topology σ which follows, see [4].) Since $\bar{\theta}$ is an isomorphism from $M(\omega_1)$ onto $M(\omega_2)$ we have

$$\bar{\theta}(\delta_{x_n}) \xrightarrow{bso} \bar{\theta}(\delta_0) = \delta_0, \tag{2}$$

whence

$$\bar{\theta}(\delta_{x_n}) \xrightarrow{\sigma} \delta_0. \tag{3}$$

[4; Lemma 1.2]. This implies $\beta(x_n) = \alpha\bar{\theta}(\delta_{x_n}) \to 0$, for otherwise there exists a positive number b such that for infinitely many values of n, $\alpha\bar{\theta}(\beta_{x_n}) > b$. Then if f is a continuous function with $\operatorname{supp} f \subset [0, b]$ and $f(0) = 1$ we get $\int_0^\infty f(t)\, d\bar{\theta}(\delta_{x_n})(t) = 0$ for infinitely many values of n, while $\int_0^\infty f(t)\, d\delta_0(t) = f(0) = 1$, and this contradicts (3). Hence β is continuous from the right, whence there exists $A_\theta \geqq 0$ such that $\alpha(\bar{\theta}(\delta_x)) = \beta(x) = A_\theta x$ for every $x \in R^+$.

Next we prove that $A_\theta > 0$. If $A_\theta = 0$, then $\alpha(\bar{\theta}(\delta_x)) = 0$ for every $x \in R^+$. We prove that this implies $\alpha(\theta(f)) = 0$ for every $f \in L^1(\omega_1)$ having a compact support and with $\alpha(f) > 0$. Suppose $f \in L^1(\omega_1)$, with $\alpha(f) = a$, $\operatorname{supp} f \subset [a, b]$ where $0 < a < b < \infty$. Then if $g = f * \delta_{-a}$, we have $g \in L^1(\omega_1)$, $\alpha(g) = 0$ and $\operatorname{supp} g \subset [0, b-a]$. Therefore, $L^1(\omega_1) * g$ is dense in $L^1(\omega_1)$ [**1**; Theorem 2]. Since θ is an isomorphism between $L^1(\omega_1)$ and $L^1(\omega_2)$, $L^1(\omega_2) * \theta(g)$ is dense in $L^1(\omega_2)$, whence $\alpha(\theta(g)) = 0$. We have $f = g * \delta_a$. Hence $\theta(f) = \theta(g) * \bar{\theta}(\delta_a)$. Thus,

$$\alpha(\theta(f)) = \alpha(\theta(g)) + \alpha(\bar{\theta}(\delta_a)) = 0.$$

To obtain a contradiction we show that there exists $f \in L^1(\omega_1)$ having a compact support with $\alpha(f) > 0$ and with $\alpha(\theta(f)) > 0$.

There exists $K \geqq 1$ and $M > 0$ such that

$$\frac{\omega_1(Kn)}{\omega_2(n)} \leqq M \qquad (n \in N), \tag{4}$$

[**2**; Theorem 4.1]. Since for each $\delta > 0$, $(1/\omega_1(\delta n))^{1/n} \to \infty$ as $n \to \infty$, by [**2**; Theorem 3.2.II] there exists f in $L^1(\omega_1)$ with $\alpha(f) = K$, with $\operatorname{supp} f \subset [K, K+1]$, and such that

$$\|f^{*n}\| < \omega_1(Kn) \qquad (n \in N). \tag{5}$$

For this f, by (4) and (5) we have

$$\|(\theta f)^{*n}\| / \omega_2(n) \leqq M\|\theta\| \qquad (n \in N)$$

and so by [**2**; Theorem 3.6], $\alpha\theta(f) \geqq 1$. From this contradiction we conclude that $A_\theta > 0$ and the lemma is proved.

The following proposition strengthens the statement of our Lemma 1.

