

# Models in Which Every Nonmeager Set is Nonmeager in a Nowhere Dense Cantor Set

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*Abstract.* We prove that it is relatively consistent with ZFC that in any perfect Polish space, for every nonmeager set  $A$  there exists a nowhere dense Cantor set  $C$  such that  $A \cap C$  is nonmeager in  $C$ . We also examine variants of this result and establish a measure theoretic analog.

## 1 Introduction

Our starting point is the following question of Laczkovich:

Does there exist (in ZFC) a nonmeager set that is relatively meager in every nowhere dense perfect set?

Note that the continuum hypothesis implies the existence of a Luzin set, *i.e.*, an uncountable set of reals which meets every nowhere dense set in a countable set. Hence, we can think of Laczkovich's question as asking whether one can construct a particular weak version of a Luzin set without any extra set theoretic assumptions.

Recall that a space  $X$  is Polish iff it is completely metrizable and separable. A subset of  $X$  is nowhere dense iff its closure has no interior and it is meager iff it is the countable union of nowhere dense sets. A subset of  $X$  is residual iff it is the complement of a meager set. A perfect set in a Polish space is a closed nonempty set without isolated points, and a Polish space is said to be perfect if it is nonempty and has no isolated points. As we shall see, the underlying space in the question of Laczkovich can be taken to be any perfect Polish space. If we ask, as is quite natural, for the nowhere dense perfect sets in the statement to be Cantor sets (*i.e.*, sets homeomorphic to the Cantor middle third set), then we do not know whether the nature of the Polish space matters. Even for various standard incarnations of the reals (the real line, the Baire space, and so on), we have only partial results on their equivalence in this context. We answered Laczkovich's question for the Cantor set in 1997 by building a model where the answer is negative. (And of course the perfect nowhere dense sets in this case are necessarily Cantor sets.) Very shortly afterwards, we noticed the more elegant solution presented here which uses a slightly stronger variant of a statement proven consistent by Shelah in [Sh1980]. We show in Section 3 that the stronger conclusion, in which for any perfect Polish space the perfect nowhere dense sets can be taken to be Cantor sets, follows from yet another variant on the same statement. The

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proof of the consistency of the variants in question is similar to the proof of Shelah. Unfortunately, the proof is quite technical and the argument in [Sh1980] is only a brief sketch, so we give the argument in some detail in Section 4 in order to be clear. An alternative model for the negative answer to Laczkovich's question for the Cantor set is provided by a paper of Ciesielski and Shelah [CS]. See Remark 3.6. In the final section of the paper, we show how a measure theoretic version of our results can be deduced from results in Roslanowski and Shelah [RS]. The authors thank Ilijas Farah for helpful discussions concerning the models constructed in [RS].

For a Polish space  $X$ , write  $\text{perfect}(X)$  to mean that for every nonmeager set  $A \subseteq X$  there is a nowhere dense perfect set  $P \subseteq X$  such that  $A \cap P$  is nonmeager relative to  $P$ . Write  $\text{cantor}(X)$  if moreover  $P$  can be taken to be a Cantor set. Note that  $\text{perfect}(X)$  and  $\text{cantor}(X)$  are trivially equivalent in spaces in which nowhere dense perfect sets are necessarily Cantor sets, e.g.,  $2^\omega$  and  $\mathbb{R}$ .

We recall for emphasis the following well-known elementary fact of which we will make frequent use without mention.

**Proposition 1.1** *If  $X$  is a topological space and  $Y$  is a dense subspace of  $X$ , then for any  $A \subseteq Y$ ,  $A$  is nowhere dense in  $Y$  if and only if  $A$  is nowhere dense in  $X$ . Similarly,  $A$  is meager in  $Y$  if and only if  $A$  is meager in  $X$ . ■*

## 2 Relationships Between Various Polish Spaces

We begin by showing that for any two perfect Polish spaces  $X$  and  $Y$ ,  $\text{perfect}(X)$  and  $\text{perfect}(Y)$  are equivalent.

