GENERAL VALUE DISTRIBUTION THEORY*

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We shall introduce the main theorems of value distribution theory in the most general case of complex dimension one: analytic mappings of arbitrary Riemann surfaces into arbitrary Riemann surfaces. The case of mappings of arbitrary Riemann surfaces into closed Riemann surfaces was discussed in [41]. Earlier literature on analytic mappings is listed in the Bibliography.

An implicit form of the second main theorem was reported in the research announcement [38]. The essence of the present paper is that the second main theorem can be given an explicit expression, even in the most general case. This will enable us to establish the *affininity relation* (10),

$$\sum \alpha(a) + \sum \beta(a) + \sum \gamma(a) \leqq 2 + \eta.$$

As special cases, this relation contains the generalized defect relation, the generalized ramification relation, and a relation pertaining to the coverage of the zeros of the metric.

1. Program. The basic tool in value distribution theory is the proximity function, that is, a function that describes the nearness of a generic point ζ on the range surface to a given point a. This function is required to have two properties: it must tend logarithmically to infinity as ζ tends to a, and it must remain positive or at least uniformly bounded below no matter how ζ and a roam on the surface. In the classical case of the plane such functions are immediately available, e.g., $\overset{+}{\log}|\zeta - a|^{-1}$ and $\log[\zeta, a]^{-1}$. In contrast, on an abstract Riemann surface the construction of a proximity function turns into an essential part of the theory.

The basic idea of our approach is as follows. On an arbitrary Riemann surface S take two points ζ_0 and ζ_1. Let $t_0 = t(\zeta, \zeta_0, \zeta_1)$ be a harmonic function

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with positive and negative singularities at ζ_0 and ζ_1 respectively, and, in a sense to be specified, constant values on the ideal boundary. The function $s_0 = \log(1+e^{t_0})$ continues to have a positive logarithmic pole at ζ_0 but it is bounded below by zero on S. It qualifies as a proximity function to ζ_0. For any other point a we could proceed in the same manner but we wish the resulting proximity function to have the same Laplacian as s_0, so that we may effectively make use of Stokes' formula. This we accomplish by adding to s_0 the harmonic function $t = t(\zeta, a, \zeta_0)$. The singularities at ζ_0 then cancel, and the sum $s = s_0 + t$ has as singularity only the positive logarithmic pole at a. The problem is to show that $s \geq O(1)$ uniformly for all ζ and a.

It will also be important that s has a symmetry property: $s(a, b) = s(b, a)$. This will be achieved by so normalizing t at ζ_0 that $t(\zeta) - 2\log|\zeta - \zeta_0| \to s_0(a)$ as $\zeta \to \zeta_0$.

Once the proximity function has been constructed, the main theorems can be established largely in the same way as for mappings into closed surfaces [41]. There is one significant difference, however. The conformal metric we use has only a finite number of zeros on a closed surface and these zeros could in [41] be conveniently added to the points $a_1, \ldots, a_q$ in the second main theorem. In the general case there are infinitely many zeros of our metric and they bring a new aspect to the second main theorem and its consequences.

That there actually exist nontrivial mappings into surfaces of infinite genus is clear. The projection mapping between two suitable covering surfaces is a simple example.

2. A lemma on harmonic functions. We start with a well known property of harmonic functions. Let $\overline{W}_0$ be a Riemann surface. It can be arbitrary, but for our purposes it suffices to consider a bordered compact surface with border α_0. In the interior W_0 of $\overline{W}_0$ we consider a compact set α_1. Again it can be arbitrary but it suffices to take a finite set of analytic Jordan curves.

LEMMA 1. *Let u be a harmonic function on $\overline{W}_0$ with* $\operatorname{sgn} u|\alpha_1 \neq \text{const.}$ *Then there exists a constant $q \in (0, 1)$, independent of u, such that*

$$|u||\alpha_1 \leqq q \max_{\alpha_0} |u|.$$

The proof is immediate. If the maximum is equal to zero, there is nothing to prove. In other cases we normalize by a multiplicative constant so as to make the maximum equal to one. We claim that $|u| \alpha_1 \leqq q < 1$. If not, there is a sequence of functions u_n which at some points ζ_n of α_1 tend to one. The family is normal, and a limiting harmonic function has absolute value one at some accumulation point of the ζ_n. This violates the maximum principle, and the lemma is proved.

