15

Clifford relations

Clifford algebras and *Clifford relations* were studied by mathematicians long before canonical anti-commutation relations were considered by physicists. Actually, the "(neutral) CAR representations" that we introduced in Def. 12.1 could be called "representations of self-adjoint Clifford relations".

We will use the name "Clifford relations" for anti-commutation relations identical to those of Def. 12.1, but without assuming that the underlying vector space is real, the corresponding operators are self-adjoint or that they even act on a Hilbert space.

In our short presentation we will restrict ourselves mostly to Clifford relations over finite-dimensional pseudo-Euclidean spaces. Our main motivation is to describe spinor representations of the *Lorentz group* (in any dimension). Nevertheless, we will consider the case of a general signature as well.

Some real Clifford algebras are closely related to the *quaternion algebra*, denoted by $\mathbb{H}$. Therefore, we devote Sect. 15.2 to a brief summary of its properties.

We will use the shorthand $\mathbb{K}(n) := L(\mathbb{K}^n)$, where $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. We will write $[x]$ for the integer part of $x \in \mathbb{R}$.

15.1 Clifford algebras

15.1.1 Representations of Clifford relations

Let $\mathbb{K}$ be an arbitrary field and $\mathcal{Y}$ a vector space over $\mathbb{K}$. We assume that $\mathcal{Y}$ is equipped with a symmetric bilinear form ν.

Let $\mathcal{V}$ be another vector space (possibly over a bigger field).

Definition 15.1 *We will say that a linear map*

$$\mathcal{Y} \ni y \mapsto \gamma^{\pi}(y) \in L(\mathcal{V}) \tag{15.1}$$

is a representation of Clifford relations *or, for brevity,* a Clifford representation over $\mathcal{Y}$ in $\mathcal{V}$ *if*

$$\left[\gamma^{\pi}(y_1), \gamma^{\pi}(y_2)\right]_+ = 2y_1 \cdot \nu y_2 \mathbb{1}, \quad y_1, y_2 \in \mathcal{Y}. \tag{15.2}$$

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15.1.2 Clifford algebras

Definition 15.2 *The* Clifford algebra $\mathrm{Cliff}(\mathcal{Y})$ *is the unital algebra over* $\mathbb{K}$ *generated by elements* $\gamma(y)$, $y \in \mathcal{Y}$, *with relations*

$$\gamma(\lambda y) = \lambda\gamma(y),\ \lambda \in \mathbb{K}, \qquad \gamma(y_1 + y_2) = \gamma(y_1) + \gamma(y_2),$$
$$\gamma(y_1)\gamma(y_2) + \gamma(y_2)\gamma(y_1) = 2y_1\cdot\nu y_2 \mathbb{1}.$$

We have the following analog of Prop. 12.31:

Proposition 15.3 *If*

$$\mathcal{Y} \ni y \mapsto \gamma^\pi(y) \in L(\mathcal{V})$$

is a representation of Clifford relations, then there exists a unique homomorphism

$$\pi : \mathrm{Cliff}(\mathcal{Y}) \to L(\mathcal{V})$$

such that $\pi(\mathbb{1}) = \mathbb{1}_\mathcal{V}$ *and* $\pi(\gamma(y)) = \gamma^\pi(y)$, $y \in \mathcal{Y}$.

Many concepts and facts described in the context of the CAR apply almost verbatim to Clifford relations and algebras. For instance, $\alpha(\phi(y)) = -\phi(y)$, $y \in \mathcal{Y}$, extends to a unique involutive automorphism α of $\mathrm{Cliff}(\mathcal{Y})$. Clifford algebras split into their even and odd parts: $\mathrm{Cliff}(\mathcal{Y}) = \mathrm{Cliff}_0(\mathcal{Y}) \oplus \mathrm{Cliff}_1(\mathcal{Y})$. $\mathrm{Cliff}_0(\mathcal{Y})$ is a sub-algebra of $\mathrm{Cliff}(\mathcal{Y})$, which differs from $\mathrm{Cliff}(\mathcal{Y})$ if the field $\mathbb{K}$ has a characteristic different from 2 (which is the case for $\mathbb{K} = \mathbb{R}, \mathbb{C}$).

There also exists a unique anti-automorphism $A \to A^\dagger$, called the *transposition*, which on products of $\gamma(y)$ equals

$$(\gamma(y_1)\cdots\gamma(y_k))^\dagger = \gamma(y_k)\cdots\gamma(y_1).$$

15.1.3 Complex Clifford algebras

Let us consider an n-dimensional space $\mathcal{Y}$ over $\mathbb{C}$ equipped with a non-degenerate form ν. All such forms are isomorphic to one another, so it is enough to assume that $\mathcal{Y} = \mathbb{C}^n$ and $z\cdot\nu z = \sum_{j=1}^{n}(z_j)^2$ for $z = (z_1, \dots, z_n) \in \mathbb{C}^n$. It is easy to see that in this case

$$\mathrm{Cliff}(\mathbb{C}^{2m}) = \mathbb{C}(2^m),$$
$$\mathrm{Cliff}(\mathbb{C}^{2m+1}) = \mathbb{C}(2^m) \oplus \mathbb{C}(2^m).$$

Thus, as an algebra, $\mathrm{Cliff}(\mathbb{C}^n)$ coincides with $\mathrm{CAR}(\mathbb{R}^n)$ defined in Def. 12.30, where the transposition † coincides with $^\#$. However, we forget about the Hermitian conjugation $*$, the complex conjugation c and the norm $\|\cdot\|$. ($\mathrm{CAR}(\mathbb{R}^n)$ is a C^*-algebra, whereas $\mathrm{Cliff}(\mathbb{C}^n)$ is not.)

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Suppose now that the space $\mathcal{Y}$ is oriented (see Subsect. 3.6.8 for the definition of an orientation of a complex space). Let $(e_1, \dots, e_n)$ be an o.n. basis of $\mathcal{Y}$ compatible with its orientation, and write γ_j for $\gamma(e_j)$.

Definition 15.4 *The* volume element *is defined as*

$$\omega := \gamma_1 \cdots \gamma_n.$$

Note that ω depends on the o.n. basis $(e_1, \dots, e_n)$ only through its orientation. Set $m := [n/2]$. The following table summarizes the form of the algebras $\mathrm{Cliff}(\mathbb{C}^n)$:

Table 15.1 *Form of* $\mathrm{Cliff}(\mathbb{C}^n)$

$n \pmod 4$	ω^2	$\mathrm{Cliff}_0(\mathbb{C}^n)$	$\mathrm{Cliff}(\mathbb{C}^n)$
0	$\mathbb{1}$	$\mathbb{C}(2^{m-1}) \oplus \mathbb{C}(2^{m-1})$	$\mathbb{C}(2^m)$
1	$\mathbb{1}$	$\mathbb{C}(2^m)$	$\mathbb{C}(2^m) \oplus \mathbb{C}(2^m)$
2	$-\mathbb{1}$	$\mathbb{C}(2^{m-1}) \oplus \mathbb{C}(2^{m-1})$	$\mathbb{C}(2^m)$
3	$-\mathbb{1}$	$\mathbb{C}(2^m)$	$\mathbb{C}(2^m) \oplus \mathbb{C}(2^m)$

15.2 Quaternions

In this section we briefly recall the properties of quaternions.

