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WEIGHTED GENERALIZED HARDY INEQUALITIES FOR NONINCREASING FUNCTIONS

Dedicated to Professor P.G. Rooney in honour of his sixty-fifth birthday.

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ABSTRACT. The nonnegative weight function pairs u, v for which the operator $Tf(x) = \int_0^\infty a(t)f(xt)\,dt$ maps the nonnegative nonincreasing functions in $L^p(v)$ boundedly into weak $L^q(u)$ are characterized. This result is used, in particular, both to generalize and to provide an alternate proof of certain strong type inequalities recently obtained by Ariño and Muckenhoupt for the Hardy averaging operator restricted to nonnegative nonincreasing functions.

1. **Introduction.** Weighted inequalities for the Hardy averaging operator P and its adjoint P' defined for locally integrable functions f on $(0,\infty)$ by

$$Pf(x) = \frac{1}{x}\int_0^x f(t)\,dt, \quad P'f(x) = \int_x^\infty f(t)\frac{dt}{t}, \quad x > 0$$

have been widely studied. Given $1 \le p \le q < \infty$, the nonnegative weight pairs u, v for which these operators are bounded from $L^p(v)$ to $L^q(u)$ were characterized in [2]; a characterization of the corresponding weak type inequalities were also given there. Maźja [9] considered the range $1 \le q < p < \infty$.

The operators P, P', and their variants when restricted to nonnegative nonincreasing functions occur naturally in certain rearrangement inequalities for other operators satisfying appropriate weak type estimates, see [4,§ 3.5]. Thus, Ariño and Muckenhoupt [3] recently obtained boundedness results for the Hardy-Littlewood maximal operator in the classical Lorentz spaces $\Lambda_p(u)$ by characterizing, for $1 \le p < \infty$, the single weight functions u for which P restricted to nonnegative nonincreasing functions maps $L^p(u)$ boundedly into itself. A different characterization for nonincreasing u had been given by Boyd [5]. Sawyer [12] and Stepanov [13] have extended these results by characterizing, for various ranges of p and q the pairs (u, v) for which P restricted to nonnegative nonincreasing functions is bounded from $L^p(v)$ to $L^q(u)$. Neugebauer [10,11] obtained weighted estimates, including some weak type inequalities, for the restriction to monotone functions of P, P', and some of their variants.

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The operators P and P' are particular members of the family of operators T with nonnegative measurable kernel $a(t)$ given by

$$Tf(x) = \int_0^\infty a(t)f(xt)\,dt, \qquad x > 0.$$

The Stieltjes transform, the fractional integrals of Riemann-Liouville and Weyl, and the Laplace transform are further examples of operators that are either of this form, or are related in a simple way to an operator of this form.

The main purpose of this paper is to obtain, given $0 < p, q < \infty$, a simple characterization of the weight pairs u, v for which T restricted to nonnegative nonincreasing functions f satisfies the weak type inequality

$$\left(\int_{\{x>0 \mid Tf(x)>\lambda\}} u(x)\,dx\right)^{1/q} \leq \frac{C}{\lambda}\left(\int_0^\infty f(x)^p v(x)\,dx\right)^{1/p} \tag{1.1}$$

for some constant C independent of f and $\lambda > 0$.

Except for special choices of the kernel $a(t)$, a simple characterization of the corresponding strong type inequalities seems not to be known. It is a pleasant occurrence therefore to find that the weak type inequalities can be given a simple characterization for arbitrary kernels. These are not only of interest in their own right, but can often be used to derive strong type inequalities by interpolation; this approach is illustrated below in Theorems 2, 3 and 4.

The main result is the following Theorem. For convenience we use the notation $U(x) = \int_0^x u(t)\,dt$, $V(x) = \int_0^x v(t)\,dt$, $A(x) = \int_0^x a(t)\,dt$ for $x > 0$ and for $1 < p < \infty$, $p' = p/(p-1)$.

THEOREM 1. *Let $q > 0$ and suppose u, v are nonnegative weight functions defined on $(0, \infty)$. If $1 < p < \infty$, there is a constant C such that (1.1) holds for all nonnegative nonincreasing f if and only if there is a constant K such that*

$$U(r)^{1/q}\left\{\left(\int_0^\infty A(x/r)^{p'} V(x)^{-p'} v(x)\,dx\right)^{1/p'} + \left(\int_0^\infty a\right)\left(\int_0^\infty v\right)^{-1/p}\right\} \leq K \tag{1.2}$$

for all $r > 0$. The smallest constants C in (1.1) and K in (1.2) satisfy $[(p')^{1/p}p+1]^{-1}K \leq C \leq \max[1, (p'/p)^{1/p'}]K$. If $0 < p \leq 1$, there is a constant C such that (1.1) holds for all nonnegative, nonincreasing f if and only if there is a constant K such that

$$U(r)^{1/q} V(s)^{-1/p} A(s/r) \leq K \tag{1.3}$$

for all $0 < s, r < \infty$. The smallest constants C in (1.1) and K in (1.3) are equal.

As usual, in (1.2) and (1.3) products of the form $0 \cdot \infty$ are taken to be zero.