Consider now a real-valued function $\varphi \in C$ on α_0. The solution u_0 of the Dirichlet problem in W_0 is obtained from φ by a linear operator, $u_0 = L_0\varphi$, and obviously the flux $\int_{\alpha_0} d{*}L_0\varphi = 0$. If $\operatorname{sgn} u_0 | \alpha_1 \neq \text{const}$, then u_0 satisfies our inequality. We now intersect W_0 by another compact surface $\overline{W}_1$ bordered by $\alpha_1 \subset W_0$ and β_1, say, with $\beta_1 \cap \overline{W}_0 = \emptyset$. Given a continuous ψ on α_1 let u_1 be the harmonic function on W_1 with $u_1 | \alpha_1 = \psi$ and $u_1 | \beta_1 = c_1$ (const.) such that $\int_{\alpha_1} d{*}u_1 = 0$. This can happen only if c_1 is between the minimum and maximum of ψ and we see that u_1 attains its extrema on α_1. Set $u_1 = L_1\psi$. If $\overline{W}_1$ is bordered with border α_1 but noncompact, extending to the ideal boundary β of the open Riemann surface $W_0 \cup W_1$, then we take for u_1 the directed limit of corresponding functions constructed on exhausting bordered subregions with α_1 fixed and β_1 tending to β The operator L_1 is in this fashion defined for a noncompact W_1. Because of uniform convergence we have $|u_1| \leqq \max|\psi|$ and $\int_{\alpha_1} d{*}L_1\psi = 0$.

We also need a composite operator: given φ on α_0 we take $L_0\varphi$ on α_1 and follow it with $L_1L_0\varphi$ on α_0. We set $K = L_1L_0$.

LEMMA 2. *If* $\operatorname{sgn} K^n\varphi | \alpha_1 \neq \text{const}$ *for* $i = 1, \ldots, n$, *then*

$$|K^n\varphi| \leqq q^n \max|\varphi|. \tag{1}$$

For $n = 1$ this follows from our inequalities for u_0 and u_1. At each iteration of K we obtain another factor q on the right.

3. Principal functions. We can now state a general existence theorem for harmonic functions (cf. [43]). Let W be an open Riemann surface and $\overline{W}_1$ a bordered boundary neighborhood with compact border α_1. On $\overline{W}_1$ let σ be a harmonic function such that $\sigma | \alpha_1 = 0$ and $\int_{\alpha_1} d{*}\sigma = 0$. It is easy to see that these conditions do not restrict generality. The problem is to construct on W a harmonic function p that imitates the behavior of σ on W_1. Let a compact bordered $\overline{W}_0$ with border $\alpha_0 \subset W_1$ intersect W_1 as above and set $M = \max_{\alpha_0}|\sigma|$.

THEOREM 1. *There exists on W a harmonic function p, unique up to an additive constant, such that*

$$p \mid W_1 = \sigma + L_1 p, \tag{2}$$

$$|p - \sigma| \leqq \frac{M}{1-q}, \tag{3}$$

$$|p| \| W_0 \leqq \frac{M}{1-q}. \tag{4}$$

The theorem states that we can always find a harmonic function p on all of W with behavior σ on W_1 such that $p-\sigma$ is bounded on W_1 and "constant on the ideal boundary". Moreover, the bounds for $p-\sigma$ and $p \mid W_0$ only depend on the geometric configuration and on M, not on σ otherwise. This makes it possible to give the bounds (3) and (4) simultaneously for uniformly bounded families of functions σ. We can briefly say that if $\sigma \mid \alpha_0 = O(1)$, then $p - \sigma = O(1)$ and $p \mid W_0 = O(1)$, all uniformly.