15.2.1 Basic definitions

Definition 15.5 *The real algebra* $\mathbb{H}$ *with basis* $1, \mathrm{i}, \mathrm{j}, \mathrm{k}$ *satisfying the relations*

$$\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = -1, \quad \mathrm{ij} = \mathrm{k}, \quad \mathrm{jk} = \mathrm{i}, \quad \mathrm{ki} = \mathrm{j}$$

is called the algebra of quaternions*. It is equipped with an involution* $*$ *acting as*

$$1^* = 1, \quad \mathrm{i}^* = -\mathrm{i}, \quad \mathrm{j}^* = -\mathrm{j}, \quad \mathrm{k}^* = -\mathrm{k}.$$

For $x \in \mathbb{H}$*, we set*

$$\mathrm{Re}\, x := \frac{1}{2}(x + x^*), \quad |x| := \sqrt{x^* x}.$$

(Note that $x^* x$ *is always real positive.)*

If $x = x_1 + x_\mathrm{i}\mathrm{i} + x_\mathrm{j}\mathrm{j} + x_\mathrm{k}\mathrm{k}$ with $x_1, x_\mathrm{i}, x_\mathrm{j}, x_\mathrm{k} \in \mathbb{R}$, then

$$\mathrm{Re}\, x = x_1, \qquad |x| = \sqrt{x_1^2 + x_\mathrm{i}^2 + x_\mathrm{j}^2 + x_\mathrm{k}^2}.$$

Note that $|\cdot|$ is a norm on the algebra $\mathbb{H}$. If $x, y \in \mathbb{H}$, then $|xy| = |x||y|$. $\mathbb{H}$ is an example of a real C^*-algebra.

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$\mathbb{H}$ is a real Hilbert space with the scalar product

$$\langle x|y\rangle := \operatorname{Re} x^* y = x_1 y_1 + x_{\mathrm{i}} y_{\mathrm{i}} + x_{\mathrm{j}} y_j + x_{\mathrm{k}} y_{\mathrm{k}}, \quad x, y \in \mathbb{H}.$$

Definition 15.6 *An algebra all of whose non-zero elements are invertible is called a* division algebra*.*

Clearly, $\mathbb{H}$ is a division algebra.

15.2.2 Quaternionic vector spaces

Quaternionic vector spaces and *finite-dimensional quaternionic vector spaces* have obvious definitions. Every finite-dimensional quaternionic vector space is isomorphic to $\mathbb{H}^n$ for some n. Note the identifications

$$\mathbb{R}^n \otimes \mathbb{C} = \mathbb{C}^n, \quad \mathbb{R}^n \otimes \mathbb{H} = \mathbb{H}^n.$$

$\mathbb{H}$-*linear transformations* on a quaternionic vector space have an obvious definition. Note the identifications

$$\mathbb{R}(n) \otimes \mathbb{C} = \mathbb{C}(n), \quad \mathbb{R}(n) \otimes \mathbb{H} = \mathbb{H}(n).$$

Definition 15.7 *Suppose that $\mathcal{X}$ is a quaternionic vector space, equipped (as a real space) with a scalar product $\langle x|y\rangle \in \mathbb{R}$, $x, y \in \mathcal{X}$. We say that this scalar product is* compatible with the quaternionic structure *if*

$$\langle \lambda x|\lambda y\rangle = |\lambda|^2 \langle x|y\rangle, \quad \lambda \in \mathbb{H}, \; x, y \in \mathcal{X}.$$

A quaternionic space with a compatible scalar product complete in the corresponding norm is called a quaternionic Hilbert space*.*

Every finite-dimensional quaternionic Hilbert space is isomorphic to $\mathbb{H}^n$ with the scalar product

$$\langle x|y\rangle := \sum \operatorname{Re} x_i^* y_i, \quad x, y \in \mathbb{H}^n.$$

15.2.3 Embedding complex numbers in quaternions

Clearly, there exists exactly one continuous injective homomorphism $\mathbb{R} \to \mathbb{H}$. However, there exist many continuous injective homomorphisms $\mathbb{C} \to \mathbb{H}$. Such a homomorphism is determined uniquely if we fix the image of $\mathrm{i} \in \mathbb{C}$ inside $\mathbb{H}$. It is natural to denote it also by i.

Let us fix such a homomorphism $\mathbb{C} \to \mathbb{H}$. Now $\mathbb{H}$ becomes a two-dimensional vector space over the field $\mathbb{C}$. The map

$$\mathbb{H} \ni x \mapsto \frac{1}{2}(x - \mathrm{i}x\mathrm{i}) \in \mathbb{C} \tag{15.3}$$

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is a projection. $\mathbb{H}$ is equipped with a sesquilinear scalar product

$$(x|y) := \frac{1}{2}(yx^* - \mathrm{i}yx^*\mathrm{i}). \tag{15.4}$$

In fact, by (15.3), the values of this scalar product are in $\mathbb{C}$. The computation

$$(x|zy) = \frac{1}{2}(zyx^* - \mathrm{i}zyx^*\mathrm{i}) = z(x|y),$$
$$(zx|y) = \frac{1}{2}(yx^*\overline{z} - \mathrm{i}yx^*\overline{z}\mathrm{i}) = (x|y)\overline{z}, \quad z \in \mathbb{C},$$

shows that (15.4) is sesquilinear.

Note that the real scalar product is compatible with the complex scalar product: $\langle x|y\rangle = \mathrm{Re}(x|y)$.

$(1, \mathrm{j})$ is an example of an o.n. basis of $\mathbb{H}$ w.r.t. (15.4).

If we fix an embedding (15.3), then quaternionic vector spaces can be reinterpreted as complex vector spaces, and quaternionic Hilbert spaces as complex Hilbert spaces.

Definition 15.8 *If $\mathcal{X}$ is a quaternionic vector space, then $\mathcal{X}_{\mathbb{C}}$ will denote the same $\mathcal{X}$ understood as a complex space. It will be called the* complex form of $\mathcal{X}$.

15.2.4 Matrix representation of quaternions

Quaternions can be represented by the Pauli matrices multiplied by i:

$$\pi(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \pi(\mathrm{i}) = \begin{bmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{bmatrix}, \quad \pi(\mathrm{j}) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \pi(\mathrm{k}) = \begin{bmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}.$$

Thus we obtain a representation of quaternions on the Hilbert space $\mathbb{C}^2$:

$$\pi : \mathbb{H} \to B(\mathbb{C}^2). \tag{15.5}$$

In this representation,

$$\pi(x^*) = \pi(x)^*, \quad |x| = \sqrt{\det \pi(x)}. \tag{15.6}$$

We have

$$\pi(\mathbb{H}) = \big\{\lambda U \ : \ U \in SU(2), \quad \lambda \in [0, \infty[\big\}.$$

Another useful relation, which depends on the representation chosen above, is

$$\pi(\mathbb{H}) = \big\{A \in B(\mathbb{C}^2) \ : \ A = R\overline{A}R^{-1}\big\}, \tag{15.7}$$

where $\overline{A}$ is the usual complex conjugation of the matrix A and $R = \pi(\mathrm{j})$. Note that $R\overline{R} = -\mathbb{1}$.

If we replace (15.5) by $W\pi(\cdot)W^*$ for some unitary W, then R is replaced by $R_W := WR\overline{W}^*$. Note that $R_W\overline{R}_W = -\mathbb{1}$ as well.