**Proposition 2.1** *We have the following implications.*

- (a) *Suppose  $X$  is a perfect Polish space and  $\text{perfect}(X)$  holds. Then  $\text{perfect}(\omega^\omega)$  holds.*
- (b) *For every perfect Polish space  $X$ ,  $\text{perfect}(\omega^\omega)$  implies  $\text{perfect}(X)$ .*

**Proof** We will use the well-known fact that every perfect Polish space  $X$  has a dense  $G_\delta$  subset  $Y$  homeomorphic to  $\omega^\omega$ . (To get  $Y$ , first remove the boundaries of the elements of a countable base for  $X$ . What remains is a zero-dimensional dense  $G_\delta$ . Remove a countable dense subset of this dense  $G_\delta$  and call the result  $Y$ . Then  $Y$  is a perfect Polish space which is zero-dimensional and has no compact open sets and hence is homeomorphic to  $\omega^\omega$ .)

(a) Let  $Y$  be a residual subspace of  $X$  homeomorphic to  $\omega^\omega$ . Let  $A$  be a nonmeager set in  $Y$ . In  $X$ ,  $A$  is nonmeager so there is a nowhere dense perfect set  $C$  so that  $A$  is nonmeager in  $C$ . By replacing  $C$  by the closure of one of its nonempty open subsets, we may assume that  $A$  is everywhere nonmeager in  $C$ . In particular,  $A \cap C$  is dense in  $C$ . Note that  $F = Y \cap C$  is closed relative to  $Y$ , is nonempty and has no isolated points (because it contains  $A \cap C$  which is dense in  $C$ ). Since  $F$  is dense in  $C$ ,  $A \cap C = (A \cap Y) \cap C = A \cap F$  is not meager in  $F$ . Also, because  $Y$  is dense in  $X$  and  $F$  is nowhere dense in  $X$ ,  $F$  is also nowhere dense in  $Y$ .

(b) Let  $X$  be a perfect Polish space. Let  $Y$  be a residual subspace of  $X$  homeomorphic to  $\omega^\omega$ . Let  $A \subseteq X$  be nonmeager. Then  $A \cap Y$  is nonmeager in  $X$  and hence in  $Y$  as well since  $Y$  is dense. By  $\text{perfect}(\omega^\omega)$ , there is a nowhere dense perfect set  $C$  in  $Y$

such that  $A \cap C$  is nonmeager in  $C$ . If  $P$  denotes the closure of  $C$  in  $X$ , then, since  $C$  is dense in  $P$ ,  $A \cap C$  is nonmeager in  $P$  and hence  $A \cap P$  is also nonmeager in  $P$ .  $P$  is perfect since it is the closure of a nonempty set without isolated points.  $P$  is nowhere dense since it is the closure of a set which is nowhere dense in  $Y$  and hence in  $X$  as well. ■

Part (b) holds for  $\text{cantor}(\cdot)$  by an easier argument.

**Proposition 2.2** For every perfect Polish space  $X$ ,  $\text{cantor}(\omega^\omega)$  implies  $\text{cantor}(X)$ .

**Proof** Similar to the proof of Proposition 2.1(b), except that this time the proof yields a nowhere dense Cantor set  $C \subseteq Y$  such that  $A \cap C$  is nonmeager in  $C$  and then we are done. ■

We do not know whether (a) holds for  $\text{cantor}(\cdot)$ .

**Problem 2.3** Does  $\text{cantor}(2^\omega)$  imply  $\text{cantor}(\omega^\omega)$ ?

**Problem 2.4** Does  $\text{cantor}([0, 1])$  imply  $\text{cantor}([0, 1] \times [0, 1])$ ?

Of course,  $\text{cantor}([0, 1]) \equiv \text{perfect}([0, 1]) \equiv \text{perfect}(2^\omega) \equiv \text{cantor}(2^\omega)$ , so these two questions have equivalent hypotheses.