The function p is called the *principal function* on W_0. Its existence has been known for some time (see e.g. [4]) and it has a variety of applications. For instance, recently my students G. Weill [54-56], B. Rodin [35], and M. Goldstein [13] looked into various aspects of the K- and L-kernels, studied thus far on certain plane regions and finite Riemann surfaces. It turned out that the beautiful theory of Bergman and Schiffer extends in its entirety to arbitrary Riemann surfaces of finite or infinite genus. This in turn gives access to constructive, and if need be, even numerical, methods in the theory of square integrable differentials. Properties (3) and (4) are new and will be the basis of establishing the uniform boundedness below of our proximity function s.

4. Proof. To establish Theorem 1 we observe that it suffices to find p on α_0. In fact, then $p = L_0 p$ on W_0 and equation (2) gives it on W_1. On combining these two equations we have on α_0:

$$p = \sigma + L_1 L_0 p. \tag{5}$$

Here $L_1 L_0$ is our operator K. It is known that the solution $L_0 p$ of the Dirichlet problem can be expressed as an integral involving the Green's function on W_0. Similarly on W_1 the operator L_1 gives a harmonic function representable as an integral in terms of a modified Green's function (cf. [43] and No. 5 below):

Thus we are dealing with a simple integral equation of the Fredholm type. It is known that its solution is given by the Neumann series $p=\sum_0^\infty K^n\sigma$ provided the series converges uniformly. Indeed, in this case the operator K can be applied term by term and we find $Kp=\sum_1^\infty K^n\sigma=p-\sigma$. This is our equation (5).

To show the uniform convergence we only have to apply Lemma 2 to obtain a majorant geometric series. That the condition of the lemma is satisfied is shown as follows: In $\overline{W}_0\cap\overline{W}_1$ take the harmonic function h with $h|\alpha_1=0$, $h|\alpha_0=$ const. such that $\int_{\alpha_1}d{*h}=1$, α_0 and α_1 being oriented so as to leave $W_0\cap W_1$ to the left and right respectively.

LEMMA 3. $$\int_{\alpha_1}K^n\sigma\, d{*h}=0.$$

For the proof take in $\overline{W}_0\cap\overline{W}_1$ any harmonic function u with $\int_{\alpha_1}d{*u}=0$. An application of Green's formula to u and h gives

$$\int_{\alpha_0}u\,d{*h}=\int_{\alpha_1}u\,d{*h}.$$

For the function u we can take σ by assumption, $L_0\varphi$ by Green's formula, $L_1\psi$ by definition, or $K^n\varphi$. For $n=0$ the lemma is trivial. Suppose now it is true for $n-1$. Then the integral is the same along α_0, but here the integrand can be replaced by L_0 acting on it. This integral equals that along α_1, and here the integrand is the same as L_1 acting on it. Since $L_1L_0=K$ we have K^n and Lemma 3 is established.

Lemma 2 now applies and we obtain $|p\|\alpha_0\leqq M(1-q)^{-1}$. By the maximum principle the same is true of p on W_0 and we have established property (4). In particular the inequality holds on α_1, but there $\dot{p}=p-\sigma$ and by the maximum principle we have property (3). The proof of Theorem 1 is complete.

An analogue of the theorem can be established for higher dimensions.

5. Generalized Green's function. Our first application of Theorem 1 is to a generalization of the Green's function. Let S be an arbitrary Riemann surface. Throughout our presentation we denote by D a parametric disk and by D' and D'' increasingly smaller concentric disks. Set $S''=S-D''$. Denote on S'' by g_a the modified Green's function which has a positive logarithmic pole at a with coefficient 2, vanishes on the curve $\partial D''$, and has L_1 behavior on the ideal boundary. The exact construction will be carried out in the course of the proof of the following auxiliary result. We also set $S' = S - D'$.

