

## ON THE CHARACTERISATION OF ALTERNATING GROUPS BY CODEGREES

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### Abstract

Let  $G$  be a finite group and  $\text{Irr}(G)$  the set of all irreducible complex characters of  $G$ . Define the codegree of  $\chi \in \text{Irr}(G)$  as  $\text{cod}(\chi) := |G : \ker(\chi)|/\chi(1)$  and let  $\text{cod}(G) := \{\text{cod}(\chi) \mid \chi \in \text{Irr}(G)\}$  be the codegree set of  $G$ . Let  $A_n$  be an alternating group of degree  $n \geq 5$ . We show that  $A_n$  is determined up to isomorphism by  $\text{cod}(A_n)$ .

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### 1. Introduction

Let  $G$  be a finite group and  $\text{Irr}(G)$  the set of all irreducible complex characters of  $G$ . For any  $\chi \in \text{Irr}(G)$ , define the codegree of  $\chi$  as  $\text{cod}(\chi) := |G : \ker(\chi)|/\chi(1)$  and the codegree set of  $G$  as  $\text{cod}(G) := \{\text{cod}(\chi) \mid \chi \in \text{Irr}(G)\}$ . We refer the reader to the authors' previous paper [8] for the current literature on codegrees.

The following conjecture appears in the *Kourovka Notebook of Unsolved Problems in Group Theory* [12, Question 20.79].

**CODEGREE VERSION OF HUPPERT'S CONJECTURE.** Let  $H$  be a finite nonabelian simple group and  $G$  a finite group such that  $\text{cod}(H) = \text{cod}(G)$ . Then  $G \cong H$ .

In [8], the authors verified the conjecture for all sporadic simple groups. In this paper, we provide a general proof verifying this conjecture for all alternating groups of degree greater than or equal to 5.

**THEOREM 1.1.** *Let  $A_n$  be an alternating group of degree  $n \geq 5$  and  $G$  a finite group. If  $\text{cod}(G) = \text{cod}(A_n)$ , then  $G \cong A_n$ .*

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Throughout the paper, we follow the notation used in Isaacs' book [10] and the ATLAS of Finite Groups [6].

## 2. Proof of Theorem 1.1

First, we note that the cases  $n = 5, 6$  and  $7$  have already been proven in [1, 2], so in the following, we always assume that  $n > 7$ . Now, let  $G$  be a minimal counterexample and  $N$  be a maximal normal subgroup of  $G$ . So  $\text{cod}(G) = \text{cod}(A_n)$  and  $G/N$  is simple. By [8, Lemma 2.5],  $\text{cod}(G/N) \subseteq \text{cod}(A_n)$ . Then, by [9, Theorem B],  $G/N \cong A_n$  since  $N \neq 1$  since  $G \not\cong A_n$ .

*Step 1:  $N$  is a minimal normal subgroup of  $G$ .*

Suppose  $L$  is a nontrivial normal subgroup of  $G$  with  $L < N$ . Then by [8, Lemma 2.6],  $\text{cod}(G/N) \subseteq \text{cod}(G/L) \subseteq \text{cod}(G)$ . However,  $\text{cod}(G/N) = \text{cod}(A_n) = \text{cod}(G)$ , so equality must be attained in each inclusion. Thus,  $\text{cod}(G/L) = \text{cod}(A_n)$  which implies that  $G/L \cong A_n$  since  $G$  is a minimal counterexample. This is a contradiction since we also have  $G/N \cong A_n$ , but  $L < N$ .