We introduce one more version of  $\text{perfect}(X)$  based on the following observation. Suppose that  $\text{perfect}(\omega^\omega)$  holds. Then for any nonmeager set  $A$ , we have a nowhere dense perfect set  $P$  such that  $A \cap P$  is nonmeager in  $P$ . Replacing  $P$  by the closure of one of its open sets, we may assume that  $A \cap P$  is everywhere nonmeager in  $P$ . Then if  $P$  has a compact open subset  $U$ , then  $U$  is a Cantor set and  $A \cap U$  is nonmeager in  $U$ . Otherwise,  $P$  itself is homeomorphic to  $\omega^\omega$ . Hence, the perfect set  $P$  in the conclusion of  $\text{perfect}(\omega^\omega)$  can always be taken to be either a closed nowhere dense copy of  $\omega^\omega$  or a Cantor set. Let  $\text{baire}(X)$  be the strengthening of  $\text{perfect}(X)$  in which we require that the perfect nowhere dense sets in the definition be homeomorphic to the Baire space  $\omega^\omega$ . Of course a Polish space need not contain any closed copies of  $\omega^\omega$ , so  $\text{baire}(X)$  can fail. However, when  $X = \omega^\omega$  it would seem reasonable that  $\text{baire}(X)$  might hold, and we will show in the next section that  $\text{baire}(\omega^\omega)$  is indeed consistent. Its relationship to  $\text{cantor}(\omega^\omega)$  is unclear to us.

**Problem 2.5** (a) Does  $\text{perfect}(\omega^\omega)$  imply that one of  $\text{baire}(\omega^\omega)$  or  $\text{cantor}(\omega^\omega)$  must hold? (b) Does either of  $\text{baire}(\omega^\omega)$  or  $\text{cantor}(\omega^\omega)$  imply the other?

### 3 Consistency Results

We now turn to the proof of the consistency of  $\text{cantor}(\omega^\omega)$  and  $\text{baire}(\omega^\omega)$ . We need a variation on the following result which forms part of the proof of [Sh1980, Theorem 4.7], which states that if ZFC is consistent, then so is  $\text{ZFC} + 2^\omega = \omega_2 + \text{“There is a universal (linear) order of power } \omega_1\text{”}$ .

**Theorem 3.1** *If ZFC is consistent, then so is ZFC + both of the following statements.*

- (a) *There is a nonmeager set in  $\mathbb{R}$  of cardinality  $\omega_1$ .*
- (b) *Let  $A$  and  $B$  be everywhere nonmeager subsets of  $\mathbb{R}$  of cardinality  $\omega_1$ . Then  $A$  and  $B$  are order-isomorphic.*

We shall need the following variant of this result.

**Theorem 3.2** *If ZFC is consistent, then so is ZFC + both of the following statements.*

- (a') *Every nonmeager set in  $\mathbb{R}$  has a nonmeager subset of cardinality  $\omega_1$ .*
- (b') *Let  $A$  and  $B$  be everywhere nonmeager subsets of  $\mathbb{R}$  of cardinality  $\omega_1$ . Suppose we are given countable dense subsets  $A_0 \subseteq A$  and  $B_0 \subseteq B$ . Then  $A$  and  $B$  are order-isomorphic by an order isomorphism taking  $A_0$  isomorphically to  $B_0$ .*

**Problem 3.3** In the presence of (a), does (b) imply (b')?

We shall in fact verify in Theorem 4.9 that in (b') we can even ask that given pairwise disjoint countable dense subsets  $A_i$ ,  $i < \omega$ , of  $A$  and pairwise disjoint countable dense subsets  $B_i$ ,  $i < \omega$ , of  $B$ , the order-isomorphism of  $A$  and  $B$  takes  $A_i$  isomorphically to  $B_i$  for each  $i < \omega$ . As explained in the introduction, the proof is similar to the one in [Sh1980], but as the proof is quite technical and the argument in [Sh1980] is only a brief sketch, we need to give the argument in some detail in order to be clear. We do that in the next section. Here, we derive the consequences of interest to us for this paper. The definition of  $\text{baire}(X)$  is given at the end of Section 2.