If $\phi(t) > 0$ and $\Phi(x) = \int_0^x \phi(t)\,dt$ satisfies $\Phi(x) < \infty$ for all $x > 0$ with $\lim_{x\to\infty} \Phi(x) = \infty$, then the generalized Hardy averaging operator P_ϕ and its adjoint P'_ϕ defined by

$$P_\phi f(x) = \frac{1}{\Phi(x)}\int_0^x f(t)\phi(t)\,dt, \quad P'_\phi f(x) = \int_x^\infty f(t)\phi(t)\,\frac{dt}{\Phi(t)}, \qquad x > 0$$

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are related to P and P' by $P_\phi f(x) = P[f \circ \Phi^{-1}](\Phi(x))$ and $P'_\phi f(x) = P'[f \circ \Phi^{-1}](\Phi(x))$. Since the operators P and P' are given by the kernels $a(t) = \chi_{(0,1]}(t)$, and $a(t) = t^{-1}\chi_{[1,\infty)}(t)$ respectively, the following corollaries for the case $\phi(t) = 1$ are immediate consequences of Theorem 1 and the general case follows from that by a change of variable.

COROLLARY 1. *The weak type inequality (1.1) for $T = P_\phi$ holds for all nonnegative nonincreasing functions f in the case $1 < p < \infty$ if and only if there is a constant K such that*

$$U(r)^{1/q}\left\{\left(\int_0^r\left(\frac{\Phi(x)}{\Phi(r)}\right)^{p'} V(x)^{-p'}v(x)\,dx\right)^{1/p'} + V(r)^{-1/p}\right\} \le K \tag{1.4}$$

for all $r > 0$ while for $0 < p \le 1$ it holds if and only if there is a constant K such that

$$U(r)^{1/q}V(s)^{-1/p}\frac{\Phi(s)}{\Phi(r)} \le K \tag{1.5}$$

for all $0 < s \le r < \infty$.

COROLLARY 2. *If $\int_0^\infty u \ne 0$ and $T = P'_\phi$, then the weak type inequality (1.1) holds for all nonnegative nonincreasing functions f in the case $1 < p < \infty$ if and only if $\int_0^\infty v(t)\,dt = \infty$ and there is a constant K such that*

$$U(r)^{1/q}\left(\int_r^\infty\left(\log\frac{\Phi(x)}{\Phi(r)}\right)^{p'} V(x)^{-p'}v(x)\,dx\right)^{1/p'} \le K \tag{1.6}$$

for all $r > 0$ while for $0 < p \le 1$ it holds if and only if there is a constant K such that

$$U(r)^{1/q}V(s)^{-1/p}\log\frac{\Phi(s)}{\Phi(r)} \le K \tag{1.7}$$

for all $0 < r \le s < \infty$.

Note that the conditions (1.2) and (1.3) depend on A rather than a. Thus, the weight pairs (u, v) satisfying (1.1) for an operator with kernel a_1 coincide with those satisfying (1.1) for an operator with kernel a_2 provided there are positive constants c_1, c_2 such that $c_1A_1(s) \le A_2(s) \le c_2A_1(s)$ for all $s > 0$. In particular, this is the case for the operator P, the Riemann-Liouville fractional integral operator R_α, $0 < \alpha < 1$, given by

$$R_\alpha f(x) = x^{-\alpha}\int_0^x (x-t)^{\alpha-1}f(t)\,dt$$

and the operator $\mathcal{L}$ associated with the Laplace Transform given by

$$\mathcal{L}f(x) = \int_0^\infty e^{-t/x}f(t)\,\frac{dt}{x}.$$

This may also be seen directly since elementary estimates show that $Pf(x) \le R_\alpha f(x) \le \alpha^{-1}Pf(x)$ and $e^{-1}Pf(x) \le \mathcal{L}f(x) \le Pf(x)$ for nonnegative nonincreasing f.

Some other features of the weight conditions may also be noted. If $\phi(t) = 1$, $q = p$ and $u(t) = v(t) = t^{\alpha-1}$, then (1.4) holds if and only if $0 < \alpha < p$ while (1.5) holds if and only if $0 < \alpha \leq p$. Thus the simpler condition (1.5) is not in general equivalent to (1.4). Hence also (1.3) is generally not equivalent to (1.2). Further, it may be noted that the weight condition (1.4) is closely related to that which characterizes certain weak type inequalities for the (unrestricted) Stieltjes transformation [1, Theorem 4]. Indeed, if $\int_0^\infty v(t)\,dt = \infty$ and $p \geqq 1$, then (1.4) with $\phi(t) = 1$ is a necessary and sufficient condition for the Stieltjes transform to map $L^p(w)$ boundedly into weak $L^q(u)$ where $w(x) = x^{-p}V(x)^p v(x)^{1-p}$. Note also that in Corollary 2, the requirement that $\int_0^\infty v(t)\,dt = \infty$ is implied by (1.7) so this is in fact a necessary condition for all $0 < p < \infty$.

Sawyer [12] obtained his strong type inequalities for P referred to above by considering the more general problem of characterizing the weight pairs (u, v) for which an operator of the form $Sf(x) = \int_0^\infty K(x,t)f(t)\,dt$ restricted to nonnegative nonincreasing functions maps $L^p(v)$ boundedly into $L^q(u)$. His main result asserts that this occurs if and only if a related operator $\tilde{S}$ satisfies a *dual* inequality with respect to a different, but related, pair of weight functions. However, except for very special kernels such as that of the operator P, this latter condition does not seem to reduce to simple conditions on the weight functions. Lai [7,8] generalized certain results of Neugebauer [10,11] to obtain conditions for the boundedness of certain operators of the form S restricted to nonnegative nonincreasing functions and has also given conditions which imply that the restricted operator is bounded if it is known to be of weak type. The operators considered there include some, but not all of those to be treated here.

As a first application of our weak type result, we will obtain strong type inequalities for a class of operators T for which $A(s)$ satisfies a submultiplicative condition. In particular, this class includes the operators P, R_α and $\mathcal{L}$.