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15.2.5 Real simple algebras

It is well known that one can classify all simple finite-dimensional algebras over $\mathbb{C}$ and $\mathbb{R}$. The complex case is particularly simple.

Theorem 15.9 *Let $\mathfrak{A}$ be a complex finite-dimensional simple algebra. Then there exists a positive integer n such that $\mathfrak{A}$ is isomorphic to $\mathbb{C}(n)$.*

The corresponding classification in the real case is more complicated.

Theorem 15.10 *Let $\mathfrak{A}$ be a real finite-dimensional simple algebra. Then there exists a positive integer n such that $\mathfrak{A}$ is isomorphic to $\mathbb{C}(n)$, $\mathbb{R}(n)$ or $\mathbb{H}(n)$.*

Moreover, suppose that $\pi : \mathfrak{A} \to L(\mathcal{V})$ is a representation of $\mathfrak{A}$ in a complex space $\mathcal{V}$. (Such a representation always exists.) Define the complex conjugate representation $\overline{\pi} : \mathfrak{A} \to L(\overline{\mathcal{V}})$ by $\overline{\pi}(A) := \overline{\pi(A)}$, $A \in \mathfrak{A}$. Then the following are true:

(1) *$\mathfrak{A} \simeq \mathbb{C}(n)$ iff there exists no $R : \overline{\mathcal{V}} \to \mathcal{V}$ linear invertible such that $\pi(A)R = R\overline{\pi}(A)$.*
(2) *$\mathfrak{A} \simeq \mathbb{R}(n)$ iff there exists $R : \overline{\mathcal{V}} \to \mathcal{V}$ linear invertible such that $\pi(A)R = R\overline{\pi}(A)$ and $R\overline{R} = \mathbb{1}$.*
(3) *$\mathfrak{A} \simeq \mathbb{H}(n)$ iff there exists $R : \overline{\mathcal{V}} \to \mathcal{V}$ linear invertible such that $\pi(A)R = R\overline{\pi}(A)$ and $R\overline{R} = -\mathbb{1}$.*

If π is irreducible, then R in (2) and (3) is defined uniquely up to a phase factor.

Remark 15.11 *Note that we have the following equivalent versions of (1), (2) and (3) of the above theorem:*

(1) *There exists no anti-linear invertible χ on $\mathcal{V}$ such that $\pi(A)\chi = \chi\pi(A)$.*
(2) *There exists an anti-linear invertible χ on $\mathcal{V}$ such that $\pi(A)\chi = \chi\pi(A)$ and $\chi^2 = \mathbb{1}$.*
(3) *There exists an anti-linear invertible χ on $\mathcal{V}$ such that $\pi(A)\chi = \chi\pi(A)$ and $\chi^2 = -\mathbb{1}$.*

We can pass from χ to R by $\chi v = R\overline{v}$.

In particular, $\mathbb{R}(n)$ can be embedded in $\mathbb{C}(n)$, and then $R = \mathbb{1}$. $\mathbb{H}(n)$ can be embedded in $\mathbb{C}(2) \otimes \mathbb{C}(n)$, so that $R = \pi(\mathrm{j}) \otimes \mathbb{1}$.

15.3 Clifford relations over $\mathbb{R}^{q,p}$

Let us consider an n-dimensional vector space over $\mathbb{R}$ equipped with a non-degenerate symmetric form ν. All such forms are determined by their signature, that is, a pair of non-negative integers q, p with $n = q + p$, so that by an

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appropriate choice of a basis the form ν can be written as

$$y{\cdot}\nu y = -\sum_{j=1}^{q}(y_j)^2 + \sum_{j=q+1}^{n}(y_j)^2. \tag{15.8}$$

Definition 15.12 *The vector space $\mathbb{R}^n$ equipped with form (15.8) will be denoted $\mathbb{R}^{q,p}$.*

In this section we will study representations of Clifford relations over $\mathbb{R}^{q,p}$.

Definition 15.13 *A representation of Clifford relations will then be called a* real, complex, *resp.* quaternionic representation*, if it acts on a real, complex, resp. quaternionic space $\mathcal{V}$. Elements of $\mathcal{V}$ will be called* real, complex, *resp.* quaternionic spinors.

Of course, the complex case is the most important.

15.3.1 Basic facts

Let

$$\mathbb{R}^{q,p} \ni y \mapsto \gamma^\pi(y) \in L(\mathcal{V}) \tag{15.9}$$

be a Clifford representation.

Definition 15.14 *We set $\gamma_i^\pi := \gamma^\pi(e_i)$, where e_i is the canonical basis of $\mathbb{R}^{q,p}$, and the* volume element of the representation *γ^π is defined as*

$$\omega^\pi = \gamma_1^\pi \cdots \gamma_n^\pi. \tag{15.10}$$

Proposition 15.15 *Consider the Clifford representation (15.9). Then*

$$\mathbb{R}^{q,p} \ni y \mapsto -\gamma^\pi(y) \in L(\mathcal{V}) \tag{15.11}$$

is also a Clifford representation. If n is even, then

$$\omega^\pi \gamma^\pi(y)(\omega^\pi)^{-1} = -\gamma^\pi(y),$$

so ω^π implements the equivalence between (15.9) and (15.11).

The following proposition is proven by mimicking the arguments of Thms. 12.27 and 12.28. Recall that $q + p = n$.

Proposition 15.16 (1) *Let n be even. Then all complex irreducible Clifford representations over $\mathbb{R}^{q,p}$ are equivalent and act on $\mathbb{C}^{n/2}$.*

(2) *Let n be odd. Then there exist exactly two inequivalent complex irreducible Clifford representations over $\mathbb{R}^{q,p}$. Moreover, if (15.9) is irreducible, then so*

is (15.11), and they are inequivalent. They act on $\mathbb{C}^{(n-1)/2}$ *and satisfy*

$$\omega^\pi = \pm \mathrm{i}^{(n-1)/2+q} \mathbb{1}. \tag{15.12}$$

(3) *If* γ^π *is an irreducible complex Clifford representation, then the complex algebra generated by* $\gamma^\pi(y)$, $y \in \mathcal{Y}$, *is isomorphic to* $\mathbb{C}(2^{[n]/2})$.

The following proposition shows that it is easy to pass from the signature q, p to p, q.

Proposition 15.17 *Suppose that* $\mathcal{V}$ *is complex. Let the linear map* $\epsilon : \mathbb{R}^{p,q} \to \mathbb{R}^{q,p}$ *be defined by* $\epsilon e_j = e_{q+j}$ *for* $1 \le j \le p$, $\epsilon e_{p+j} = e_j$ *for* $1 \le j \le q$, *where* $e_1, \dots, e_n$ *is the canonical basis. Then*

$$\mathbb{R}^{p,q} \ni y \mapsto \mathrm{i}\gamma^\pi(\epsilon y) \in L(\mathcal{V}) \tag{15.13}$$

is a representation of Clifford relations.

15.3.2 Charge reversal

In this section we consider a representation (15.9) of Clifford relations in a complex space $\mathcal{V}$. For simplicity, we drop the superscript π.

Definition 15.18 *Suppose that* χ_+ *and* χ_- *are anti-linear operators on* $\mathcal{V}$.