*Step 2:  $N$  is the only nontrivial, proper normal subgroup of  $G$ .*

Otherwise, we assume  $M$  is another proper nontrivial normal subgroup of  $G$ . If  $N$  is included in  $M$ , then  $M = N$  or  $M = G$  since  $G/N$  is simple, which is a contradiction. Then  $N \cap M = 1$  and  $G = N \times M$ . Since  $M$  is also a maximal normal subgroup of  $G$ , we have  $N \cong M \cong A_n$ . Choose  $\psi_1 \in \text{Irr}(N)$  and  $\psi_2 \in \text{Irr}(M)$  such that  $\text{cod}(\psi_1) = \text{cod}(\psi_2) = \max(\text{cod}(A_n))$ . Set  $\chi = \psi_1 \cdot \psi_2 \in \text{Irr}(G)$ . Then  $\text{cod}(\chi) = (\max(\text{cod}(A_n)))^2 \notin \text{cod}(G)$ , which is a contradiction.

*Step 3:  $\chi$  is faithful, for each nontrivial  $\chi \in \text{Irr}(G|N) := \text{Irr}(G) - \text{Irr}(G/N)$ .*

From the proof of [8, Lemma 2.5],

$$\text{Irr}(G/N) = \{\hat{\chi}(gN) = \chi(g) \mid \chi \in \text{Irr}(G) \text{ and } N \leq \ker(\chi)\}.$$

By the definition of  $\text{Irr}(G|N)$ , it follows that if  $\chi \in \text{Irr}(G|N)$ , then  $N \not\leq \ker(\chi)$ . Thus, since  $N$  is the unique nontrivial, proper, normal subgroup of  $G$ ,  $\ker(\chi) = G$  or  $\ker(\chi) = 1$ . Therefore,  $\ker(\chi) = 1$  for all nontrivial  $\chi \in \text{Irr}(G|N)$ .

*Step 4:  $N$  is an elementary abelian group.*

Suppose that  $N$  is not abelian. Since  $N$  is a minimal normal subgroup, by [7, Theorem 4.3A(iii)],  $N = S^n$ , where  $S$  is a nonabelian simple group and  $n \in \mathbb{Z}^+$ . By [14, Lemma 4.2] and [11, Theorem 4.3.34], there is a nontrivial character  $\chi \in \text{Irr}(N)$  which extends to some  $\psi \in \text{Irr}(G)$ . Now,  $\ker(\psi) = 1$  by Step 3, so  $\text{cod}(\psi) = |G|/\psi(1) = |G/N| \cdot |N|/\chi(1)$ . However, by assumption,  $\text{cod}(G) = \text{cod}(A_n) = \text{cod}(G/N)$ . Thus,  $\text{cod}(\psi) \in \text{cod}(G) = \text{cod}(G/N)$ , so  $\text{cod}(\psi) = |G/N|/\phi(1)$  for some  $\phi \in \text{Irr}(G/N)$ . Hence,  $|G/N|$  is divisible by  $\text{cod}(\psi)$  which contradicts the fact that  $\text{cod}(\psi) = |G/N| \cdot |N|/\chi(1)$ , as  $\chi(1) \neq |N|$ . Thus,  $N$  must be abelian.

Now to show that  $N$  is elementary abelian, let a prime  $p$  divide  $|N|$ . Then  $N$  has a  $p$ -Sylow subgroup  $K$ , and  $K$  is the unique  $p$ -Sylow subgroup of  $N$  since  $N$  is abelian,

so  $K$  is characteristic in  $N$ . Thus,  $K$  is a normal subgroup of  $G$ , so  $K = N$  as  $N$  is minimal. Thus,  $|N| = p^n$ . Now, take the subgroup  $N^p = \{n^p \mid n \in N\}$  of  $N$ , which is proper by Cauchy's theorem. Since  $N^p$  is characteristic in  $N$ , it must be normal in  $G$ , so  $N^p$  is trivial by the uniqueness of  $N$ . Thus, every element of  $N$  has order  $p$  and  $N$  is elementary abelian.

*Step 5:*  $C_G(N) = N$ .

First note that since  $N$  is normal,  $C_G(N) \leq G$ . Additionally, since  $N$  is abelian by Step 4,  $N \leq C_G(N)$ . By the maximality of  $N$ , we must have  $C_G(N) = N$  or  $C_G(N) = G$ . If  $C_G(N) = N$ , we are done.