**Theorem 3.4** *Assume (a') and (b'). Then  $\text{cantor}(\omega^\omega)$  and  $\text{baire}(\omega^\omega)$  both hold.*

**Proof** We will use the following elementary fact.

**Fact 3.5** *If  $K, L \subseteq \mathbb{R}$  are dense and  $h: K \rightarrow L$  is an order isomorphism, then  $h$  extends to an order isomorphism of  $\mathbb{R}$ .*

Suppose that  $A \subseteq \mathbb{R} \setminus \mathbb{Q}$  is not meager. We wish to find a Cantor set  $C \subseteq \mathbb{R} \setminus \mathbb{Q}$  such that  $A \cap C$  is nonmeager relative to  $C$ . By (a'), we may assume that  $A$  has cardinality exactly  $\omega_1$ .  $A$  is everywhere nonmeager in some open interval  $(a, b)$ . Let  $C \subseteq (a, b) \setminus \mathbb{Q}$  be a Cantor set, and, by (a'), let  $B \subseteq C$  be a set of cardinality  $\omega_1$  which is nonmeager relative to  $C$ . Then  $(A \cup \mathbb{Q}) \cap (a, b)$  and  $(A \cup B \cup \mathbb{Q}) \cap (a, b)$  are both everywhere nonmeager in  $(a, b)$  and both have cardinality  $\omega_1$ . By (b'), there is an order-isomorphism  $h: (A \cup \mathbb{Q}) \cap (a, b) \rightarrow (A \cup B \cup \mathbb{Q}) \cap (a, b)$  such that  $h[\mathbb{Q} \cap (a, b)] = \mathbb{Q} \cap (a, b)$ . Extend  $h$  to  $(a, b)$  and denote the extension also by  $h$ . Since  $h$  is a homeomorphism,  $h^{-1}[C]$  is a Cantor set and  $h^{-1}[B]$  is nonmeager relative to  $h^{-1}[C]$ . Since  $h^{-1}[C] \subseteq \mathbb{R} \setminus \mathbb{Q}$  and  $h^{-1}[B] \subseteq A$ , we are done.

To get  $\text{baire}(\omega^\omega)$ , we make a different choice of  $C$  in the preceding argument. This time, choose  $C$  to be any Cantor set so that  $C \cap \mathbb{Q}$  is dense in  $C$ . Then  $h^{-1}[C]$  will have the same property, so  $h^{-1}[C \setminus \mathbb{Q}] = h^{-1}[C] \setminus \mathbb{Q}$  is closed nowhere dense in  $\mathbb{R} \setminus \mathbb{Q}$  and homeomorphic to  $\omega^\omega$ . (Also, this time choose  $B \subseteq C \setminus \mathbb{Q}$ .) ■

**Remark 3.6** The reader can easily verify that a similar but simpler argument yields that (a') and (b) imply  $\text{perfect}(\mathbb{R})$ . An alternative proof of the consistency of  $\text{perfect}(\mathbb{R})$  can be had by using Theorem 2 of [CS], which states that the following statement is consistent relative to ZFC:

For every  $A \subseteq 2^\omega \times 2^\omega$  for which the sets  $A$  and  $A^c = (2^\omega \times 2^\omega \setminus A)$  are nowhere meager in  $2^\omega \times 2^\omega$ , there is a homeomorphism  $f: 2^\omega \rightarrow 2^\omega$  such that the set  $\{x \in 2^\omega : (x, f(x)) \in A\}$  does not have the Baire property in  $2^\omega$ .

(A set has the Baire property if it has the form  $U \Delta M$  where  $U$  is open and  $M$  is meager.) Note that the map  $2^\omega \rightarrow 2^\omega$  given by  $x \mapsto (x, f(x))$  is a homeomorphism. Hence the conclusion could be stated as “ $f \cap A$  does not have the Baire property in  $f$ ”. Since  $2^\omega \times 2^\omega$  is homeomorphic to  $2^\omega$  and the graph of a homeomorphism of  $2^\omega$  is a perfect nowhere dense set in  $2^\omega \times 2^\omega$ , the statement above implies the following special case of  $\text{perfect}(2^\omega)$  (which is equivalent to  $\text{perfect}(\mathbb{R})$ ).

For every  $A \subseteq 2^\omega$  for which the sets  $A$  and  $A^c$  are both nowhere meager in  $2^\omega$ , there is a perfect nowhere dense set  $P$  such that the set  $A \cap P$  is not meager in  $P$ .