(1) χ_+ *is called a* real charge reversal *if*

$$\chi_+ \gamma(y) \chi_+^{-1} = \gamma(y), \quad \chi_+^2 = \mathbb{1}.$$

(2) χ_+ *is called a* quaternionic charge reversal *if*

$$\chi_+ \gamma(y) \chi_+^{-1} = \gamma(y), \quad \chi_+^2 = -\mathbb{1}.$$

(3) χ_- *is called a* pseudo-real charge reversal *if*

$$\chi_- \gamma(y) \chi_-^{-1} = -\gamma(y), \quad \chi_-^2 = \mathbb{1}.$$

(4) χ_- *is called a* pseudo-quaternionic charge reversal *if*

$$\chi_- \gamma(y) \chi_-^{-1} = -\gamma(y), \quad \chi_-^2 = -\mathbb{1}.$$

In the case of an irreducible representation, the operators $\chi_\pm$ are determined uniquely up to a phase factor.

Theorem 15.19 *A complex irreducible representation of Clifford relations over* $\mathbb{R}^{q,p}$ *possesses a charge reversal of the following types:*

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$p-q \pmod 8$		
0	real	pseudo-real
1	real	
2	real	pseudo-quaternionic
3		pseudo-quaternionic
4	quaternionic	pseudo-quaternionic
5	quaternionic	
6	quaternionic	pseudo-real
7		pseudo-real

If both χ_- and χ_+ exist (which is the case for all even n), then $\chi_+\chi_-$ is proportional to ω (see Def. 15.14).

Proof Prop. 15.17 shows that it is enough to prove the real and quaternionic parts of Thm. 15.19. In fact, (15.9) is irreducible iff (15.13) is. Moreover, (15.9) possesses a real, resp. quaternionic charge reversal iff (15.13) possesses a pseudo-real, resp. pseudo-quaternionic charge reversal.

For the proof of Thm. 15.19, it is convenient to use *real Pauli matrices*, that is,

$$\theta_1 := \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \theta_2 := \frac{1}{\mathrm{i}}\sigma_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \theta_3 := \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Note that $\theta_1^2 = -\theta_2^2 = \theta_3^2 = \mathbb{1}$, and

$$\begin{aligned} \theta_1\theta_2 &= -\theta_2\theta_1 = \theta_3, \\ \theta_2\theta_3 &= -\theta_3\theta_2 = \theta_1, \\ \theta_3\theta_1 &= -\theta_1\theta_3 = \theta_2. \end{aligned}$$

Moreover, $\mathbb{R}(2)$ is generated by θ_1, θ_2.

Let us now start the main part of the proof. Recall that $n = q + p$. For any (q, p) with $m = [(q+p)/2]$, we will construct a family of matrices in $\mathbb{R}(2^m)$,

$$\gamma_1^{q,p}, \dots, \gamma_{q+p}^{q,p},$$

such that

$$[\gamma_i^{q,p}, \gamma_j^{q,p}]_+ = 0, \quad 0 \le i < j \le n,$$

$(\gamma_j^{q,p})^2 = -\mathbb{1}$ for q distinct j and $(\gamma_j^{q,p})^2 = \mathbb{1}$ for p distinct j. If possible, we will also construct a real matrix $R_+^{q,p}$ such that $R_+^{q,p}\gamma_j^{q,p}(R_+^{q,p})^{-1} = \gamma_j^{q,p}$ and $(R_+^{q,p})^2 = \pm\mathbb{1}$.

First assume that $q+p$ is even. The case $q=p$ is particularly easy. We set

$$
\begin{aligned}
\gamma_{2j-1}^{q,q} &:= \theta_3^{\otimes(j-1)} \otimes \theta_1, \\
\gamma_{2j}^{q,q} &:= \theta_3^{\otimes(j-1)} \otimes \theta_2, \quad j=1,\ldots,q, \\
R_+^{q,q} &:= \mathbb{1}^{\otimes q}.
\end{aligned}
\tag{15.14}
$$

For $q<p$, we set

$$
\begin{aligned}
\gamma_{2j}^{q,p} &:= \mathrm{i}\gamma_{2j}^{\frac{q+p}{2},\frac{q+p}{2}}, \quad j=1,\ldots,\frac{p-q}{2}; \\
\gamma_k^{q,p} &:= \gamma_k^{\frac{q+p}{2},\frac{q+p}{2}}, \quad \text{for remaining } k; \\
R_+^{q,p} &:= (\theta_1 \otimes \theta_2)^{\frac{p-q}{4}}, \quad \text{for even } \frac{p-q}{2}, \text{ then } (R_+^{q,p})^2 = (-\mathbb{1})^{\frac{p-q}{4}}; \\
R_+^{q,p} &:= (\theta_1 \otimes \theta_2)^{\frac{p-q-2}{4}} \otimes \theta_1 \otimes \theta_3^p, \quad \text{for odd } \frac{p-q}{2}, \text{ then } (R_+^{q,p})^2 = (-\mathbb{1})^{\frac{p-q-2}{4}}.
\end{aligned}
$$

For $q>p$, we define

$$
\begin{aligned}
\gamma_{2j-1}^{q,p} &:= \mathrm{i}\gamma_{2j-1}^{\frac{q+p}{2},\frac{q+p}{2}}, \quad j=1,\ldots,\frac{q-p}{2}; \\
\gamma_k^{q,p} &:= \gamma_k^{\frac{q+p}{2},\frac{q+p}{2}}, \quad \text{for remaining } k; \\
R_+^{q,p} &:= (\theta_2 \otimes \theta_1)^{\frac{q-p}{4}}, \quad \text{for even } \frac{p-q}{2}, \text{ then } (R_+^{q,p})^2 = (-\mathbb{1})^{\frac{p-q}{4}}; \\
R_+^{q,p} &:= (\theta_2 \otimes \theta_1)^{\frac{p-q+2}{4}} \otimes \theta_2 \otimes \theta_3^p, \quad \text{for odd } \frac{p-q}{2}, \text{ then } (R_+^{q,p})^2 = (-\mathbb{1})^{\frac{p-q-2}{4}}.
\end{aligned}
$$

This ends the proof of the real and quaternionic cases for $q+p$ even.

Next assume that $q+p$ is odd. This time, the case $q+1=p$ is particularly easy. We set

$$
\begin{aligned}
\gamma_{2j-1}^{q,q+1} &:= \theta_3^{\otimes(j-1)} \otimes \theta_1, \\
\gamma_{2j}^{q,q+1} &:= \theta_3^{\otimes(j-1)} \otimes \theta_2, \quad j=1,\ldots,q, \\
\gamma_{2q+1}^{q,q+1} &:= \theta_3^{\otimes q}, \\
R_+^{q,q+1} &:= \mathbb{1}^{\otimes q}.
\end{aligned}
\tag{15.15}
$$

For $q<p-1$, we set

$$
\begin{aligned}
\gamma_{2j}^{q,p} &:= \mathrm{i}\gamma_{2j}^{\frac{q+p-1}{2},\frac{q+p+1}{2}}, \quad j=1,\ldots,\frac{p-q-1}{2}; \\
\gamma_k^{q,p} &:= \gamma_k^{\frac{q+p-1}{2},\frac{q+p+1}{2}}, \quad \text{for remaining } k; \\
R_+^{q,p} &:= (\theta_1 \otimes \theta_2)^{\frac{p-q-1}{4}}, \quad \text{for even } \frac{p-q-1}{2}, \text{ then } (R_+^{q,p})^2 = (-\mathbb{1})^{\frac{p-q-1}{4}}; \\
R_+^{q,p} &\quad \text{does not exist for odd } \frac{p-q-1}{2}.
\end{aligned}
$$