Proposition 2. *Suppose θ is an isomorphism from $L^1(\omega_1)$ onto $L^1(\omega_2)$ and $\bar{\theta}$ is its extension as described in Proposition 1. Then there exists a constant $A_\theta > 0$, such that $\alpha(\bar{\theta}(\mu)) = A_\theta \alpha(\mu)$, for every $\mu \in M(\omega_1)$.*

Proof. By Lemma 1 there exists $A_\theta > 0$ such that

$$\alpha(\bar{\theta}(\delta_x)) = A_\theta x \qquad (x \in R^+).$$

Suppose

$$\mu=\sum_{i=1}^{N} a_i\delta_{x_i},$$

where $x_1<\ldots<x_N$ and $a_i\neq 0$, $i=1,\ldots,N$. Then

$$\bar{\theta}(\mu)=\sum_{i=1}^{N} a_i\bar{\theta}(\delta_{x_i}),$$

and we have

$$\alpha\bar{\theta}(\delta_{x_1})=A_\theta x_1<\ldots<\alpha\bar{\theta}(\delta_{x_N})=A_\theta x_N.$$

Hence,

$$\alpha(\bar{\theta}(\mu))=A_\theta x_1=A_\theta\alpha(\mu). \tag{1}$$

For a general $\mu\in M(\omega_1)$, we first prove that $\alpha(\bar{\theta}(\mu))\geqq A_\theta\alpha(\mu)$. Let $(\mu_i)\subset M(\omega_1)$ be a net such that $\mu_i\xrightarrow{bso}\mu$, $\alpha(\mu_i)\geqq\alpha(\mu)$ and such that each μ_i is a finite linear combination of point masses [4; Lemma 1.3]. Since $\bar{\theta}$ is an isomorphism we have $\bar{\theta}(\mu_i)\xrightarrow{bso}\bar{\theta}(\mu)$, whence $\bar{\theta}(\mu_i)\xrightarrow{\sigma}\bar{\theta}(\mu)$, [4; Lemma 1.2]. If $\alpha(\bar{\theta}(\mu))<A_\theta\alpha(\mu)$, then we choose b such that $\alpha\bar{\theta}(\mu)<b<A_\theta\alpha(\mu)$ and we let g be a continuous function with $\operatorname{supp} g\subset[\alpha\bar{\theta}(\mu),b]$ and with $\int_0^\infty g(x)\,d\bar{\theta}(\mu)(x)\neq 0$. Since $A_\theta\alpha(\mu)\leqq\alpha\bar{\theta}(\mu_i)$, we have $\int_0^\infty g(x)\,d\bar{\theta}(\mu_i)(x)=0$. Then

$$0\neq\int_0^\infty g(x)\,d\bar{\theta}(\mu)(x)=\lim\int_0^\infty g(x)\,d\bar{\theta}(\mu_i)(x)=0.$$

From this contradiction we conclude

$$\alpha\bar{\theta}(\mu)\geqq A_\theta\alpha(\mu). \tag{2}$$

Now, let $f\in L^1(\omega_1)$ have compact support and let $\alpha(f)=a$. Then $h=f*\delta_{-a}\in L^1(\omega_1)$, and $\alpha(h)=0$. Thus, $L^1(\omega_1)*h$ is dense in $L^1(\omega_1)$ [1; Theorem 2]. Hence, $L^1(\omega_2)*\theta(h)$ is dense in $L^1(\omega_2)$. Therefore, $\alpha\theta(h)=0$. We have $\theta(f)=\theta\,(h*\delta_a)=\theta(h)*\bar{\theta}(\delta_a)$. Hence,

$$\alpha(\theta(f))=\alpha(\theta(h))+\alpha(\bar{\theta}(\delta_a))=0+A_\theta\alpha=A_\theta\alpha(h). \tag{3}$$