To reduce  $\text{perfect}(2^\omega)$  to this special case, consider a nonmeager set  $A \subseteq 2^\omega$ .  $A$  is everywhere nonmeager in some clopen set  $U$ . If  $A$  is comeager in some clopen set, then it contains a nowhere dense perfect set and we are done. Hence we may assume that, relative to  $U$ ,  $A$  and  $A^c$  are both everywhere nonmeager. The clopen set  $U$  is homeomorphic to  $2^\omega$ , so we now find ourselves in the special case described above.

#### 4 Order-Isomorphisms of Everywhere Nonmeager Sets

We now turn to the proof of the consistency of (a') and (b'). We begin by recalling the basic properties of oracle-cc forcing. See [Sh1998, Chapter IV] for the details. A version of this material is also explained in [Bu, Sections 4–6].

**Definition 4.1** A sequence

$$\overline{M} = \langle M_\delta : \delta \text{ is a limit ordinal } < \omega_1 \rangle$$

is called an *oracle* if each  $M_\delta$  is a countable transitive model of a sufficiently large fragment of ZFC,  $\delta \in M_\delta$  and for each  $A \subseteq \omega_1$ ,  $\{\delta : A \cap \delta \in M_\delta\}$  is stationary in  $\omega_1$ .

The meaning of “sufficiently large” depends on the context. In a particular proof, some fragment of ZFC for which models can be produced in ZFC must suffice for all the oracles in the proof. The existence of an oracle is equivalent to  $\diamond$ , (see [Ku, Theorem II 7.14]) and hence implies CH. We limit the definition of the  $\overline{M}$ -chain condition to partial orders of cardinality  $\omega_1$ . This covers our present needs.

Associated with an oracle  $\overline{M}$ , there is a filter  $\text{Trap } \overline{M}$  generated by the sets

$$\{\delta < \omega_1 : \delta \text{ is a limit ordinal and } A \cap \delta \in M_\delta\}, \quad A \subseteq \omega_1.$$

This is a proper normal filter containing all closed unbounded sets.

**Definition 4.2** If  $P$  is any partial order,  $P' \subseteq P$ , and  $\mathfrak{D}$  is any class of sets, then we write  $P' <_{\mathfrak{D}} P$  to mean that every predense subset of  $P'$  which belongs to  $\mathfrak{D}$  is predense in  $P$ .

**Definition 4.3** A partial order  $P$  satisfies the  $\overline{M}$ -chain condition, or simply is  $\overline{M}$ -cc, if there is a one-to-one function  $f: P \rightarrow \omega_1$  such that

$$\{\delta < \omega_1 : \delta \text{ is a limit ordinal and } f^{-1}(\delta) <_{M_{\delta,f}} P\}$$

belongs to  $\text{Trap } \overline{M}$ , where  $M_{\delta,f} = \{f^{-1}(A) : A \subseteq \delta, A \in M_{\delta}\}$ .

It is not hard to verify that if  $P$  is  $\overline{M}$ -cc, then  $P$  is ccc, i.e., all antichains in  $P$  are countable. Also, any one-to-one function  $g: P \rightarrow \omega_1$  can replace  $f$  in the definition.

**Proposition 4.4** The  $\overline{M}$ -cc satisfies the following properties.

- (1) If  $\alpha < \omega_2$  is a limit ordinal,  $\langle \langle P_{\beta} \rangle_{\beta < \alpha}, \langle \dot{Q}_{\beta} \rangle_{\beta < \alpha} \rangle$  is a finite-support  $\alpha$ -stage iteration of partial orders, and for each  $\beta < \alpha$ ,  $P_{\beta}$  is  $\overline{M}$ -cc, then  $P_{\alpha}$  is  $\overline{M}$ -cc.
- (2) If  $P$  is  $\overline{M}$ -cc, then there is a  $P$ -name  $\overline{M}^*$  for an oracle such that for each  $P$ -name  $\dot{Q}$  for a partial order, if  $\Vdash_P$  “ $\dot{Q}$  is  $\overline{M}^*$ -cc” then  $P * \dot{Q}$  is  $\overline{M}$ -cc.
- (3) If  $\overline{M}_{\alpha}$ ,  $\alpha < \omega_1$ , are oracles, then there is an oracle  $\overline{M}$  such that for any partial order  $P$ , if  $P$  is  $\overline{M}_{\alpha}$ -cc, then  $P$  is  $\overline{M}$ -cc for all  $\alpha < \omega_1$ .