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For $q > p-1$, we define

$$\begin{aligned}
\gamma_{2j-1}^{q,p} &:= \mathrm{i}\gamma_{2j-1}^{\frac{q+p-1}{2},\frac{q+p+1}{2}}, \quad j = 1, \dots, \frac{q-p+1}{2};\\
\gamma_k^{q,p} &:= \gamma_k^{\frac{q+p-1}{2},\frac{q+p+1}{2}}, \quad \text{for remaining } k;\\
R_+^{q,p} &:= (\theta_2 \otimes \theta_1)^{\frac{q-p+1}{4}}, \quad \text{for even } \frac{p-q-1}{2}, \text{ then } (R_+^{q,p})^2 = (-\mathbb{1})^{\frac{p-q-1}{4}};\\
R_+^{q,p} &\quad \text{does not exist for odd } \frac{p-q-1}{2}.
\end{aligned}$$

This ends the proof of the real and quaternionic cases for $q+p$ odd. □

15.3.3 Real spinors

In this subsection we consider real representations of Clifford relations.

Note that if we have a Clifford representation on a real space, then by replacing this space with its complexification we obtain a complex Clifford representation.

Conversely, if we have a Clifford representation on a complex space $\mathcal{V}$ equipped with a charge reversal χ_+ of real type, then we can decompose $\mathcal{V}$ into a direct sum of real subspaces, $\mathcal{V} = \mathcal{V}^{\chi_+} \oplus \mathcal{V}^{-\chi_+}$, where

$$\mathcal{V}^{\chi_+} := \{v \in \mathcal{V} \ : \ \chi_+ v = v\}, \quad \mathcal{V}^{-\chi_+} := \{v \in \mathcal{V} \ : \ \chi_+ v = -v\}.$$

Clearly, we can restrict the representation of Clifford relations to real spaces $\mathcal{V}^{\chi_+}$ and $\mathcal{V}^{-\chi_+}$.

Suppose that $p-q$ equals 0, 1 or 2 modulo 8. Recall that in this case irreducible complex Clifford representations are equipped with a real type charge conjugation. Therefore, there exists a real representation of Clifford relations over $\mathbb{R}^{q,p}$ in $\mathbb{R}^{2^{[n/2]}}$. If γ^π is such a representation, then the real algebra generated by $\gamma^\pi(y)$, $y \in \mathcal{Y}$, equals $\mathbb{R}(2^{[n/2]})$.

Clifford representations possessing a real type charge reversal that appeared in the proof of Thm. 15.19 used complex matrices. It is possible to redefine those representations so that they involve purely real matrices. Such representations are often more complicated than those appearing in the proof of Thm. 15.19. In what follows we will construct such Clifford representations for all real cases of (q,p). They will be generalizations of the *Majorana representation*, well known in physics in the case $(1,3)$.

First recall that for $q = p$ the representation described in (15.14) involved only real matrices. Then we describe real representations with one of q,p equal to zero and the other ≤ 8. First we consider the Euclidean case:

$$\begin{aligned}
\gamma_1^{0,1} &:= \mathbb{1}, \ \gamma_1^{0,2} := \theta_1, \ \gamma_1^{0,8} := \theta_1 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}, \\
\gamma_2^{0,2} &:= \theta_3, \ \gamma_2^{0,8} := \theta_3 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}, \\
\gamma_3^{0,8} &:= \theta_2 \otimes \theta_2 \otimes \theta_1 \otimes \mathbb{1}, \\
\gamma_4^{0,8} &:= \theta_2 \otimes \theta_2 \otimes \theta_3 \otimes \mathbb{1}, \\
\gamma_5^{0,8} &:= \theta_2 \otimes \mathbb{1} \otimes \theta_2 \otimes \theta_1, \\
\gamma_6^{0,8} &:= \theta_2 \otimes \mathbb{1} \otimes \theta_2 \otimes \theta_3, \\
\gamma_7^{0,8} &:= \theta_2 \otimes \theta_1 \otimes \mathbb{1} \otimes \theta_2, \\
\gamma_8^{0,8} &:= \theta_2 \otimes \theta_3 \otimes \mathbb{1} \otimes \theta_2,
\end{aligned}$$

$$\omega^{0,1} := \mathbb{1}, \ \omega^{0,2} := \theta_2, \ \omega^{0,8} := \theta_2 \otimes \theta_2 \otimes \theta_2 \otimes \theta_2.$$

Next we consider the anti-Euclidean case:

$$\begin{array}{lll}
\gamma_1^{6,0} := \theta_2 \otimes \mathbb{1} \otimes \mathbb{1}, & \gamma_1^{7,0} := \theta_2 \otimes \mathbb{1} \otimes \mathbb{1}, & \gamma_1^{8,0} := \theta_2 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}, \\
\gamma_2^{6,0} := \theta_1 \otimes \theta_2 \otimes \mathbb{1}, & \gamma_2^{7,0} := \theta_1 \otimes \theta_2 \otimes \mathbb{1}, & \gamma_2^{8,0} := \theta_3 \otimes \theta_2 \otimes \theta_2 \otimes \theta_2, \\
\gamma_3^{6,0} := \theta_1 \otimes \theta_1 \otimes \theta_2, & \gamma_3^{7,0} := \theta_1 \otimes \theta_1 \otimes \theta_2, & \gamma_3^{8,0} := \theta_3 \otimes \theta_2 \otimes \theta_1 \otimes \mathbb{1}, \\
\gamma_4^{6,0} := \theta_1 \otimes \theta_3 \otimes \theta_2, & \gamma_4^{7,0} := \theta_1 \otimes \theta_3 \otimes \theta_2, & \gamma_4^{8,0} := \theta_3 \otimes \theta_2 \otimes \theta_3 \otimes \mathbb{1}, \\
\gamma_5^{6,0} := \theta_3 \otimes \mathbb{1} \otimes \theta_2, & \gamma_5^{7,0} := \theta_3 \otimes \mathbb{1} \otimes \theta_2, & \gamma_5^{8,0} := \theta_3 \otimes \mathbb{1} \otimes \theta_2 \otimes \theta_1, \\
\gamma_6^{6,0} := \theta_3 \otimes \theta_2 \otimes \theta_1, & \gamma_6^{7,0} := \theta_3 \otimes \theta_2 \otimes \theta_1, & \gamma_6^{8,0} := \theta_3 \otimes \mathbb{1} \otimes \theta_2 \otimes \theta_3, \\
 & \gamma_7^{7,0} := \theta_3 \otimes \theta_2 \otimes \theta_3, & \gamma_7^{8,0} := \theta_3 \otimes \theta_1 \otimes \mathbb{1} \otimes \theta_2, \\
 & & \gamma_8^{8,0} := \theta_3 \otimes \theta_3 \otimes \mathbb{1} \otimes \theta_2,
\end{array}$$

$$\omega^{6,0} := \theta_2 \otimes \mathbb{1} \otimes \mathbb{1}, \ \omega^{7,0} := \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}, \ \omega^{8,0} := \theta_1 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}.$$