THEOREM 2. *Let* $0 < p < \infty$ *and suppose* u *is a nonnegative weight function defined on* $(0,\infty)$. *Suppose the kernel* $a(t)$ *is integrable on* $(0,\infty)$ *and satisfies* $A(st) \leq BA(s)A(t)$ *for a constant* B *and all* $0 < s, t \leq 1$. *The following statements are equivalent.*

(a) There is a constant C depending on p, a and u such that

$$\int_0^\infty [Tf(x)]^p u(x)\,dx \leq C \int_0^\infty f(x)^p u(x)\,dx \tag{1.8}$$

for all nonnegative nonincreasing f.

(b) There is a constant K depending on p, a and u such that

$$\int_r^\infty A(r/x)^p u(x)\,dx \leq K\,U(r) \tag{1.9}$$

for all $r > 0$.

(c) There is $0 < \gamma < 1$ *and a constant* K *depending on* p, γ, *a and u such that*

$$A(s)^{\gamma p} U(t) \leq K\,U(st) \tag{1.10}$$

for all $0 < s \leq 1$ *and* $t > 0$.

(d) There is $0 < p_1 < p$ *so that for all* $q > p_1$ *there is a constant C depending on q, a and u such that*

$$\int_{\{x>0 \mid Tf(x)>\lambda\}} u(x)\,dx \le \frac{C}{\lambda^q}\int_0^\infty f(x)^q u(x)\,dx \tag{1.11}$$

for all nonnegative nonincreasing f and all $\lambda > 0$.

Parts of Theorem 2 are known in the special case that $a(t)$ is nonincreasing. In that case, the nonincreasing equimeasurable rearrangement $(Tf)^*$ of $|Tf|$ does not exceed Tf^*, so (a) is equivalent to the statement that T is bounded on the classical Lorentz space $\Lambda_p(u)$. Boyd [5; Theorems 3.1 and 4.1] has shown that if u is nonincreasing and $1 \le p < \infty$, a sufficient condition for this boundedness is that $\int_0^\infty a(s)h(s)\,ds < \infty$ where $h(s) = [\sup_{t>0} U(t)/U(st)]^{1/p}$. Hence, if (1.10) holds, then

$$\int_0^1 a(s)h(s)\,ds \le K^{1/p}\int_0^1 a(s)A(s)^{-\gamma}\,ds = K^{1/p}(1-\gamma)^{-1}A(1)^{1-\gamma} < \infty$$

so it follows from Boyd's result that (c) implies (a) in this case.

The condition (b) was introduced by Braverman [6, Theorem 3] who proved the equivalence of (a) and (b) for $1 \le p < \infty$ for kernels $a(t)$ supported and nonincreasing on $(0,1)$ satisfying $a(st) \le Ba(s)a(t)$ for all $0 < s, t, < 1$. Since in this case we have

$$\begin{aligned} A(st) &= \int_0^s a(tx)t\,dx = \int_0^s\int_0^t a(tx)\,dy\,dx \le \int_0^s\int_0^t a(yx)\,dy\,dx \\ &\le B\int_0^s\int_0^t a(y)a(x)\,dy\,dx = BA(s)A(t) \end{aligned}$$

for all $0 < s, t \le 1$, it follows that these kernels satisfy the hypothesis of Theorem 2.

For some operators of the form T, the equivalence of (a) and (b) for $1 \le p < \infty$ and that of (a) and (d) for $1 < p < \infty$ may be deduced from results of Lai, [7, Theorems 3.1 and 3.2] and [8, Theorem 5.3].

On the other hand, for $0 < \alpha < \beta < 1$ the kernel $a(t) = \chi_{(\alpha,\beta)}(t)$ satisfies the hypothesis of Theorem 2 with $B = (\beta-\alpha)^{-1}$, but satisfies neither Braverman's hypothesis nor those of Lai.

As a second application of the weak type inequalities, we will prove the following strong type results for P_ϕ and P'_ϕ.

THEOREM 3. *Let* $0 < p < \infty$ *and suppose u is a nonnegative weight function defined on* $(0,\infty)$. *The following statements are equivalent.*

(a) There is a constant C depending on p, ϕ *and u such that*

$$\int_0^\infty [P_\phi f(x)]^p u(x)\,dx \le C\int_0^\infty f(x)^p u(x)\,dx \tag{1.12}$$

for all nonnegative nonincreasing f.

(b) There is a constant K depending on p, ϕ *and u such that*

$$\int_r^\infty \frac{u(x)}{\Phi(x)^p}\,dx \le K\Phi(r)^{-p}U(r) \tag{1.13}$$

for all $r > 0$.
(c) There is $0 < \gamma < 1$ *and a constant K depending on p,* γ*,* ϕ *and u such that*

$$\left(\frac{\Phi(st)}{\Phi(t)}\right)^{\gamma p} U(t) \leq K\, U(st) \tag{1.14}$$

for all $0 < s \leq 1$ *and* $t > 0$.
(d) There is $0 < p_1 < p$ *so that for all* $q > p_1$ *there is a constant C depending on q, a and u such that*

$$\int_{\{x>0 \mid P_\phi f(x) > \lambda\}} u(x)\, dx \leq \frac{C}{\lambda^q} \int_0^\infty f(x)^q u(x)\, dx \tag{1.15}$$

for all nonnegative nonincreasing f and all $\lambda > 0$.