We will need the following lemmas.

**Lemma 4.5** Let  $\overline{M} = \langle M_{\delta} : \delta < \omega_1 \rangle$  be an oracle and let  $A$  and  $B$  be everywhere non-meager subsets of  $\mathbb{R}$ . Suppose we are given pairwise disjoint countable dense subsets  $A_i$ ,  $i < \omega$ , of  $A$  and pairwise disjoint countable dense subsets  $B_i$ ,  $i < \omega$ , of  $B$ . Then there is a forcing notion  $P$  satisfying the  $\overline{M}$ -cc such that for every  $G \subseteq P$  generic over  $V$ ,  $V[G] \models A$  and  $B$  are order-isomorphic by an order isomorphism taking  $A_i$  isomorphically to  $B_i$  for each  $i < \omega$ .

**Proof** Fix well-orderings of  $A$  and  $B$  in type  $\omega_1$ . (CH holds because there is an oracle.) We will inductively define one-to-one enumerations  $\langle a_{\alpha} : \alpha < \omega_1 \rangle$  of  $A$  and  $\langle b_{\alpha} : \alpha < \omega_1 \rangle$  of  $B$  and functions  $f_{\delta}$ ,  $\delta < \omega_1$ . We let  $A_{\delta} = \{a_{\alpha} : \omega\delta \leq \alpha < \omega(\delta + 1)\}$  and  $B_{\delta} = \{b_{\alpha} : \omega\delta \leq \alpha < \omega(\delta + 1)\}$  for  $\delta < \omega_1$ . For  $A' \subseteq A$  and  $B' \subseteq B$ , let  $P(A', B')$  denote the set of finite partial order-preserving maps  $p: A' \rightarrow B'$  such that  $p[A_{\delta}] \subseteq B_{\delta}$  for all  $\delta < \omega_1$ . We also use the notation

$$A \upharpoonright \alpha = \{a_{\beta} : \beta < \alpha\}, \quad B \upharpoonright \alpha = \{b_{\beta} : \beta < \alpha\}.$$

We will arrange that the following conditions hold.

- (1) The sets  $A_{\delta}$  and  $B_{\delta}$  are dense in  $\mathbb{R}$ .
- (2) For  $\delta < \omega$ , the sets  $A_{\delta}$  and  $B_{\delta}$  are as in the hypothesis.
- (3) For each  $\delta < \omega_1$ ,  $f_{\delta}$  is a bijective map of  $P(A \upharpoonright \omega\delta, B \upharpoonright \omega\delta)$  onto  $\omega\delta$ .
- (4) For each  $\delta < \delta' < \omega_1$ ,  $f_{\delta} \subseteq f_{\delta'}$ .

- (5) For each infinite  $\delta < \omega_1$ , the predense subsets of  $P(A \upharpoonright \omega\delta, B \upharpoonright \omega\delta)$  which have the form  $f_\delta^{-1}[S]$  for some  $S \in \bigcup_{\eta \leq \delta} M_\eta$  remain predense in  $P(A \upharpoonright \omega(\delta + 1), B \upharpoonright \omega(\delta + 1))$ .

To do this, we proceed as follows. The construction of the functions  $f_\delta$  is dictated by (4) at limit stages, and  $f_{\delta+1}$  is an arbitrary extension of  $f_\delta$  satisfying (3). The elements of  $A_\delta$  and  $B_\delta$  for  $\delta < \omega$  are given by (2). For  $\delta \geq \omega$ , by induction on  $\delta$  we choose the elements of  $A_\delta$  and  $B_\delta$  by alternately defining  $a_{\omega\delta+n}$  and  $b_{\omega\delta+n}$ , beginning with  $a_{\omega\delta}$  when  $\delta$  is even and with  $b_{\omega\delta}$  when  $\delta$  is odd. Let us illustrate the construction with the case where  $\delta$  is even. Fix an enumeration  $\langle I_m : 0 < m < \omega \rangle$  of the nonempty open intervals with rational endpoints. The first element  $a_{\omega\delta}$  is simply the least element, in the well-ordering of  $A$  fixed at the beginning of the proof, which is different from any of the elements of  $A$  chosen so far. We now choose  $b_{\omega\delta}, a_{\omega\delta+1}, b_{\omega\delta+1}, a_{\omega\delta+2}, b_{\omega\delta+2}, \dots$ , in that order. For  $n > 0$ , we pick  $a_{\omega\delta+n}$  and  $b_{\omega\delta+n}$  from  $I_n$  to ensure  $A_\delta$  and  $B_\delta$  will be dense.