If not, then  $C_G(N) = G$ , so  $N$  must be in the centre of  $G$ . Then since  $N$  is the unique minimal normal subgroup of  $G$  by Step 2,  $|N|$  must be prime. If not, there always exists a proper nontrivial subgroup  $K$  of  $N$ , and  $K$  is normal since it is contained in  $Z(G)$ , contradicting the minimality of  $N$ . Hence, we have  $N \leq Z(G)$  which implies that  $Z(G) \cong N$ . This is because  $N$  is a maximal normal subgroup of  $G$  so if not, we would have  $Z(G) = G$ , implying  $G$  is abelian which is a contradiction. Thus,  $N$  is isomorphic to a subgroup of the Schur multiplier of  $G/N$  by [10, Corollary 11.20].

Now, we note that it is well known that for  $n > 7$ , the Schur multiplier of  $A_n$  is  $\mathbb{Z}_2$ , so  $G \cong 2.A_n$  [17]. From [3, Theorem 4.3],  $2.A_n$  always has a faithful irreducible character  $\chi$  of degree  $2^{\lfloor (n-2)/2 \rfloor}$ . Recall that by Step 2, there is only one nontrivial proper normal subgroup of  $G \cong 2.A_n$ . In particular,  $N \cong \mathbb{Z}_2$  is the only nontrivial proper normal subgroup of  $G$ . Thus,  $|\ker(\chi)| = 1$  or  $2$ . Then  $\text{cod}(\chi) = |2.A_n : \ker(\chi)|/\chi(1)$ . If  $|\ker(\chi)| = 1$ , then  $\text{cod}(\chi) = n!/2^{\lfloor (n-2)/2 \rfloor}$ , and if  $|\ker(\chi)| = 2$ , then  $\text{cod}(\chi) = (n!/2)/2^{\lfloor (n-2)/2 \rfloor} = n!/2^{\lfloor n/2 \rfloor}$ . In either case, for any prime  $p \neq 2$ ,  $|\text{cod}(\chi)|_p = |n!|_p = |A_n|_p$ . However,  $\text{cod}(\chi) \in \text{cod}(A_n)$  since  $\text{cod}(G) = \text{cod}(A_n)$ . Therefore, there is a character degree of  $A_n$  which is a power of 2.

However, from [13], for  $n > 7$ ,  $A_n$  only has a character degree equal to a power of 2 when  $n = 2^d + 1$  for some positive integer  $d$ . In this case,  $2^d = n - 1 \in \text{cd}(A_n)$  so we need  $|A_n|/n - 1 = |2.A_n|/2^{\lfloor (n-2)/2 \rfloor}$  or  $|2.A_n|/2^{\lfloor n/2 \rfloor}$ . Hence,

$$\frac{1}{n - 1} = \frac{2}{2^{\lfloor (n-2)/2 \rfloor}} = \frac{1}{2^{\lfloor (n-2)/2 \rfloor - 1}} \quad \text{or} \quad \frac{1}{2^{\lfloor n/2 \rfloor - 1}}$$

so  $n - 1 = 2^{\lfloor (n-2)/2 \rfloor - 1}$  or  $2^{\lfloor n/2 \rfloor - 1}$ . However, the only integer solution to either of these equations occurs when  $n = 9$  and  $9 - 1 = 8 = 2^3 = 2^{\lfloor 9/2 \rfloor - 1}$ . In this case, we check the ATLAS [6] to find that the codegree sets of  $A_9$  and  $2.A_9$  do not have the same order. This is a contradiction, so  $C_G(N) = N$ .