For $1 \leq p < \infty$, the equivalence of (a) and (b) was proved by Ariño and Muckenhoupt [3, Theorem 1.7] for the case $\phi(t) = 1$ and by Neugebauer [11, Theorem 2.3] for the case $\phi(t) = t^{-\alpha}$, $0 \leq \alpha < 1$. Boyd [5; Theorem 4.1 and Lemma 3.6] proved the equivalence of (a) and (c) for $\phi(t) = 1$, $1 \leq p < \infty$, under the additional assumption that u is nonincreasing. Neugebauer [10, Theorem 7.2] proved the equivalence of (a) and (d) for $\phi(t) = 1$, $1 < p < \infty$. The equivalence of (a) and (b) for $1 \leq p < \infty$ and that of (a) and (d) for $1 < p < \infty$ may be deduced from results of Lai [7, Theorems 3.1 and 3.2] and [8, Theorem 5.3].

In view of our earlier remarks, the analogue of Theorem 3 for R_α or $\mathcal{L}$ may be obtained by setting $\phi(t) = 1$ and replacing P_ϕ throughout by R_α or $\mathcal{L}$.

The analogue of Theorem 3 for P'_ϕ is given by the following theorem which shows that when $v = u$ and $q = p$, not only do the weight functions for the weak type and the strong type inequalities coincide, but the weight classes are independent of $p > 0$.

THEOREM 4. *Suppose u is a nonnegative weight function defined on* $(0, \infty)$. *If for some* $0 < p < \infty$ *one of the following statements holds, then they all hold for all* $0 < p < \infty$.
(a) There is a constant K depending on p, ϕ *and u such that*

$$\int_0^r \left(\log \frac{\Phi(r)}{\Phi(x)}\right)^p u(x)\, dx \leq K U(r) \tag{1.16}$$

for all $r > 0$.
(b) There is a constant K depending on p, ϕ *and u such that*

$$U(r)^{1/p} U(s)^{-1/p} \log \frac{\Phi(s)}{\Phi(r)} \leq K \tag{1.17}$$

for all $0 < r \leq s < \infty$.
(c) There is a constant C depending on p, ϕ *and u such that*

$$\int_{\{x>0 \mid P'_\phi f(x) > \lambda\}} u(x)\, dx \leq \frac{C}{\lambda^p} \int_0^\infty f(x)^p u(x)\, dx \tag{1.18}$$

for all nonegative nonincreasing functions f and all $\lambda > 0$.
(d) There is a constant C depending on p, ϕ *and u such that*

$$\int_0^\infty [P'_\phi f]^p u(x)\,dx \le C \int_0^\infty f(x)^p u(x)\,dx \tag{1.19}$$

for all nonnegative nonincreasing functions f.

For $\phi(t) = 1$, Neugebauer [11, Theorem 3.3] proved that (1.19) holds for $1 \le p < \infty$ if and only if (1.16) holds for $p = 1$. For related results, see Lai [7,8].

The proof of Theorem 1 is given in Sections two and three; Theorems 2 and 3 are proved in Section four and Theorem 4 is proved in Section five.

2. **Proof of Theorem 1 (Necessity).** Consider first the case $p > 1$. If $\int_0^x v(t)\,dt = \infty$ for all $x > 0$ then the second factor in (1.2) is zero so (1.2) holds. Thus we may assume that $\int_0^{x_0} v(t)\,dt < \infty$ for some $x_0 > 0$. We further assume, temporarily, that $\int_0^\infty v(t)\,dt = \infty$ and that the kernel a is integrable with compact support in $(0,\infty)$. Fix $r > 0$ and set

$$f(t) = \left(\int_t^\infty A(y/r)^{p'} V(y)^{-p'-1} v(y)\,dy\right)^{1/p}.$$

Then $f(t) \ge 0$ is nonincreasing and Fubini's Theorem shows that

$$\int_0^\infty f(t)^p v(t)\,dt = \int_0^\infty A(y/r)^{p'} V(y)^{-p'} v(y)\,dy. \tag{2.1}$$

Note that our temporary assumptions about v and a ensure that (2.1) is finite. On the other hand, since $A(y/r)$ is nondecreasing in y

$$\begin{aligned} Tf(r) = \int_0^\infty a(t) f(rt)\,dt &\ge \int_0^\infty a(t) A(t)^{p'/p} \left(\int_{rt}^\infty V(y)^{-p'-1} v(y)\,dy\right)^{1/p} dt \\ &= (p')^{-1/p} \int_0^\infty a(t) A(t)^{p'/p} V(rt)^{-p'/p}\,dt \end{aligned}$$

and upon expressing $V(rt)^{-p'/p}$ as

$$\frac{p'}{p}\int_{rt}^\infty V(y)^{-p'} v(y)\,dy$$

and applying Fubini's Theorem, it follows that

$$\begin{aligned} Tf(r) &\ge (p')^{1/p'} p^{-1} \int_0^\infty V(y)^{-p'} v(y) \left(\int_0^{y/r} a(t) A(t)^{p'/p}\,dt\right) dy \\ &= (p')^{-1/p} p^{-1} \int_0^\infty A(y/r)^{p'} V(y)^{-p'} v(y)\,dy. \end{aligned} \tag{2.2}$$

Thus, with

$$\lambda = (1+\delta)^{-1} (p')^{-1/p} p^{-1} \int_0^\infty A(y/r)^{p'} V(y)^{-p'} v(y)\,dy$$

in (1.1), (2.1) and (2.2) yield (1.2) with $K = (p')^{1/p} pC$ since $\delta > 0$ is arbitrary. This completes the proof for v and a satisfying our temporary assumptions.