To choose one of these elements, say  $b_{\omega\delta+n}$ , let  $N$  be a countable elementary submodel of  $H_\theta$ , for a suitably large  $\theta$ , such that  $A, B, f_\delta$ , the sequences  $\langle a_\alpha : \alpha \leq \omega\delta+n \rangle$  and  $\langle b_\alpha : \alpha < \omega\delta+n \rangle$ , and  $\bigcup_{\eta \leq \delta} M_\eta$  are all elements of  $N$ . Choose  $b_{\omega\delta+n}$  to be a member of  $B$  which is a Cohen real over  $N$ .

We must check that the construction gives (5). Let  $D$  be a predense subset of  $P(A \upharpoonright \omega\delta, B \upharpoonright \omega\delta)$  of the appropriate form. In particular, we have  $D \in N$ . We will show that  $D$  remains predense in  $P(A \upharpoonright \omega\delta + n + 1, B \upharpoonright \omega\delta + n + 1)$ .

**Remark 4.6** We are showing by induction on  $n$  that  $D$  remains predense in  $P(A \upharpoonright \omega\delta + n + 1, B \upharpoonright \omega\delta + n)$  and then in  $P(A \upharpoonright \omega\delta + n + 1, B \upharpoonright \omega\delta + n + 1)$ . (This establishes (5) since each member of  $P(A \upharpoonright \omega(\delta + 1), B \upharpoonright \omega(\delta + 1))$  belongs to  $P(A \upharpoonright \omega\delta + n, B \upharpoonright \omega\delta + n)$  for some  $n < \omega$ .) Our current stage has the second form. Note that at the stage  $n = 0$ , we first consider the passage from  $P(A \upharpoonright \omega\delta, B \upharpoonright \omega\delta)$  to  $P(A \upharpoonright \omega\delta + 1, B \upharpoonright \omega\delta)$ . But these two partial orders are equal because there is no legal target value for  $a_{\omega\delta}$  until  $b_{\omega\delta}$  is chosen. So the preservation of the predense sets trivially holds at that stage. In particular, it does not matter that  $a_{\delta\omega}$  is not Cohen generic over the previous construction.

Let

$$p \in P(A \upharpoonright \omega\delta + n + 1, B \upharpoonright \omega\delta + n + 1) \setminus P(A \upharpoonright \omega\delta + n + 1, B \upharpoonright \omega\delta + n).$$

Then  $p$  has the form  $q \cup \{(a, b_{\omega\delta+n})\}$  for some  $q \in P(A \upharpoonright \omega\delta + n + 1, B \upharpoonright \omega\delta + n)$  and  $a \in \{a_{\omega\delta+m} : m \leq n\}$ . Fix  $r \in D$ . The set

$$\{b \in \mathbb{R} : q \cup \{(a, b)\} \text{ is compatible with } r\} \in N$$

(“compatible with” here means only that  $q \cup \{(a, b)\} \cup r$  is a finite order isomorphism) is open and hence its complement  $C_r$  is closed, as is the set  $C_D = \bigcap_{r \in D} C_r$  of  $b$  for which  $q \cup \{(a, b)\}$  is incompatible with every member of  $D$ . Since  $p$  is a partial order isomorphism, there are open rational intervals  $J_1$  and  $J_2$  such that  $J_1 \cap \text{dom } p = \{a\}$ ,  $J_2 \cap \text{ran } p = \{b_{\omega\delta+n}\}$ . Note that whenever  $x \in J_1$  and  $b \in J_2$ ,  $q \cup \{(x, b)\}$  is a partial isomorphism.