LEMMA 4. *Let E be a compact set in S'' and $O \subset S''$ an open set containing E. Then*

$$g_a | E = O(1) \text{ uniformly for } a \in S'' - O. \tag{6}$$

For the proof we make use of the symmetry property $g_a(t) = g_t(a)$, established in the same fashion as in the classical case. We must show that

$$g_t | S'' - O = O(1) \text{ uniformly for } t \in E.$$

Cover E with a finite set of disks K_i in O such that slightly smaller concentric disks K_i' already cover E. We decompose E into compact subsets E_i contained in K_i'. Obviously it suffices to show the uniform boundedness for $t \in E_i$. In applying Theorem 1 we take $W = S'' - t$ and let W_1 consist of three components $K_i - t$, $D - D''$, and a bordered neighborhood D_β, with compact ∂D_β, of the ideal boundary β of S. For W_0 choose $S' - K_i' - D_\beta'$, where D_β' is a bordered neighborhood of β with compact $\partial D_\beta' \subset D_\beta$. In $K_i - t$ take $\sigma = 2 \log(|1 - \zeta \bar{t}|/|\zeta - t|)$, and in $D - D''$, $\sigma = 2 \log r$. Here and later r stands for the distance of the generic point ζ from the center of the parametric disk in question. In D_β we set $\sigma = 0$. The conditions $\sigma | \partial W_0 = 0$ and $\int_{\alpha_0} d{*\sigma} = 0$ are obviously satisfied, $\sigma | \partial W_0 = O(1)$ uniformly, and consequently $p - \sigma$ and $p | W_0$ are $O(1)$ uniformly. We normalize on $\partial D''$. Since p on it is c, a constant, we take $g_t = p - c$ and have $g_t | S'' - O = O(1)$. In fact, this is true on $(D - D'') \cup D_\beta$, and the rest of the set $S'' - O$ is a subset of W_0.

6. Proximity function. We are now ready to carry out the construction of the proximity function according to our program in No. 1. To form t_0 take $W = S - \zeta_0 - \zeta_1$ and $W_1 = (D_0 - \zeta_0) \cup (D_1 - \zeta_1)$, where the parametric disks D_0, D_1 are centered at ζ_0, ζ_1. In $D_0 - \zeta_0$ choose $\sigma = -2 \log r$, in $D_1 - \zeta_1$, $\sigma = 2 \log r$. We tacitly have also a neighborhood of the ideal boundary β, but since we can choose $\sigma = 0$ there we no longer write it down here or in later applications of the theorem. For W_0 take $S - D_0' - D_1'$. Since $\sigma | \partial W_0 = O(1)$ and the flux vanishes, we infer that $p - \sigma = O(1)$ and $p | W_0 = O(1)$. The principal function p is taken as t_0. Then $t_0 | D_0 = -2 \log r + O(1)$ and it follows that $s_0 | D_0 = \log(1 + r^{-2} e^{O(1)})$ $= -2 \log r + \log(r^2 + O(1))$.

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LEMMA 5. $s_0|D_0 = -2\log r + O(1)$ *and* $s_0 \geqq 0$ *on* S.

The construction of s will depend on the location of a. Let D and $\tilde{D}$ be disks disjoint from each other and from D_0''. Consider three cases: (I) $a \in D_0''$, (II) $a \in S - D_0'' - D$, (III) $a \in S - D_0'' - \tilde{D}$. The union of the three sets is obviously S, and it suffices to establish a uniform lower bound for s separately in each of the three cases. The third case can be dispensed with since it is the same as the second case.

Case I. $a \in D_0''$. Take $W = S - \zeta_0 - a$, $W_1 = D_0 - \zeta_0 - a$, and $W_0 = S - D_0'$. Set $\sigma = 2\log(r|1-\zeta\bar{a}|/|\zeta - a|)$. Then $\sigma|\partial W_0$ is $O(1)$ and so are $p - \sigma$ and $p|W_0$. The normalization is at ζ_0 where $p - 2\log r \to -2\log|a| + c(a)$ with $c(a) = O(1)$ uniformly for $a \in D_0''$. By Lemma 5 this limit is $s_0(a) + c_1(a)$ where again $c_1(a) = O(1)$. The function $t = p - c_1$ has the required normalization. Moreover, since $|1-\zeta\bar{a}|/|\zeta - a| > 1$, we have $t|D_0 > 2\log r + O(1)$ and $t|S - D_0 = O(1)$. On combining this with Lemma 5 we obtain $s|D_0 > O(1)$, $s|S - D_0 > O(1)$, hence $s > O(1)$ uniformly for $a \in D_0''$.