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We remove the assumption on v first. Note that if $\int_0^\infty v(t)\,dt < \infty$, (1.1) with $f(t) = 1$ shows that

$$\left(\int_0^\infty u(x)\,dx\right)^{1/q}\left(\int_0^\infty a(t)\,dt\right) \leq C\left(\int_0^\infty v(t)\,dt\right)^{1/p}. \tag{2.3}$$

Then, since (1.1) holds with the same constant C if v is replaced by $v_\epsilon(x) = v(x) + \epsilon$, $\epsilon > 0$, and since v_ϵ satisfies $\int_0^\infty v_\epsilon = \infty$, the above argument and (2.3) shows that

$$U(r)^{1/q}\left\{\left(\int_0^\infty A(x/r)^{p'}\left(\int_0^x v_\epsilon\right)^{-p'} v_\epsilon(x)\,dx\right)^{1/p'} + \left(\int_0^\infty a\right)\left(\int_0^\infty v\right)^{-1/p}\right\} \tag{2.4}$$

does not exceed $[(p')^{1/p}p + 1]C$. Since (2.4) is not increased if the term $v_\epsilon(x)$ is replaced by $v(x)$, (1.2) follows from the monotone convergence theorem upon letting $\epsilon \longrightarrow 0$.

The assumption on a may now be removed by applying the above argument to the operator T_n with kernel $a_n(t) = \min[a(t), n]\chi_{[1/n,n]}(t)$ and then letting $n \longrightarrow \infty$. This completes the proof for $p > 1$.

Suppose now that $0 < p \leqq 1$ and fix $r, s > 0$. If $V(s) = \infty$ or $A(s/r) = 0$ then (1.3) holds by convention, so we assume $V(s) < \infty$ and $A(s/r) > 0$. Set $f(t) = \chi_{(0,s)}(t)$. Then $Tf(x) \geq Tf(r) = A(s/r)$ for $x \in (0, r)$. Hence (1.1) shows that $U(r) = 0$ if $A(s/r) = \infty$ and

$$U(r)^{1/q} \leq \frac{C(1+\delta)}{A(s/r)}\left(\int_0^\infty f(t)^p v(t)\,dt\right)^{1/p} = \frac{C(1+\delta)}{A(s/r)}V(s)^{1/p}$$

for $\delta > 0$ if $A(s/r) < \infty$. Since $\delta > 0$ is arbitrary, it follows that (1.3) holds with $K = C$. This completes the proof for $0 < p \leq 1$.

3. **Proof of Theorem 1 (Sufficiency).** The monotone convergence theorem shows that we need only prove (1.1) for bounded nonnegative nonincreasing f with $f(t) = 0$ for large t. Further, since $Tf(x)$ is nonincreasing, it suffices to show that

$$U(r)^{1/q} \leq \frac{C}{\lambda}\left(\int_0^\infty f(t)^p v(t)\,dt\right)^{1/p} \tag{3.1}$$

for all r with $(0, r] \subset \{\, x > 0 \mid Tf(x) > \lambda \,\}$. Thus, we may fix $r > 0$ with $U(r) > 0$ and $Tf(r) > \lambda$.

Suppose $1 < p < \infty$ and (1.2) holds. We may assume that $\int_0^\infty a \neq 0$ for otherwise $Tf = 0$. Set $t_0 = \inf\{\, t > 0 \mid A(t) > 0 \,\}$. Then (1.2) shows that

$$\left(\int_0^\infty a\right)^{p'-1}\left(\int_0^\infty v\right)^{-p'/p} < \infty$$

and

$$\int_{rt}^\infty A(y/r)^{p'-1}V(y)^{-p'}v(y)\,dy < \infty, \qquad t > t_0.$$

Hence

$$h(t) = \left\{\int_{rt}^\infty A(y/r)^{p'-1}V(y)^{-p'}v(y)\,dy + (p/p')\left(\int_0^\infty a\right)^{p'-1}\left(\int_0^\infty v\right)^{-p'/p}\right\}^{1/p'}$$

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is nonincreasing and finite for all $t > t_0$. Fubini's Theorem shows

$$\begin{aligned}\int_{t_0}^{\infty} a(t)h(t)^{p'}\,dt &= \int_{t_0}^{\infty} a(t)\Big(\int_{rt}^{\infty} A(y/r)^{p'/p}V(y)^{-p'}v(y)\,dy\Big)\,dt \\ &\quad + (p/p')\Big(\int_0^\infty a\Big)^{p'}\Big(\int_0^\infty v\Big)^{-p'/p} \\ &= \int_0^\infty A(y/r)^{p'}V(y)^{-p'}v(y)\,dy \\ &\quad + (p/p')\Big(\int_0^\infty a\Big)^{p'}\Big(\int_0^\infty v\Big)^{-p'/p} \\ &\leq \max[1,p/p']\Big\{\Big(\int_0^\infty A(y/r)^{p'}V(y)^{-p'}v(y)\,dy\Big)^{1/p'} \\ &\quad + \Big(\int_0^\infty a\Big)\Big(\int_0^\infty v\Big)^{-1/p}\Big\}^{p'}\end{aligned}$$

and hence

$$\Big(\int_{t_0}^{\infty} a(t)h(t)^{p'}\,dt\Big)^{1/p'} \leqq \max[1,(p/p')^{1/p'}]KU(r)^{-1/q} \tag{3.2}$$

by (1.2). Since A is nondecreasing, for $t > t_0$ we have

$$\begin{aligned}h(t)^{-p} &\leq A(t)^{-1}\Big\{\int_{rt}^{\infty} V(y)^{-p'}v(y)\,dy + (p/p')\Big(\int_0^\infty v\Big)^{-p'/p}\Big\}^{-p/p'} \\ &= (p'/p)^{p/p'}A(t)^{-1}V(rt).\end{aligned} \tag{3.3}$$