Now let us consider a pair $q < p$. Let $p = q + 8r + u$, $0 \leq u < 8$. Clearly, $u = 0, 1$ or 2. Then we set (where we drop the factors of $\mathbb{1}$ tensor multiplied on the right)

$$\begin{aligned}
\gamma_k^{q,p} &:= \gamma_k^{q,q}, \qquad k = 1, \ldots, 2q; \\
\gamma_{2q+8i+j}^{q,p} &:= \omega^{q,q} \otimes (\omega^{0,8})^{\otimes i} \otimes \gamma_j^{0,8}, \quad i = 0, \ldots, r-1, \quad j = 1, \ldots, 8; \\
\gamma_{2q+8r+j}^{q,p} &:= \omega^{q,q} \otimes (\omega^{0,8})^{\otimes r} \otimes \gamma_j^{0,u}, \quad j = 1, \ldots, u; \\
R_+^{q,p} &:= \mathbb{1}^{\otimes 4q} \otimes (R_+^{0,8})^{\otimes r} \otimes R_+^{0,u}.
\end{aligned} \tag{15.16}$$

Similarly, for a pair $q > p$, we write $q = p + 8r + u$, $0 \leq u < 8$. We have $u = 0, 6$ or 7. We set

$$\begin{aligned}
\gamma_k^{q,p} &:= \gamma_k^{p,p}, \qquad k = 1, \ldots, 2p; \\
\gamma_{2p+8i+j}^{q,p} &:= \omega^{p,p} \otimes (\omega^{8,0})^{\otimes i} \otimes \gamma_j^{8,0}, \quad i = 0, \ldots, r-1, \quad j = 1, \ldots, 8; \\
\gamma_{2q+8r+j}^{q,p} &:= \omega^{p,p} \otimes (\omega^{8,0})^{\otimes r} \otimes \gamma_j^{u,0}, \quad j = 1, \ldots, u; \\
R_+^{q,p} &:= \mathbb{1}^{\otimes 4p} \otimes (R_+^{8,0})^{\otimes r} \otimes R_+^{u,0}.
\end{aligned} \tag{15.17}$$

$\square$

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15.3.4 Quaternionic spinors

In this subsection we consider quaternionic representations of Clifford relations.

Recall that a quaternionic vector space, after embedding $\mathbb{C}$ in $\mathbb{H}$, can be interpreted as a complex vector space. Therefore, every Clifford representation on a quaternionic vector space $\mathcal{V}$ can be interpreted as a complex Clifford representation on $\mathcal{V}_{\mathbb{C}}$.

Conversely, if we have a complex Clifford representation with a quaternionic charge reversal χ_+, then setting $\mathrm{j} := \chi_+$ we can consider $\mathcal{V}$ as a vector space over $\mathbb{H}$. The Clifford representation then becomes $\mathbb{H}$-linear.

Suppose that $p-q$ equals 4, 5 or 6 modulo 8. Recall that in this case irreducible complex representations possess a charge conjugation of quaternionic type. Therefore, there exists a quaternionic representation of Clifford relations over $\mathbb{R}^{q,p}$ in $\mathbb{H}^{2^{[n/2]-1}}$. If γ^π is such a representation, then the real algebra generated by $\gamma^\pi(y)$, $y \in \mathcal{Y}$, equals $\mathbb{H}(2^{[n/2]-1})$.

It is instructive to construct representations of Clifford relations for all quaternionic cases of (q,p) by matrices in $\mathbb{H}\big(2^{[n/2]}\big)$.

Note that the matrices $\mathbb{1}$, $\mathrm{i}\theta_1$, θ_2, $\mathrm{i}\theta_3$ can be viewed as the generators of quaternions. Moreover, a real matrix tensored with a quaternion is a quaternionic matrix.

Let us first describe quaternionic representations with one of q,p equal to zero and the other ≤ 8. First we consider the Euclidean case:

$$
\begin{array}{lll}
\gamma_1^{0,4} := \theta_1\otimes\mathbb{1}, & \gamma_1^{0,5} := \theta_1\otimes\mathbb{1}, & \gamma_1^{0,6} := \theta_1\otimes\mathbb{1}\otimes\mathbb{1},\\
\gamma_2^{0,4} := \theta_3\otimes\mathbb{1}, & \gamma_2^{0,5} := \theta_3\otimes\mathbb{1}, & \gamma_2^{0,6} := \theta_3\otimes\mathbb{1}\otimes\mathbb{1},\\
\gamma_3^{0,4} := \theta_2\otimes\mathrm{i}\theta_1, & \gamma_3^{0,5} := \theta_2\otimes\mathrm{i}\theta_1, & \gamma_3^{0,6} := \theta_2\otimes\theta_1\otimes\theta_2,\\
\gamma_4^{0,4} := \theta_2\otimes\mathrm{i}\theta_3, & \gamma_4^{0,5} := \theta_2\otimes\mathrm{i}\theta_3, & \gamma_4^{0,6} := \theta_2\otimes\theta_3\otimes\theta_2,\\
 & \gamma_5^{0,5} := \theta_2\otimes\theta_2, & \gamma_5^{0,6} := \theta_2\otimes\mathbb{1}\otimes\mathrm{i}\theta_1,\\
 & & \gamma_6^{0,6} := \theta_2\otimes\mathbb{1}\otimes\mathrm{i}\theta_3,\\[1ex]
R_+^{0,4} := \mathbb{1}\otimes\theta_2, & R_+^{0,5} := \mathbb{1}\otimes\theta_2, & R_+^{0,6} := \mathbb{1}\otimes\mathbb{1}\otimes\theta_2.
\end{array}
$$

Next we consider the anti-Euclidean case:

$$
\begin{array}{lll}
\gamma_1^{2,0} := \theta_2, & \gamma_1^{3,0} := \theta_2, & \gamma_1^{4,0} := \theta_2\otimes\mathbb{1},\\
\gamma_2^{2,0} := \mathrm{i}\theta_1, & \gamma_2^{3,0} := \mathrm{i}\theta_1, & \gamma_2^{4,0} := \theta_3\otimes\theta_2,\\
 & \gamma_3^{3,0} := \mathrm{i}\theta_3, & \gamma_3^{4,0} := \theta_3\otimes\mathrm{i}\theta_1,\\
 & & \gamma_4^{4,0} := \theta_3\otimes\mathrm{i}\theta_3,\\[1ex]
R_+^{2,0} := \theta_2, & R_+^{3,0} := \theta_2, & R_+^{4,0} := \mathbb{1}\otimes\theta_2.
\end{array}
$$

The case of arbitrary q,p is dealt with as in the case of real spinors; see (15.16) and (15.17). □

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15.3.5 Representations of Clifford relations on pseudo-unitary spaces

Let $\mathcal{V}$ be a finite-dimensional complex vector space and

$$\mathbb{R}^{q,p} \ni y \mapsto \gamma(y) \in L(\mathcal{V}) \tag{15.18}$$

be a Clifford representation. Recall that $\mathcal{V}^*$ denotes the space of anti-linear functionals on $\mathcal{V}$. Clearly,

$$\mathbb{R}^{q,p} \ni y \mapsto \pm\gamma(y)^* \in L(\mathcal{V}^*) \tag{15.19}$$

are also Clifford representations. It is natural to ask when (15.18) and (15.19) are equivalent. The following proposition answers this question for irreducible representations.

Proposition 15.20 *Let (15.18) be irreducible.*

(1) *There exists an invertible $\lambda_+ \in L_{\mathrm{h}}(\mathcal{V}, \mathcal{V}^*)$ such that*

$$\gamma(y)^* = \lambda_+ \gamma(y) \lambda_+^{-1}$$

iff p is odd or q is even.