Case II. $a \in S - D_0'' - D$. On $S'' = S - D''$ we have $-g_{\zeta_0}|D_0 = 2\log r + O(1)$, $-g_{\zeta_0}|S'' - D_0 = O(1)$. On applying Lemma 4 to $E = \partial D' \cup \zeta_0$, $O = (D - D'') \cup D_0''$ we obtain $g_a|S'' \geqq 0$, $g_a|\partial D' = O(1)$, $g_a|\zeta_0 = O(1)$. Consequently the restriction of $g_a - g_{\zeta_0}$ to D_0 is $> 2\log r + O(1)$; to $S'' - D_0$, $> O(1)$; to $\partial D'$, $O(1)$; and at ζ_0 we have $g_a - g_{\zeta_0} - 2\log r \to c(a) = O(1)$ as $\zeta \to \zeta_0$, uniformly in a.

As the last application of Theorem 1 we take $W = S - \zeta_0 - a$, $W_1 = S'' - \zeta_0 - a$, $W_0 = D'$, and $\sigma = g_a - g_{\zeta_0}$ in W_1. Then $\sigma|\partial W_0 = O(1)$. The normalization is at ζ_0 where $p - 2\log r \to c_1(a) = O(1)$. Take $t = p + s_0(a) - c_1(a)$. Since $s_0(a) - c_1(a) > O(1)$, we conclude that $t|D_0 > 2\log r + O(1)$, $t|S'' - D_0 > O(1)$, and $t|D'' = O(1)$. Adding t to s_0 gives $s|D_0 > O(1)$, $s|S - D_0 > O(1)$.

This completes the construction of the proximity function $s(\zeta, a)$ that is uniformly bounded below for all ζ, $a \in S$.

7. First main theorem. Having thus formed the proximity function we introduce a conformal metric with area element $d\omega = \lambda^2 dS$, where $\lambda^2 = \Delta s = \Delta s_0$ and dS is the Euclidean area element in the parametric disk. A direct computation gives

$$\lambda^2 = \frac{e^{t_0}|\text{grad } t_0|^2}{(1+e^{t_0})^2}. \tag{7}$$

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We see that our metric has zeros, namely those of grad t_0. These zeros, however, turn out to be helpful and in fact constitute an essential part of the theory.

For the Gaussian curvature of our metric we obtain

$$K = -\frac{\Delta \log \lambda}{\lambda^2} = 1.$$

In fact, the logarithm of the numerator in (7) is harmonic while the denominator gives $\Delta \log(1+e^{t_0})$, hence again $-\lambda^2$. As a by-product we have, on an arbitrary Riemann surface, a conformal metric of finite total area (which can be seen to be 4π) and constant Gaussian curvature.

We are now ready to give the first main theorem. Consider an analytic mapping f of an arbitrary Riemann surface R into another arbitrary Riemann surface S. Remove from R a parametric disk R_0 with border β_0 and consider an adjacent regular region Ω with border $\beta_0 \cap \beta_\Omega$. On Ω let u be the harmonic function with $u \mid \beta_0 = 0$, $u \mid \beta_\Omega = k(\Omega)$ = const. such that $\int_{\beta_0} du^* = 1$. For $h \in [0, k]$ consider the level line $\beta_h = u^{-1}(h)$ and the region $\Omega_h = u^{-1}((0, h))$ between β_0 and β_h.

For a given $a \in S$ let $\nu(h, a)$ be the number of inverse images of a in Ω_h. We choose the counting function

$$A(h, a) = 4\pi \int_0^h \nu(h, a)\, dh.$$

It reflects the frequency of a-points of f. In particular, it vanishes identically for a Picard point a.

For the proximity function we take

$$B(h, a) = \int_{\beta_h} s(f(z), a)\, du^*.$$

Here the integrand is the proximity of the image of $z \in \beta_h$ to a, and B is the mean proximity of the image $f(\beta_h)$ of β_h to a.