Now, integrating by parts and discarding negative terms we have

$$\int_{t_0}^{\infty} a(t)f(rt)^p h(t)^{-p}\,dt \leq -\int_{t_0}^{\infty} A(t)h(t)^{-p}\,df(rt)^p$$

and then (3.3) shows that this does not exceed

$$\begin{aligned}-(p'/p)^{p/p'}\int_{t_0}^{\infty} V(rt)\,df(rt)^p &\leq -(p'/p)^{p/p'}\int_0^\infty V(rt)\,df(rt)^p \\ &= (p'/p)^{p/p'}\int_0^\infty f(t)^p v(t)\,dt.\end{aligned}$$

Hence

$$\Big(\int_{t_0}^{\infty} a(t)f(rt)^p h(t)^{-p}\,dt\Big)^{1/p} \leq (p'/p)^{1/p'}\Big(\int_0^\infty f(t)^p v(t)\,dt\Big)^{1/p}. \tag{3.4}$$

Hölder's inequality, (3.2) and (3.4) then show

$$Tf(r) \leq \max[1,(p'/p)^{1/p'}]KU(r)^{-1/q}\Big(\int_0^\infty f(t)^p v(t)\,dy\Big)^{1/p}.$$

Using this estimate in

$$U(r)^{1/q} \leq U(r)^{1/q}\frac{Tf(r)}{\lambda}$$

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yields (3.1) with $C \le \max[1, (p'/p)^{1/p'}]K$. This completes the proof for $p > 1$.

Now suppose $0 < p \le 1$ and (1.3) holds. Integration by parts and (1.3) show that

$$Tf(r) = \int_0^\infty a(t)f(rt)\,dt \le -\int_0^\infty A(t)\,df(rt)$$
$$\le KU(r)^{-1/q}\int_0^\infty \left(\int_0^{rt} v(y)\,dy\right)^{1/p}[-df(rt)]$$

and Minkowski's inequality for integrals shows that this does not exceed

$$KU(r)^{-1/q}\left(\int_0^\infty f(y)^p v(y)\,dy\right)^{1/p}.$$

Thus

$$U(r)^{1/q} \le U(r)^{1/q}\frac{Tf(r)}{\lambda} \le \frac{K}{\lambda}\left(\int_0^\infty f(y)^p v(y)\,dy\right)^{1/p}$$

so we have (3.1) with $C = K$.

This completes the proof.

4. **Proofs of Theorems 2 and 3.** The proofs of Theorems 2 and 3 require the following lemma, the first part of which was proved for the particular case $H(r,t) = r/t$, $1 \le p < \infty$, in [3, Lemma 2.1]; a different proof was given in [11, Theorem 2.4] for $0 < p < \infty$. The proof given here is quite different than either of these and seems to be more direct.

LEMMA 1. *Suppose $H(r,t) \ge 0$ is defined and measurable on $\{(r,t) \mid t \ge r > 0\}$ and satisfies:*

(i) $H(t,t) \le H_1$ for a constant H_1 and all $t > 0$,

(ii) $H(r,t)$ is nonincreasing in t for $t \ge r$,

(iii) $H(r,t) \le H_2 H(r,s)H(s,t)$ for a constant H_2 and all $0 < r \le s \le t$.

If $0 < p < \infty$ and u is a nonnegative weight function defined on $(0,\infty)$, then the following statements are equivalent.

(a) There is a constant K depending on p, H and u such that

$$\int_r^\infty H(r,t)^p u(t)\,dt \le K\,U(r) \tag{4.1}$$

for all $r > 0$.

(b) There is $\delta > 0$ such that for all $p_1 > p - \delta$ there is a constant K depending on p_1, H and u such that

$$\int_r^\infty H(r,t)^{p_1} u(t)\,dt \le K\,U(r) \tag{4.2}$$

for all $r > 0$.

(c) There is $0 < \gamma < 1$ and a constant K depending on p, γ, H and u such that

$$H(r,t)^{\gamma p} U(t) \le K\,U(r) \tag{4.3}$$

for all $t > r > 0$.

PROOF OF LEMMA 1. We begin by showing that (a) implies (b). Suppose (a) holds. Then, if $p_1 > p$, since (i) and (ii) show that $H(r,t) \leq H_1$ for $t \geq r$, it follows from (4.1) that

$$\int_r^\infty H(r,t)^{p_1} u(t)\,dt \leq H_1^{p_1-p} K\, U(r)$$

for all $r > 0$ so (4.2) holds for $p_1 > p$. It remains to show that (4.2) holds for some $p_1 < p$. Fix $r > 0$ with $U(r) < \infty$. Then for $0 < \delta < p$ (4.1) and (ii) show that $\int_r^R H(r,x)^{p-\delta} u(x)\,dx < \infty$ for all $R > r$ with $H(r,R) > 0$. Integration by parts shows

$$\begin{aligned}\int_r^R & H(r,x)^{p-\delta} u(x)\,dx \\ &= H(r,r)^{-\delta} \int_r^R H(r,t)^p u(t)\,dt \\ &\qquad + \int_r^R \left(\int_x^R H(r,t)^p u(t)\,dt \right) dH(r,x)^{-\delta}\end{aligned}$$

and in view of (iii), after replacing $H(r,t)$ in the inner integral by $H_2\, H(r,x)H(x,t)$, (4.1) shows that this is bounded by

$$K \left\{ H(r,r)^{-\delta}\, U(r) + H_2^p \int_r^R U(x) H(r,x)^p \, dH(r,x)^{-\delta} \right\}$$

and a further integration by parts in the last term shows that this does not exceed

$$K \left\{ \left(1 + \frac{\delta H_2^p H(r,r)^p}{p-\delta} \right) H(r,r)^{-\delta}\, U(r) + \frac{H_2^p \delta}{p-\delta} \int_r^R H(r,x)^{p-\delta} u(x)\,dx \right\}.$$