*Step 6.* Let  $\lambda$  be a nontrivial character in  $\text{Irr}(N)$  and  $\vartheta \in \text{Irr}(I_G(\lambda)|\lambda)$ , the set of irreducible constituents of  $\lambda^{I_G(\lambda)}$ , where  $I_G(\lambda)$  is the inertia group of  $\lambda$  in  $G$ . Then  $|I_G(\lambda)|/\vartheta(1) \in \text{cod}(G)$ . Also,  $\vartheta(1)$  divides  $|I_G(\lambda)/N|$  and  $|N|$  divides  $|G/N|$ . Lastly,  $I_G(\lambda) < G$ , that is,  $\lambda$  is not  $G$ -invariant.

Let  $\lambda$  be a nontrivial character in  $\text{Irr}(N)$  and  $\vartheta \in \text{Irr}(I_G(\lambda)|\lambda)$ . Let  $\chi$  be an irreducible constituent of  $\vartheta^G$ . By [10, Corollary 5.4],  $\chi \in \text{Irr}(G)$ , and by [10, Definition 5.1], we have  $\chi(1) = (|G|/|I_G(\lambda)|) \cdot \vartheta(1)$ . Moreover,  $\ker(\chi) = 1$  by Step 2, and thus

$\text{cod}(\chi) = |G|/\chi(1) = |I_G(\lambda)|/\vartheta(1)$ , so  $|I_G(\lambda)|/\vartheta(1) \in \text{cod}(G)$ . Now, since  $N$  is abelian,  $\lambda(1) = 1$ , so we have  $\vartheta(1) = \vartheta(1)/\lambda(1)$  which divides  $|I_G(\lambda)|/|N|$ , so  $|N|$  divides  $|I_G(\lambda)|/\vartheta(1)$ . Moreover,  $\text{cod}(G) = \text{cod}(G/N)$ , and all elements in  $\text{cod}(G/N)$  divide  $|G/N|$ , so  $|N|$  divides  $|G/N|$ .

Next, we want to show  $I_G(\lambda)$  is a proper subgroup of  $G$ . To reach a contradiction, assume  $I_G(\lambda) = G$ . Then  $\ker(\lambda) \leq G$ . From Step 2,  $\ker(\lambda) = 1$ , and from Step 4,  $N$  is a cyclic group of prime order. Thus, by the Normaliser–Centraliser theorem,  $G/N = \mathbf{N}_G(N)/\mathbf{C}_G(N) \leq \text{Aut}(N)$  so  $G/N$  is abelian, which is a contradiction.

*Step 7: Final contradiction.*

From Step 4,  $N$  is an elementary abelian group of order  $p^m$  for some prime  $p$  and integer  $m \geq 1$ . By the Normaliser–Centraliser theorem,  $A_n \cong G/N = \mathbf{N}_G(N)/\mathbf{C}_G(N) \leq \text{Aut}(N)$  and  $m > 1$ . Note that in general,  $\text{Aut}(N) \cong \text{GL}(m, p)$ . By Step 6,  $|N|$  divides  $|G/N|$ , so  $|N| = p^m$  divides  $|A_n|$  and  $G/N \cong A_n \leq \text{GL}(m, p)$ . We prove by contradiction that this cannot occur.

First, we claim that if  $p^m$  divides  $|A_n|$  and  $A_n \leq \text{GL}(m, p)$ , then  $p$  must equal 2. To show this, we note that for  $p > 2$ , by [4], if  $p^m$  divides  $|A_n|$ , then  $m < n/2$ . However, by [16, Theorem 1.1], if  $n > 6$ , the minimal faithful degree of a modular representation of  $A_n$  over a field of characteristic  $p$  is at least  $n - 2$ . Since embedding  $A_n$  as a subgroup of  $\text{GL}(m, p)$  is equivalent to giving a faithful representation of degree  $m$  over a field of characteristic  $p$ , we have  $m \geq n - 2$ . This is a contradiction since  $n/2 > n - 2$  implies  $n < 4$ . Therefore,  $p = 2$ .