The characteristic is chosen to be

$$C(h) = \int_{\Omega_h} (h - u(z))\, d\omega(f(z)).$$

In contrast with the classical theory the integrand depends on the region Ω. This, however, causes no difficulty. The characteristic is obviously independent of a, and a simple application of Stokes' formula to the differential ds^* shows that it retains its Shimizu-Ahlfors nature: the derivative

$$C'(h) = \int_{\Omega_h} d\omega(f(z))$$

is the total area of the multi-sheeted image under f of Ω_h over S.

We apply the Green's formula to the functions $s(f(z), a)$ and $h - u(z)$ over Ω_h less small disks about the $f^{-1}(a)$ which we then let shrink to their centers. For h we take its maximum k and obtain the first main theorem:

THEOREM 2. *Under an analytic mapping of an arbitrary Riemann surface R into an arbitrary Riemann surface S, and for any regular region $\Omega \subset R$,*

$$A(k, a) + B(k, a) = C(k) + D(k, a).$$

Here D is the integral along β_0 and is seen to be $B(0, a) + kB'(0, a)$. We observe that it is $O(k)$. For functions of any interest it turns out that the characteristic C grows more rapidly than k. Thus D is an insignificant remainder and we conclude that the beautiful classical balance prevails in the present most general situation: the $(A+B)$-affinity, so to speak, of the mapping f is the same for *all* points a. In particular, for a Picard point we have a strong proximity B of the image curve $f(\beta_h)$.

8. Estimation of B. The main question is: how many Picard points can there exist. The answer is given by estimating $\sum_i B(k, a_i)$ for any $a_1, \ldots, a_q$. This estimate is known as the second main theorem. But even in the classical case of the plane no estimate is known for the remainder in the second main theorem that would be valid for all values of the variable r. It is the integral of the integral of the remainder that can be given a universally valid bounding function. The remainder itself can behave arbitrarily wildly in certain exceptional intervals whose length can be estimated but which must be omitted in stating the second main theorem. When one then takes the defect relation from the second main theorem these exceptional intervals prevent the use of directed limits. But ordinary limits cannot be used on arbitrary Riemann surfaces for there is no one single parameter that would give an exhaustion of the entire surface. Thus these exceptional intervals block every attempt at extending the classical form of the value distribution theory to the general case.

This difficulty can be overcome by the following simple device. We replace the mean proximity to a of the image curve by what is just as natural if not more so, by the mean proximity to a of the entire image region and then integrate this. Analytically this means that, in a sense, we bring all quantities involved to the same level of integration. The remainder in the second main theorem can then be given an estimate valid for all regions, directed limits can be used, and the theory established on arbitrary Riemann surfaces.

To carry out this program we introduce, for any function defined in $[0, k]$,

$$\varphi_1(h) = \int_0^h \varphi(h)\, dh,$$

$$\varphi_2(h) = \int_0^h \varphi_1(h)\, dh.$$

We choose q points $a_1, \ldots, a_q$ in S and set, for any function $\psi(h, a)$,

$$\psi(h) = \sum_1^q \psi(h, a_i).$$

When these two operations are applied to the terms in the first main theorem one obtains

$$A_2(h) + B_2(h) = qC_2(h) + D_2(h). \tag{8}$$

It remains to estimate B_2.

To this end we set

$$\sigma(z) = \exp \sum_1^q s(f(z), a_i)$$

plus a negligible normalizing term we shall disregard in this paper (for details see [41]). We then introduce the mass distribution $dm = \sigma(z) d\omega$ on S. Its density on S is $\sigma\lambda^2$, which in the $u + iu^*$-plane induces the density $\sigma\mu^2$. Here

$$\mu(z) = \lambda(z) \,|f'(z)|\, |\operatorname{grad} u(z)|^{-1}.$$

In terms of μ we set

$$F(h) = \int_{\beta_h} \log \sigma\mu^2 du^*,$$

and

$$G(h) = -\int_{\beta_h} \log \mu^2 du^*.$$

Then

$$B_2 = F_2 + G_2.$$

The effect of this decomposition is that the components can be separately estimated.