Now, suppose $f\in(L^1(\omega_1)\backslash\{0\})$, $\alpha(f)=c$. Let $f_1=f\chi_{[c,c+1]}$, $f_2=f\chi_{(c+1,\infty)}$. Then $f=f_1+f_2$. Therefore, $\theta(f)=\theta(f_1)+\theta(f_2)$. By the conclusion of the previous paragraph we have $\alpha\theta(f_1)=A_\theta c$, and by (2) we have $\alpha\theta(f_2)\geqq A_\theta(c+1)>A_\theta c$. Therefore

$$\alpha(\theta(f))=\min\{\alpha(\theta(f_1)),\alpha(\theta(f_2))\}=A_\theta c=A_\theta\alpha(f). \tag{4}$$

Finally, if $\mu\in(M(\omega_1)\backslash\{0\})$, then for $f\in(L^1(\omega_1)\backslash\{0\})$ by (4) we have

$$\alpha(\bar{\theta}(\mu))+\alpha(\theta(f))=\alpha(\theta(\mu*f))=A_\theta\alpha(\mu*f)=A_\theta\alpha(\mu)+A_\theta\alpha(f). \tag{5}$$

Cancelling $\alpha\theta(f)=A_\theta\alpha(f)$ from both sides of (5) we obtain $\alpha\bar{\theta}(\mu)=A_\theta\alpha(\mu)$, and the proof is complete.

Corollary 1. *If θ is an isomorphism from $L^1(\omega_1)$ onto $L^1(\omega_2)$ and θ^{-1} is its inverse, then $A_{\theta^{-1}}=1/A_\theta$.*

Proof. We have $\alpha(\bar{\theta}(\bar{\theta})^{-1}(\delta_x))=\alpha(\delta_x)=x$, and by Proposition 2 we have $\alpha(\bar{\theta}(\bar{\theta})^{-1}(\delta_x)) =A_\theta\alpha((\bar{\theta})^{-1}(\delta_x))=A_\theta A_{\theta-1}x$, and the result follows.

Since $A_\theta>0$, we also have:

Corollary 2. *The function $\alpha\bar{\theta}$ is strictly increasing in the sense that if $\alpha(\mu)<\alpha(\nu)$, then $\alpha(\bar{\theta}(\mu))<\alpha(\bar{\theta}(\nu))$.*

We need the following two lemmas for the proof of our main theorem.

Lemma 2. *Suppose that μ and ν are any two locally finite measures on R^+. Then $\mu*\nu$ has a non-zero mass at $\alpha(\mu*\nu)$ if and only if μ has a non-zero mass at $\alpha(\mu)$ and ν has a non-zero mass at $\alpha(\nu)$.*

Proof. We have $(\mu*\nu)(\{\alpha(\mu*\nu)\})=\mu(\{\alpha(\mu)\})\nu(\{\alpha(\nu)\})$, and the lemma is proved.

Lemma 3. *Suppose θ is an isomorphism from $L^1(\omega_1)$ onto $L^1(\omega_2)$ and $\bar{\theta}$ is its extension as described in Propositon 1. Then for every $x\in R^+$, $\bar{\theta}(\delta_x)$ has a non-zero mass at $\alpha(\bar{\theta}(\delta_x))$.*

Proof. Let $x\in R^+$ and suppose that $\bar{\theta}(\delta_x)$ has a zero mass at $\alpha(\bar{\theta}(\delta_x))$. Then, we first prove that $\bar{\theta}(\delta_y)$ has a zero mass at $\alpha(\bar{\theta}(\delta_y))$, for every $y>0$. If $y>x$, then $\bar{\theta}(\delta_y) =\bar{\theta}(\delta_x)*\bar{\theta}(\delta_{y-x})$, whence by Lemma 2, $\bar{\theta}(\delta_y)$ has a zero mass at $\alpha(\bar{\theta}(\delta_y))$. On the other hand, if $0<y<x$, let n be a positive integer such that $x<ny$. Then $\bar{\theta}(\delta_{ny}) =\bar{\theta}(\delta_{ny-x})*\bar{\theta}(\delta_x)$, and again Lemma 2 implies that $\bar{\theta}(\delta_{ny})$ has a zero mass at $\alpha(\bar{\theta}(\delta_{ny}))$. Since $\bar{\theta}(\delta_{ny})=(\bar{\theta}(\delta_y))^{*n}$, another application of Lemma 2 implies that $\bar{\theta}(\delta_y)$ has a zero mass at $\alpha(\bar{\theta}(\delta_y))$.