(2) *There exists an invertible $\lambda_- \in L_{\mathrm{h}}(\mathcal{V}, \mathcal{V}^*)$ such that*

$$-\gamma(y)^* = \lambda_- \gamma(y) \lambda_-^{-1}$$

iff q is odd or p is even.

Proof Let $\gamma_1, \dots, \gamma_n$ be an irreducible Clifford representation in the canonical basis of $\mathbb{R}^{q,p}$. Then writing $\gamma_j = \mathrm{i}\phi_j$, $j = 1, \dots, q$ and $\gamma_j = \phi_j$, $j = q+1, \dots, n$, we obtain an irreducible Clifford representation over $\mathbb{R}^n$, $\phi_1, \dots, \phi_n$. On the space $\mathcal{V}$ we can fix a scalar product such that $\phi_i = \phi_i^*$, so that we obtain a CAR representation. This scalar product allows us to identify the space $\mathcal{V}$ with $\mathcal{V}^*$.

Obviously, $\gamma_j^* = -\gamma_j$, $j = 1, \dots, q$, and $\gamma_j^* = \gamma_j$, $j = q+1, \dots, n$.

Now set

$$\begin{aligned}
\lambda_+ &:= \pm\mathrm{i}^{q/2}\gamma_1 \cdots \gamma_q, && \text{even } q;\\
\lambda_- &:= \pm\mathrm{i}^{(q+1)/2}\gamma_1 \cdots \gamma_q, && \text{odd } q;\\
\lambda_- &:= \pm\mathrm{i}^{p/2}\gamma_{q+1} \cdots \gamma_n, && \text{even } p;\\
\lambda_+ &:= \pm\mathrm{i}^{(p-1)/2}\gamma_{q+1} \cdots \gamma_n, && \text{odd } p.
\end{aligned}$$

We check that $\lambda_\pm^* = \lambda_\pm$, $\lambda_\pm^2 = \mathbb{1}$ and $\lambda_\pm \gamma_i = \pm\gamma_i^* \lambda_\pm$.

Note that if n is odd, then we obtain two distinct formulas for λ_+ or λ_-. Using (15.12), we easily see that they define the same operator. □

If the assumptions of Prop. 15.20 (1) are satisfied, so that λ_+ exists, we endow the space $\mathcal{V}$ with a non-degenerate Hermitian form

$$\mathcal{V} \times \mathcal{V} \ni (v_1, v_2) \mapsto \overline{v}_1 \cdot \lambda_+ v_2.$$

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Definition 15.21 *For every $A \in L(\mathcal{V})$, we define its λ_+-adjoint, denoted $A^\dagger$, by*

$$\overline{v}_1 \cdot \lambda_+ A v_2 = \overline{A^\dagger v_1} \cdot \lambda_+ v_2.$$

We have

$$\gamma(y)^\dagger = \gamma(y), \quad y \in \mathcal{Y}. \tag{15.20}$$

If $\pi : \mathrm{Cliff}(\mathbb{R}^{q,p}) \to L(\mathcal{V})$ is a representation, then

$$\pi(A)^\dagger = \pi(A^\dagger), \ A \in \mathrm{Cliff}(\mathbb{R}^{q,p}). \tag{15.21}$$

If we replace λ_+ with λ_-, then instead of (15.20) we have

$$\gamma(y)^\dagger = -\gamma(y), \quad y \in \mathcal{Y}.$$

Instead of (15.21), we have:

$$\pi(A)^\dagger = \pi(A^\dagger), \ A \in \mathrm{Cliff}_0(\mathbb{R}^{q,p}). \tag{15.22}$$

15.4 Clifford algebras over $\mathbb{R}^{q,p}$

In this section we continue to study Clifford relations over $\mathbb{R}^{q,p}$. We adopt the representation-independent point of view: we concentrate on the Clifford algebra $\mathrm{Cliff}(\mathbb{R}^{q,p})$.

For $n = 0, 1, 2$, $\mathrm{Cliff}(\mathbb{R}^{n,0})$ are division algebras. In fact, $\mathrm{Cliff}(\mathbb{R}^{0,0}) = \mathbb{R}$, $\mathrm{Cliff}(\mathbb{R}^{1,0}) = \mathbb{C}$ and $\mathrm{Cliff}(\mathbb{R}^{2,0}) = \mathbb{H}$.

15.4.1 Form of Clifford algebras for a general signature

Let q, p be arbitrary non-negative integers, $n = q + p$ and $m := [(q + p)/2]$. Let us consider the real algebra $\mathrm{Cliff}(\mathbb{R}^{q,p})$.

We have the following counterpart of Def. 15.14:

Definition 15.22 *We will write $\gamma_i := \gamma(e_i)$, where e_i is the canonical basis of $\mathbb{R}^{q,p}$. The* volume element *of $\mathrm{Cliff}(\mathbb{R}^{q,p})$ will be denoted by*

$$\omega = \gamma_1 \cdots \gamma_n. \tag{15.23}$$

Remark 15.23 *In the case $n = 4$ with the Lorentz signature, particle physicists often denote the operator ω by γ_5. This notation is so popular that it is sometimes used in the case of a dimension different from 4.*

It is possible to describe $\mathrm{Cliff}(\mathbb{R}^{q,p})$ for an arbitrary q, p. Table 15.2, a well-known table of real Clifford algebras, should be compared with the analogous table for the complex case (see Table 15.1, Subsect. 15.1.3).

In the case of n odd all the algebras $\mathrm{Cliff}(\mathbb{R}^{q,p})$ have a non-trivial center spanned by $\mathbb{1}, \omega$.

If $\omega^2 = \mathbb{1}$, which corresponds to cases 1 and 5, $\mathrm{Cliff}(\mathbb{R}^{q,p})$ splits into a direct sum and $\omega \simeq \mathbb{1} \oplus (-\mathbb{1})$.

Table 15.2 *Form of* $\mathrm{Cliff}(\mathbb{R}^{q,p})$

$p-q \pmod 8$	ω^2	$\mathrm{Cliff}_0(\mathbb{R}^{q,p})$	$\mathrm{Cliff}(\mathbb{R}^{q,p})$
0	$\mathbb{1}$	$\mathbb{C}(2^{m-1})$	$\mathbb{R}(2^m)$
1	$\mathbb{1}$	$\mathbb{R}(2^m)$	$\mathbb{R}(2^m) \oplus \mathbb{R}(2^m)$
2	$-\mathbb{1}$	$\mathbb{R}(2^{m-1}) \oplus \mathbb{R}(2^{m-1})$	$\mathbb{R}(2^m)$
3	$-\mathbb{1}$	$\mathbb{R}(2^m)$	$\mathbb{C}(2^m)$
4	$\mathbb{1}$	$\mathbb{C}(2^{m-1})$	$\mathbb{H}(2^{m-1})$
5	$\mathbb{1}$	$\mathbb{H}(2^{m-1})$	$\mathbb{H}(2^{m-1}) \oplus \mathbb{H}(2^{m-1})$
6	$-\mathbb{1}$	$\mathbb{H}(2^{m-2}) \oplus \mathbb{H}(2^{m-2})$	$\mathbb{H}(2^m)$
7	$-\mathbb{1}$	$\mathbb{H}(2^{m-1})$	$\mathbb{C}(2^m)$

If $\omega^2 = -\mathbb{1}$, which corresponds to cases 3 and 7, the algebras are complex and $\omega = \mathrm{i}\mathbb{1}$.