Now, let  $p = 2$ . As above, from [4], we obtain  $|n!|_2 \leq 2^{n-1}$ . Thus, if  $2^m$  divides  $|A_n|$ , then  $2^m \leq |A_n|_2 \leq 2^{n-2}$  so  $m \leq n - 2$ . We will deal first with  $n > 8$  and then treat the case  $n = 8$  later. For  $n > 8$ , [15, Theorem 1.1] shows that the minimal faithful degree of a modular representation of  $A_n$  over a field of characteristic 2 is at least  $n - 2$ . Therefore, we must have  $m \geq n - 2$ , so we have equality,  $m = n - 2$ .

Let  $\lambda \in \text{Irr}(N)$ ,  $\vartheta \in \text{Irr}(I_G(\lambda)|\lambda)$  and  $T := I_G(\lambda)$ . Then  $1 < |G : T| < |N| = 2^{n-2}$  for  $|G : T|$  is the number of all conjugates of  $\lambda$ . By Step 5,  $|T|/\vartheta(1) \in \text{cod}(G)$  and moreover  $|N|$  divides  $|T|/\vartheta(1)$ . Since  $|N| = |N|_2 = |A_n|_2$  and  $\text{cod}(G) = \text{cod}(A_n)$ , it follows that  $\|T|/\vartheta(1)|_2 = |N|$ . Thus,  $\|T/N|/\vartheta(1)|_2 = 1$  so the 2-parts of  $|T/N|$  and  $\vartheta(1)$  are equal. Thus, for every  $\vartheta \in \text{Irr}(T | \lambda)$ , we have  $|\vartheta(1)|_2 = |T/N|_2$ . However,  $|T/N| = \sum_{\vartheta \in \text{Irr}(T|\lambda)} \vartheta(1)^2$ . Hence, if  $|\vartheta(1)|_2 = 2^k \geq 2$  for every  $\vartheta \in \text{Irr}(T | \lambda)$ , we would have  $|T/N|_2 = 2^{2k}$ , which contradicts the fact that  $|\vartheta(1)|_2 = |T/N|_2$ . Therefore,  $|T/N|_2 = 1$ . Thus, since  $|G/N|_2 \geq |N| = 2^{n-2}$ , we have  $|G : T|_2 = |G/N : T/N|_2 \geq 2^{n-2}$ , so  $|G : T| \geq 2^{n-2} = |N|$ , which is a contradiction.

Now we turn to the case  $n = 8$ . We have  $p = 2$  and  $m = 4, 5$  or  $6$ . In this case,  $A_8 \cong \text{GL}(4, 2)$  and  $2^6$  divides  $|A_8|$ . We look at each possibility for  $m$  in turn. If  $m = 6$ , then  $|N|_2 = |A_8|_2$ . For this case, the same argument as above holds since  $6 = 8 - 2$ , and we reach a contradiction.

Second, let  $m = 5$ . As above,  $|G : T| < |N| = 2^5$  and  $|T|/\vartheta(1) \in \text{cod}(G)$  such that  $2^5$  divides  $|T|/\vartheta(1)$ . Further,  $\|T/N|/\vartheta(1)|_2 \leq 2$  so  $|T/N|_2 \leq 4$  and  $|G/N : T/N|_2 \geq 16$ . Thus, 16 divides  $|G/N : T/N|$  and  $|G/N : T/N| < 32$ . However, we may check the index

of all subgroups of  $G/N \cong A_8$  using [6] and find that none of them satisfy these two properties.

Third, let  $m = 4$ . Then  $G/N \cong A_8 \cong \text{GL}(4, 2)$  and  $N = (\mathbb{Z}_2)^4$  so  $G$  is an extension of  $\text{GL}(4, 2)$  by  $N$ . We may computationally calculate the codegree set for any such group using MAGMA [5]. There are only four such nonisomorphic extensions and we find that none of them have the same codegree set as  $A_8$ . (The MAGMA code is available at <https://github.com/zachslonim/Characterizing-Alternating-Groups-by-Their-Codegrees>.) In every case,  $|N| = p^m$  produces a contradiction, so  $N = 1$  and  $G \cong A_n$ .

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