Next we prove that this implies $\bar{\theta}(\mu)$ has a zero mass at $\alpha(\bar{\theta}(\mu))$, for every $\mu\in M(\omega_1)$ having a compact support and with $\alpha(\mu)>0$. Let $\mu\in M(\omega_1)$ with $\operatorname{supp}\mu\subset[a,b]$, $0<a<b<\infty$. Then $\nu=\mu*\beta_{-a/2}\in M(\omega_1)$. We have $\mu=\nu*\delta_{a/2}$, whence $\bar{\theta}(\mu)=\bar{\theta}(\nu)*\bar{\theta}(\delta_{a/2})$, and the discussion in the above paragraph together with Lemma 2 implies that $\bar{\theta}(\mu)$ has a zero mass at $\alpha(\bar{\theta}(\mu))$.

If $\lambda\in M(\omega_1)$ does not have compact support and $\alpha(\lambda)=k>0$, then we decompose λ into $\lambda=\lambda_1+\lambda_2$, with $\operatorname{supp}\lambda_1\subset[k,k+1]$, $\alpha(\lambda_1)=k$, and $\alpha(\lambda_2)\geqq k+1$, so that $\bar{\theta}(\lambda_1)$ has a zero mass at $\alpha(\bar{\theta}(\lambda_1))$. Since $\alpha(\bar{\theta}(\lambda_2))\geqq A_\theta(k+1)>A_\theta k=\alpha(\bar{\theta}(\lambda_1))$, the measure $\bar{\theta}(\lambda)=\bar{\theta}(\lambda_1) +\bar{\theta}(\lambda_2)$, has a zero mass at $\alpha(\bar{\theta}(\lambda))=\alpha(\bar{\theta}(\lambda_1))$.

Now, $\alpha((\bar{\theta})^{-1}(\delta_1))=A_{\theta^{-1}}>0$. Hence $\delta_1=\bar{\theta}(\bar{\theta})^{-1}(\delta_1)$ has a zero mass at $\alpha((\bar{\theta}(\bar{\theta})^{-1})(\delta_1)) =\alpha(\delta_1)=1$. From this contradiction we conclude that $\bar{\theta}(\delta_x)$ has a non-zero mass at $\alpha(\bar{\theta}(\delta_x))$ for every $x\in R^+$.

Theorem 1. *A necessary and sufficient condition for $L^1(\omega_1)$ and $L^1(\omega_2)$ to be isomorphic is the existence of positive numbers a, b, m and M such that,*

$$m \leqq \frac{\omega_2(ax)}{\omega_1(x)} b^x \leqq M \qquad (x \in R^+). \tag{1}$$

Proof. Suppose that there exist $a>0$, $b>0$, $m>0$ and $M>0$ for which (1) is fulfilled. Define $\theta: L^1(\omega_1) \to L^1(\omega_2)$ by

$$(\theta f)(x) = 1/ab^{x/a} f(x/a) \qquad (f \in L^1(\omega_1), \quad \text{a.e.} \quad x \in R^+). \tag{2}$$

Then θ is an isomorphism from $L^1(\omega_1)$ onto $L^1(\omega_2)$.