In the case $p - q \equiv 0, 1, 2 \pmod 8$, $\mathrm{Cliff}(\mathbb{R}^{q,p})$ can be represented as real matrices, which will correspond to the real type in Thm. 15.19. In the case $p - q \equiv 4, 5, 6 \pmod 8$, $\mathrm{Cliff}(\mathbb{R}^{q,p})$ can be represented as quaternionic matrices, which corresponds to the quaternionic type in Thm. 15.19.

$\mathbb{C} \otimes \mathrm{Cliff}(\mathbb{R}^{q,p})$ coincides with the algebra $\mathrm{Cliff}(\mathbb{C}^n)$. In addition, it is equipped with a unique complex conjugation such that $\mathrm{Cliff}(\mathbb{R}^{q,p})$ consists of elements in $\mathbb{C} \otimes \mathrm{Cliff}(\mathbb{R}^{q,p})$ fixed by this conjugation.

There exists a unique isomorphism of complex algebras $\rho : \mathbb{C} \otimes \mathrm{Cliff}(\mathbb{R}^{q,p}) \to \mathbb{C} \otimes \mathrm{Cliff}(\mathbb{R}^{p,q})$ satisfying

$$\rho(\gamma(y)) = \mathrm{i}\gamma(y), \quad y \in \mathcal{Y}. \tag{15.24}$$

(Note that on the left $\gamma(y)$ is an element of $\mathbb{C} \otimes \mathrm{Cliff}(\mathbb{R}^{q,p})$, and on the right of $\mathbb{C} \otimes \mathrm{Cliff}(\mathbb{R}^{p,q})$.) Under this isomorphism we have

$$\begin{aligned} \rho\big(\mathrm{Cliff}_0(\mathbb{R}^{q,p})\big) &= \mathrm{Cliff}_0(\mathbb{R}^{p,q}), \\ \rho\big(\mathrm{Cliff}_1(\mathbb{R}^{q,p})\big) &= \mathrm{i}\mathrm{Cliff}_1(\mathbb{R}^{p,q}). \end{aligned}$$

15.4.2 Pseudo-Euclidean group

Recall that we can define the group $O(\mathbb{R}^{q,p})$ of linear transformations that preserve the form (15.8). Obviously, we have a natural isomorphism $O(\mathbb{R}^{q,p}) \simeq O(\mathbb{R}^{p,q})$. The determinant defines a homomorphism of $O(\mathbb{R}^{q,p})$ into $\{1, -1\}$. Elements of $O(\mathbb{R}^{q,p})$ with the determinant 1 form a subgroup $SO(\mathbb{R}^{q,p}) \simeq SO(\mathbb{R}^{p,q})$. We have the exact sequence

$$1 \to SO(\mathbb{R}^{q,p}) \to O(\mathbb{R}^{q,p}) \to \mathbb{Z}_2 \to 1. \tag{15.25}$$

Definition 15.24 *For any $r \in O(\mathbb{R}^{q,p})$,*

$$\hat{r}\big(\gamma(y)\big) = \gamma(ry), \quad y \in \mathcal{Y},$$

defines a unique automorphism $\hat{r}$ of $\mathrm{Cliff}(\mathbb{R}^{q,p})$.

We have a homomorphism

$$O(\mathbb{R}^{q,p}) \ni r \mapsto \hat{r} \in \mathrm{Aut}\big(\mathrm{Cliff}(\mathbb{R}^{q,p})\big).$$

15.4.3 Pin group for a general signature

Definition 15.25 *We define $Pin(\mathbb{R}^{q,p})$ as the set of all $U \in \mathrm{Cliff}(\mathbb{R}^{q,p})$ such that $UU^\dagger = \mathbb{1}$ or $UU^\dagger = -\mathbb{1}$, and*

$$\big\{U\gamma(y)U^{-1} \ : \ y \in \mathcal{Y}\big\} = \big\{\gamma(y) \ : \ y \in \mathcal{Y}\big\}.$$

We set

$$Spin(\mathbb{R}^{q,p}) := Pin(\mathbb{R}^{q,p}) \cap \mathrm{Cliff}_0(\mathbb{R}^{q,p}).$$

Proposition 15.26 *Let $U \in Pin(\mathbb{R}^{q,p})$. Then there exists a unique $r \in O(\mathbb{R}^{q,p})$ such that*

$$U\gamma(y)U^{-1} = \det(r)\gamma(ry), \quad y \in \mathcal{Y}. \tag{15.26}$$

The map $Pin(\mathbb{R}^{q,p}) \to O(\mathbb{R}^{q,p})$ obtained this way is a surjective homomorphism of groups.

Definition 15.27 *If (15.26) is satisfied, we say that U* det-implements *r.*

Theorem 15.28 *Let $r \in O(\mathbb{R}^{q,p})$.*

(1) *The set of elements of* $\mathrm{Cliff}(\mathbb{R}^{q,p})$ *det-implementing r consists of a pair of operators differing by sign, $\pm U_r = \{U_r, -U_r\}$.*
(2) *$r \in SO(\mathbb{R}^{q,p})$ iff U_r is even; $r \in O(\mathbb{R}^{q,p})\backslash SO(\mathbb{R}^{q,p})$ iff U_r is odd.*
(3) *If $r_1, r_2 \in O(\mathbb{R}^{q,p})$, then $U_{r_1}U_{r_2} = \pm U_{r_1 r_2}$.*

The above statements can be summarized by the following commuting diagram of Lie groups and their continuous homomorphisms, where all vertical and horizontal sequences are exact:

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \to & \mathbb{Z}_2 & \to & \mathbb{Z}_2 & \to & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & Spin(\mathbb{R}^{q,p}) & \to & Pin(\mathbb{R}^{q,p}) & \to & \mathbb{Z}_2 & \to & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & SO(\mathbb{R}^{q,p}) & \to & O(\mathbb{R}^{q,p}) & \to & \mathbb{Z}_2 & \to & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array} \tag{15.27}$$

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Moreover, $Spin(\mathbb{R}^{q,p})$ coincides with $Spin(\mathbb{R}^{p,q})$ in the sense that if $U_r^{q,p} \in \mathrm{Cliff}_0(\mathbb{R}^{q,p})$ and $U_r^{p,q} \in \mathrm{Cliff}_0(\mathbb{R}^{p,q})$ both implement $r \in O(\mathbb{R}^{q,p}) = O(\mathbb{R}^{p,q})$, then $U_r^{q,p} = \pm U_r^{p,q}$, where we use the isomorphism described at the end of Subsect. 15.4.1.

15.5 Notes

The so-called spinor representations of orthogonal groups were studied by Cartan (1938) and Brauer–Weyl (1935).

In quantum physics, Clifford relations and spinor representations appear in the description of spin $\frac{1}{2}$ particles. In the non-relativistic case, where the group $Spin(3) \simeq SU(2)$ replaces the group of rotations $SO(3)$, this is due to Pauli (1927). In the relativistic case, where the group $Spin^{\uparrow}(1,3) \simeq SL(2,\mathbb{C})$ replaces the Lorentz group, this is due to Dirac (1928).

Introductions to Clifford algebras can be found in Lawson–Michelson (1989) and Trautman (2006).