9. Estimation of F_2. We start with F. Let

$$H(h) = \int_{\beta_h} \sigma\mu^2 du^*.$$

By the convexity of the logarithm we have

$$\frac{1}{h} F_1(h) \leqq \log\left(\frac{1}{h} H_1(h)\right),$$

hence

$$F_1(h) < h \log H_1(h) + O(h \log h).$$

Another integration gives similarly

$$F_2(h) < h^2 \log H_2(h) + O(h^2 \log h).$$

To estimate H_2 we note that

$$H_1(h) = \int_S \nu(h, a)\, dm(a).$$

In fact, on the left the density $\sigma\mu^2$ is first integrated along the level line β_h, then this integral from 0 to h, and we obtain the total mass on $f(\Omega_h)$. On the right each mass element is multiplied by the number of times it is covered, and the integral again gives the total mass on the multi-sheeted image of Ω_h over S. The integrand is the same as in $A(h, a)$, and we conclude that H_2 is obtained by integrating the first main theorem with respect to $dm(a)$ over S. On the right C gives mC, where m is the total mass of the distribution on S. D is $O(h)$ and, by virtue of the symmetry of s, remains so after the integration over the finite mass. We have shown in Nos. 1-6 that the integrand s in B is uniformly bounded below. When B is transposed to the right we thus obtain $< O(1)$ which is subsumed under $O(h)$ and we have

$$4\pi H_2(h) < mC(h) + O(h).$$

The substitution into F_2 gives

$$F_2(h) < h^2 \log [(C(h) + O(h)],$$

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where the last term contains the original $h^2 \log h$-term.

10. Evaluation of G_2. We first consider $G' = -2\int_{\beta_h} d*\log\mu$. We apply Stokes' formula to the differential $d*\log\mu$ over the region Ω_h from which we first remove small disks about the zeros of μ and then let the disks shrink to their centers. Each zero of μ gives a flux 2π and we obtain 4π times the negative of the number $\nu(h, f')$ of the zeros of f' minus the number $\nu(h, \lambda)$ of the zeros of our metric $\lambda|d\zeta|$ plus the number of the zeros of grad u. By the Lefschetz fixed point theorem the sum of the indices of the singularities of the differentiable vector field grad u, that is, the number of zeros of grad u in Ω_h, is the Euler characteristic $e(h)$ of Ω_h. The integral along β_0 gives a constant $O(1)$. In the area integral the integrand is $\Delta\log\mu = \Delta\log\lambda$, for $\log|f'(\text{grad } u)^{-1}|$ is harmonic. We divide $\Delta\log\lambda$ by λ^2 to get $-K = -1$, and multiply the area element by λ^2 to obtain the integral $\int_{\Omega_h} d\omega(f(z))$ which we know to be $C'(h)$. The result is

$$G'(h) = 4\pi[-\nu(h, f') - \nu(h, \lambda) + e(h)] + 2C'(h) + O(1).$$

From a differential geometric view point, what we have done is to apply Stokes' formula or, what is the same thing, the Gauss-Bonnet formula, to a cross-section of the tangent bundle on S, namely that extracted by the differentiable vector field $f'(z)/\text{grad } u(z)$.

11. Second main theorem. For consistency of notations we set

$$E(h) = 4\pi\int_0^h e(h)\,dh,$$

integrate G' thrice, substitute G_2 and F_2 into B_2, this into (8), and arrive at our main result, the second main theorem in its most general setting:

THEOREM 3. *For an analytic mapping f of an arbitrary open Riemann surface R into an arbitrary open or closed Riemann surface S, and for any regular subregion Ω of R,*

$$(9)\quad (q-2)C_2(k) < \sum_1^q A_2(k, a_i) - A_2(k, f') - A_2(k, \lambda) + E_2(k) + O(k^3 + k^2\log C(k)).$$

Here k^3 came from $O(1)$ in G', and all earlier remainders are accounted for.