Conversely, let θ be an isomorphism from $L^1(\omega_1)$ onto $L^1(\omega_2)$ and let $\bar{\theta}$ be its extension. For simplicity we write a for A_θ which is in fact $\alpha(\bar{\theta}(\delta_1))$. For every rational $x \in R^+$, by Proposition 2 and Lemma 3 we have

$$\bar{\theta}(\delta_x) = k(x)\delta_{ax} + \mu_x, \tag{3}$$

where $\alpha(\mu_x) \geqq ax$, $\mu_x(\{ax\}) = 0$, and $k(x) \neq 0$. Now, suppose $x, y \in R^+$ are any two rationals. Then

$$\bar{\theta}(\delta_{x+y}) = k(x+y)\delta_{a(x+y)} + \mu_{x+y}. \tag{4}$$

Also,

$$\begin{aligned}\bar{\theta}(\delta_{x+y}) &= \bar{\theta}(\delta_x) * \bar{\theta}(\delta_y) = (k(x)\delta_{ax} + \mu_x) * (k(y)\delta_{ay} + \mu_y)\\ &= k(x)k(y)\delta_{ax+ay} + k(x)\delta_{ax} * \mu_y + k(y)\delta_{ay} * \mu_x + \mu_x * \mu_y,\end{aligned} \tag{5}$$

where the measure $k(x)\delta_{ax} * \mu_y + k(y)\delta_{ay} * \mu_x + \mu_x * \mu_y$ has a zero mass at $ax+ay$, by Lemma 3. Comparing equations (4) and (5) we get

$$k(x+y) = k(x)k(y). \tag{6}$$

Therefore, there exists $b>0$, such that for every rational $x \in R^+$, we have $|k(x)| = b^x$.

From (3), for every rational $x \in R^+$, we get

$$\begin{aligned}\|\bar{\theta}(\delta_x)\|_{M(\omega_2)} &= \|k(x)\delta_{ax}\|_{M(\omega_2)} + \|\mu_x\|_{M(\omega_2)} \geqq \|k(x)\delta_{ax}\|_{M(\omega_2)}\\ &= |k(x)|\,\|\delta_{ax}\|_{M(\omega_2)} = b^x \omega_2(ax).\end{aligned} \tag{7}$$

Now,

$$\|\bar{\theta}\| \geqq \frac{\|\bar{\theta}(\delta_x)\|_{M(\omega_2)}}{\|\delta_x\|_{M(\omega_1)}} = \frac{b^x \omega_2(ax)}{\omega_1(x)}. \tag{8}$$

On the other hand, by Corollary 1 and Lemma 3 we have,

$$(\bar{\theta})^{-1}(\delta_{ax})=l(x)\delta_x+v_x, \tag{9}$$

where $l(x)\neq 0$, $\alpha(v_x)\geqq x$ and v_x has a zero mass at x. Arguing as we did in the previous paragraph, we find $c>0$ such that $|l(x)|=c^x$, for every rational $x\in R^+$. We prove $c=1/b$, or equivalently $|k(1)||l(1)|=1$. We have

$$\bar{\theta}(\delta_1)=k(1)\delta_a+\mu_1, \tag{10}$$

and,

$$(\bar{\theta})^{-1}(\delta_a)=l(1)\delta_1+v_1, \tag{11}$$

where $\alpha(\mu_1)\geqq a$, $\mu_1(\{a\})=0$ and $\alpha(v_1)\geqq 1$, $v_1(\{1\})=0$. Let $\varepsilon>0$. Since the measure μ_1 introduced in (10) has a zero mass at a and is regular, there exists $\eta>0$, and a decomposition $\mu_1=\mu_1'+\mu_1''$, with $\operatorname{supp}\mu_1'\subset[a,a+\eta]$, $\operatorname{supp}\mu_1''\subset[a+\eta,\infty)$ and $\|\mu_1'\|_{M(\omega_2)}<\varepsilon$. Then from (10) and (11) we get,

$$\begin{aligned}\delta_1&=k(1)(\bar{\theta})^{-1}(\delta_a)+(\bar{\theta})^{-1}(\mu_1')+(\bar{\theta})^{-1}(\mu_1'')\\&=k(1)(l(1)\delta_1+v_1)+(\bar{\theta})^{-1}(\mu_1')+(\bar{\theta})^{-1}(\mu_1'')\\&=k(1)l(1)\delta_1+(\bar{\theta})^{-1}(\mu_1')+k(1)v_1+(\bar{\theta})^{-1}(\mu_1'').\end{aligned} \tag{12}$$