Hence (i) and (iii) yield

$$\begin{aligned}\int_r^R & H(r,x)^{p-\delta} u(x)\,dx \\ &\leq K \left\{ \left(1 + \frac{\delta H_2^p H_1^p}{p-\delta} \right) H_2^\delta\, U(r) + \frac{H_2^p \delta}{p-\delta} \int_r^R H(r,x)^{p-\delta} u(x)\,dx \right\}\end{aligned}$$

since $H(r,r) \geq H(r,R) > 0$ and (iii) show that $H(r,r) \geqq H_2^{-1}$. Transposing the last term and letting $R \longrightarrow \infty$ yields (4.2) with $p_1 = p - \delta$ provided $0 < \delta < p/(1 + H_2^p K)$. This completes the proof that (a) implies (b).

Now for $t > r > 0$, (ii) and (i) show that

$$\begin{aligned} H(r,t)^{p_1} U(t) &\leq H(r,t)^{p_1} U(r) + H(r,t)^{p_1} \int_r^t u(x)\,dx \\ &\leq H(r,r)^{p_1} U(r) + \int_r^t H(r,x)^{p_1} u(x)\,dx. \end{aligned}$$

Thus (i) and (4.2) yield

$$H(r,t)^{p_1} U(t) \leq (H_1^{p_1} + K) U(r)$$

so (b) implies (c).

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Finally, if (c) holds, then

$$\int_r^\infty H(r,t)^p u(t)\,dt \le [KU(r)]^{1/\gamma} \int_r^\infty U(t)^{-1/\gamma} u(t)\,dt \le K^{1/\gamma} \frac{\gamma}{1-\gamma} U(r)$$

so (a) holds.

This completes the proof of the lemma.

PROOF OF THEOREM 2. Note first that $H(r,t) = A(r/t)$, $t \ge r > 0$, satisfies the hypothesis of Lemma 1 with $H_1 = A(1)$ and $H_2 = B$.

Now, if (a) holds and $r > 0$, we obtain (b) by choosing $f(t) = \chi_{(0,r]}(t)$ and reducing the range of integration on the left of (1.8) to (r, ∞).

Lemma 1 shows that (b) and (c) are equivalent.

Suppose (c) holds. We shall show that (d) holds with $p_1 = \gamma p$. Suppose $q > \max[1, \gamma p]$. Then

$$\int_0^r A(x/r)^{q'} U(x)^{-q'} u(x)\,dx \le [KU(r)^{-1}]^{q'/(\gamma p)} \int_0^r U(x)^{(-1+1/\gamma p)q'} u(x)\,dx = K^{q'/(\gamma p)} \gamma p(q-1)/(q-\gamma p) U(r)^{1-q'}$$

so

$$U(r)^{1/q} \left\{ \left(\int_0^\infty A(x/r)^{q'} U(x)^{-q'} u(x)\,dx \right)^{1/q'} + \left(\int_0^\infty a \right) \left(\int_0^\infty u \right)^{-1/q} \right\}$$

is bounded by

$$\left(K^{q'/(\gamma p)} \gamma p(q-1)/(q-\gamma p) + \left(\int_0^\infty a \right)^{q'} / (q'-1) \right)^{1/q'} + \int_0^\infty a.$$

Hence Theorem 1 shows that T satisfies (d) in this case. On the other hand, if $\gamma p < q \le 1$, Lemma 1 shows that (1.10) holds with γp replaced by q. Hence Theorem 1 yields (d) in this case also. Thus, (c) implies (d).

Finally, if (d) holds, then (a) holds by an application of the Marcinkiewicz interpolation theorem.

This completes the proof of Theorem 2.

PROOF OF THEOREM 3. The proof is entirely similar to that of Theorem 2 except that now in the proof that (b) implies (c), Lemma 1 is applied to the function $H(r,t) = \Phi(r)/\Phi(t)$, $t \ge r > 0$, and in the proof that (c) implies (d) an appeal is made to Corollary 1 rather than to Theorem 1. The details are omitted.

5. **Proof of Theorem 4.**

The main ingredient of the proof is the following lemma.

LEMMA 2. *If for some* $0 < p < \infty$ *(1.16) holds, then (1.16) holds for all* p *with* $0 < p < \infty$.