Significant consequences from the theorem can be drawn only for functions for whih $O(\ )/C_2 \to 0$ as Ω exhausts $R - R_0$. This condition was given the following elegant formulation by my student K. V. R. Rao: there must exist a constant $c \in (0, 1)$ such that $\log C(k)/C(ck) \to 0$. In view of the slow growth of the logarithm this condition is natural and obviously satisfied already by such slowly growing functions as k^m and e^{nk}. We shall only consider non-degenerate functions so defined.

12. Affinity relation. We now introduce what we shall call the *lower defect of a*,

$$\alpha(a) = 1 - \overline{\lim} \frac{A_2(k, a)}{C_2(k)},$$

where, as in the sequel, the limit is a directed limit as the regular region Ω exhausts $R - R_0$. Similarly we take the *upper defect*

$$\bar{\alpha}(a) = 1 - \underline{\lim} \frac{A_2(k, a)}{C_2(k)}.$$

For a Picard point both upper and lower defects are 1. For other points they may differ (see below).

Our *ramification index* is defined by

$$\beta(a) = \underline{\lim} \frac{A_2(k, f'_a)}{C_2(k)},$$

where the numerator counts the orders of branch-points above a. Since $f(\Omega_h)$ only has a finite number of branch-points it is meaningful to take the *total ramification*

$$\sum \beta(a) \leqq \underline{\lim} \frac{A_2(k, f')}{C_2(k)},$$

where the sum is extended over all points a of S. The sole motivation of using lower rather than upper limits in the ramification index is that of expediency.

For the zeros of λ we set

$$\gamma(a) = \underline{\lim} \frac{A_2(k, \lambda_a)}{C_2(k)},$$

where the numerator counts the points of $f(\Omega_h)$ covering a zero of λ at a. Again, only a finite number of zeros is covered by $f(\Omega_h)$ and we can consider the sum

$$\sum \gamma(a) \leqq \underline{\lim} \frac{A_2(k, \lambda)}{C_2(k)}$$

extended over all a on S.

Finally we introduce the *Euler index*

$$\eta = \underline{\lim} \frac{E_2(k)}{C_2(k)}.$$

On dividing (9) by C_2 and on taking suitable limits we obtain the following comprehensive defect, ramification, and covering relations for mappings between arbitrary Riemann surfaces.

AFFINITY RELATION.

$$\sum \alpha(a) + \sum \beta(a) + \sum \gamma(a) \leqq 2 + \eta. \tag{10}$$

All terms are obviously positive, and for $\eta < \infty$ we obtain directly estimates for each of the three terms. First, the lower defect sum and consequently the number of Picard values is dominated by $2+\eta$. The same bound is valid for the total ramification; in particular, there can be at most $4+2\eta$ totally ramified points, i.e., points covered exclusively with branch points. Finally, the sum of the γ-indices is bounded by $2+\eta$.

This latter property gains in meaning if we choose for a_i the zeros of our metric: $\lambda(a_i) = 0$. Then for $i \to \infty$, $\gamma(a_i) \to 0$, hence $\bar{\alpha}(a_i) \to 1$, while $\alpha(a_i) \to 0$. Thus the zeros of our metric are covered in an interesting manner: the coverage ratio A_2/C_2 for any zero sufficiently far in the sequence oscillates between nearly 1 and nearly 0 even if R is exhausted sequentially.

13. Closed range surfaces. As a special case we consider a closed image surface S. Since grad t_0 forms a vector field on $S - \zeta_0 - \zeta_1$, the number of its zeros, i.e., zeros of λ, is the Euler characteristic $e_S + 2$. We now add to our arbitrarily chosen points $a_1, \ldots, a_q$ these $2g$ zeros, g the genus, replace q in (9) by $q + e_S + 2$, and obtain

$$\sum \alpha + \sum \beta \leqq \eta - e_S.$$

In particular, the number of Picard values is at most the excess of the Euler index η over the Euler characteristic of the image surface S. For meromorphic functions on an arbitrary Riemann surface or for Gaussian mappings of arbitary minimal surfaces the bound for the defect sum is $2+\eta$. The bound was shown to be sharp by an interesting example of my student B. Rodin. In the most special case of meromorphic functions in the plane, $\eta=0$ and we are back in the classical bound 2 for the defect sum and the total ramification.

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