The measure $(\bar{\theta})^{-1}(\mu_1'')$ has a zero mass at 1, since

$$\alpha((\bar{\theta})^{-1}(\mu_1''))=A_{\theta-1}\alpha(\mu_1'')\geqq\frac{1}{a}(a+\eta)>1.$$

Also the measure $k(1)v_1$ has a zero mass at 1, since v_1 already had this property. The measure $(\bar{\theta})^{-1}(\mu_1')$ might have a non-zero mass at 1. Suppose

$$(\bar{\theta})^{-1}(\mu_1')=p\delta_1+\lambda, \tag{13}$$

where $\alpha(\lambda)\geqq 1$, $\lambda(\{1\})=0$, From (12) and (13) we obtain

$$\delta_1=k(1)l(1)\delta_1+(p\delta_1+\lambda)+k(1)v_1+(\bar{\theta})^{-1}(\mu_1''). \tag{14}$$

On equating the coefficients of δ_1 in both sides of (14) we obtain

$$1=k(1)l(1)+p. \tag{15}$$

On the other hand, by (13) we have,

$$\|(\bar{\theta})^{-1}(\mu_1')\|_{M(\omega_1)}\geqq\|p\delta_1\|_{M(\omega_1)}=|p|\omega_1(1), \tag{16}$$

and,

$$\|(\bar{\theta})^{-1}(\mu_1')\|_{M(\omega_1)} \leqq \|(\bar{\theta})^{-1}\| \, \|\mu_1'\|_{M(\omega_2)} \leqq \varepsilon \|(\bar{\theta})^{-1}\|. \tag{17}$$

From (16) and (17) we obtain

$$|p| \leqq \frac{\|(\bar{\theta})^{-1}\|}{\omega_1(1)} \varepsilon. \tag{18}$$

Since ε was arbitrary (15) and (18) imply

$$1 = k(1)l(1). \tag{19}$$

Thus, $c = 1/b$, whence $l(x) = 1/b^x$, for every rational $x \in R^+$ and by (9) for every rational $x \in R^+$, we have

$$\frac{\omega_1(x)}{\omega_2(ax)b^x} \leqq \|(\bar{\theta})^{-1}\|. \tag{20}$$

Combining (8) and (20) we obtain

$$\|(\bar{\theta})^{-1}\|^{-1} \leqq \frac{\omega_2(ax)b^x}{\omega_1(x)} \leqq \|\bar{\theta}\|, \tag{21}$$

for every rational $x \in R^+$. Now, continuity of ω_1 and ω_2 implies that (21) holds for every $x \in R^+$, and the proof is complete.

Corollary 3. *Suppose θ is an automorphism of $L^1(\omega)$, then $\alpha(\theta(f)) = \alpha(f)$ $(f \in L^1(\omega))$.*

Proof. By Theorem 1 there exist $a > 0$, $b > 0$, $m > 0$, and $M > 0$ such that $m \leqq [\omega(ax)/\omega(x)]b^x \leqq M$ $(x \in R^+)$. The proof of Theorem 1 shows that a can be chosen to be equal to A_θ. If $a > 1$, then

$$m \leqq \frac{\omega(ax)}{\omega(x)} b^x \leqq \frac{\omega((a-1)x)\omega(x)}{\omega(x)} b^x = \omega((a-1)x)b^x,$$

whence $m^{1/x} \leqq (\omega((a-1)x))^{1/x} b$. Now we let $x \to \infty$ to obtain $1 \leqq 0$, a contradiction. Similarly, the inequality $[\omega(ax)/\omega(x)]b^x \leqq M$, rules out the possibility $a < 1$. Hence $A_\theta = a = 1$.

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