PROOF OF LEMMA 2. Suppose (1.16) holds for $p = p_0 > 0$. Then for $0 < p < p_0$

$$\begin{aligned}\int_0^r \left(\log \frac{\Phi(r)}{\Phi(x)}\right)^p u(x)\,dx &\le \int_0^r \left[1 + \left(\log \frac{\Phi(r)}{\Phi(x)}\right)^{p_0}\right] u(x)\,dx \\ &\le (C+1)\int_0^r u(x)\,dx\end{aligned} \tag{5.1}$$

so (1.16) holds for $0 < p < p_0$. Now, let $0 < \delta < \min[1, p_0]$. For $0 < x < r$ a change of variable shows that

$$\int_x^r \left(\log \frac{\Phi(r)}{\Phi(t)}\right)^{\delta-1} \left(\log \frac{\Phi(t)}{\Phi(x)}\right)^{p_0} \phi(t)\,\frac{dt}{\Phi(t)} = B(\delta, p_0+1)\left(\log \frac{\Phi(r)}{\Phi(x)}\right)^{p_0+\delta}$$

where the Beta function $B(\delta, p_0+1) = \Gamma(\delta)\Gamma(p_0+1)/\Gamma(\delta + p_0 + 1)$. Hence Fubini's Theorem and (1.16) show that

$$\begin{aligned}&\int_0^r \left(\log \frac{\Phi(r)}{\Phi(x)}\right)^{p_0+\delta} u(x)\,dx \\ &\quad = B(\delta, p_0+1)^{-1} \int_0^r \left(\log \frac{\Phi(r)}{\Phi(t)}\right)^{\delta-1} \left(\int_0^t \left(\log \frac{\Phi(t)}{\Phi(x)}\right)^{p_0} u(x)\,dx\right) \phi(t)\,\frac{dt}{\Phi(t)} \\ &\quad \le B(\delta, p_0+1)^{-1} C \int_0^r \left(\log \frac{\Phi(r)}{\Phi(t)}\right)^{\delta-1} \left(\int_0^t u(x)\,dx\right) \phi(t)\,\frac{dt}{\Phi(t)} \\ &\quad = [\delta B(\delta, p_0+1)]^{-1} C \int_0^r \left(\log \frac{\Phi(r)}{\Phi(x)}\right)^{\delta} u(x)\,dx \\ &\quad \le [\delta B(\delta, p_0+1)]^{-1} C(C+1) \int_0^r u(x)\,dx\end{aligned}$$

where we have used (5.1) with $p = \delta$ to obtain the last inequality. Thus (1.16) holds for $p = p_0 + \delta$. Iterating this argument shows that (1.16) holds for all $p > p_0$ and completes the proof of the lemma.

PROOF OF THEOREM 4. We first prove the equivalence of (a) and (b). Fix $p > 0$ and suppose (1.16) holds. Then for $0 < r < s$

$$\begin{aligned}\left(\log \frac{\Phi(s)}{\Phi(r)}\right)^p \int_0^r u(x)\,dx &\le \int_0^r \left(\log \frac{\Phi(s)}{\Phi(x)}\right)^p u(x)\,dx \\ &\le \int_0^s \left(\log \frac{\Phi(s)}{\Phi(x)}\right)^p u(x)\,dx \\ &\le C\int_0^s u(x)\,dx\end{aligned}$$

so (1.17) holds. Conversely, if $0 < \epsilon < p$, (1.17) shows

$$\begin{aligned}\int_0^r \left(\log \frac{\Phi(r)}{\Phi(x)}\right)^{p-\epsilon} u(x)\,dx &\le C^{(p-\epsilon)/p} \int_0^r [U(r)/U(x)]^{(p-\epsilon)/p} u(x)\,dx \\ &= (p/\epsilon) C^{(p-\epsilon)/p} U(r)\end{aligned}$$

so (1.16) holds for $p-\epsilon$, and hence by Lemma 2, also for p. Thus (a) holds if and only if (b) holds.

Now suppose (b) holds. If $0 < p \leq 1$, (1.17) and Corollary 2 show that (c) holds. If $1 < p < \infty$, then since (1.17) also holds for $p_1 > p$,

$$\int_r^\infty \left(\log \frac{\Phi(t)}{\Phi(r)}\right)^{p'} U(t)^{-p'}u(t)\,dt \leq C^{p'/p_1} \int_r^\infty [U(t)/U(r)]^{p'/p_1} U(t)^{-p'}u(t)\,dt$$
$$= (1-p'/p_1')^{-1}C^{p'/p_1}U(r)^{-p'/p}$$

so Corollary 2 shows that (c) holds in this case also. This completes the proof that (b) implies (c). Conversely, if $1 < p < \infty$, $0 < t < r$, and (c) holds, then Corollary 2 shows $\int_0^\infty u(x)\,dx = \infty$ and hence

$$\begin{aligned}\left(\log \frac{\Phi(r)}{\Phi(t)}\right)^{p} U(t) &= \left(\frac{p'}{p}\int_r^\infty \left(\log \frac{\Phi(r)}{\Phi(t)}\right)^{p'} U(x)^{-p'}u(x)\,dx\right)^{p/p'} U(r)U(t)\\ &\leq \left(\frac{p'}{p}\int_r^\infty \left(\log \frac{\Phi(x)}{\Phi(t)}\right)^{p'} U(x)^{-p'}u(x)\,dx\right)^{p/p'} U(r)U(t)\\ &\leq \left(\frac{p'}{p}\int_t^\infty \left(\log \frac{\Phi(x)}{\Phi(t)}\right)^{p'} U(x)^{-p'}u(x)\,dx\right)^{p/p'} U(t)U(r)\\ &\leq (p'/p)^{p/p'}K^pU(r)\end{aligned}$$

again by Corollary 2, so (1.17) holds. Since for $0 < p \leq 1$ (1.17) and (1.7) with $v = u$, $q = p$ coincide, this completes the proof that (c) implies (b).

Finally, since we have shown that (a) and (c) are equivalent, if (c) holds then by Lemma 2 it also holds for all $p > 0$ and hence the Marcinkiewicz Theorem shows that (d) holds. Since (d) always implies (c), the proof is complete.

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