**Claim 4.7**  $C_D$  is nowhere dense in  $J_2$ .

**Proof** Fix a nonempty open subinterval  $J$  of  $J_2$ . There is an extension of  $q$  by members of  $A_0 \times B_0$ —the point of using  $A_0$  and  $B_0$  being simply that they are dense and contained in  $A(\omega\delta + n + 1)$  and  $B(\omega\delta + n)$ , respectively—which adds two points in  $J_1 \times J$  straddling the line  $x = a$ . So this extension has the form

$$q' = q \cup \{(x_1, y_1), (x_2, y_2)\}, \quad x_1 < a < x_2, \quad y_1 < y_2$$

where  $[x_1, x_2] \subseteq J_1$  and  $[y_1, y_2] \subseteq J$ . Since  $q' \in P(A \upharpoonright \omega\delta + n + 1, B \upharpoonright \omega\delta + n)$ , by the induction hypothesis  $D$  must have an element  $r$  compatible with this extension. Since  $a \notin A(\omega\delta)$ , we have  $a \notin \text{dom } r$ . Let  $x'_1, x'_2$  be the closest members of  $\text{dom}(q' \cup r)$  to the left and right of  $a$ , respectively. Write  $y'_1 = r(x'_1), y'_2 = r(x'_2)$ . Then  $(y'_1, y'_2) \subseteq (y_1, y_2) \subseteq J$  and for any choice of  $b \in (y'_1, y'_2)$ ,  $q \cup \{(a, b)\}$  is compatible with  $r$ . Hence  $(y'_1, y'_2)$  is disjoint from  $C_r$  and hence from  $C_D$ . This proves the claim. ■

Thus,  $b_{\omega\delta+n} \notin C_D$  and hence  $p$  is compatible with some member of  $D$ . This establishes (5).

Now take  $P = P(A, B)$ . The fact that  $P$  forces the desired order-isomorphism of  $A$  and  $B$  is clear from (1) and (2). To see that  $P$  is  $\overline{M}$ -cc, let  $f = \bigcup_{\delta < \omega_1} f_\delta : P \rightarrow \omega_1$ . For any  $\delta < \omega_1$  we have  $f^{-1}[\omega\delta] = P(A(\omega\delta), B(\omega\delta))$  and for each  $S \subseteq \omega\delta$  whenever a set  $D$  of the form  $f^{-1}[S] = f_\delta^{-1}[S]$  belongs to  $M_\delta$  and is predense in  $P(A(\omega\delta), B(\omega\delta))$ , a simple induction on  $\delta'$  using (5) shows that if  $\delta$  is infinite and  $\delta < \delta' \leq \omega_1$ , then  $D$  is predense in  $P(A(\omega\delta'), B(\omega\delta'))$ . In particular,  $D$  is predense in  $P = P(A(\omega_1), B(\omega_1))$ . For a club of  $\delta < \omega_1$  we have  $\omega\delta = \delta$ , so this shows that  $P$  satisfies the  $\overline{M}$ -cc. ■

**Lemma 4.8** Assume  $\diamond$ . Let  $A$  be a nonmeager subset of  $\mathbb{R}$ . Then there is an oracle  $\overline{M} = \langle M_\delta : \delta < \omega_1 \rangle$  such that if  $P$  is any partial order satisfying the  $\overline{M}$ -cc, then  $\Vdash_P$  “ $A$  is nonmeager”.

**Proof** This is [Sh1998, Example IV 2.2]. ■

**Theorem 4.9** If ZFC is consistent, then so is ZFC + both of the following statements.

- (a) Every nonmeager set in  $\mathbb{R}$  has a nonmeager subset of cardinality  $\omega_1$ .
- (b) Let  $A$  and  $B$  be everywhere nonmeager subsets of  $\mathbb{R}$  of cardinality  $\omega_1$ . Suppose we are given pairwise disjoint countable dense subsets  $A_i, i < \omega$ , of  $A$  and pairwise disjoint countable dense subsets  $B_i, i < \omega$ , of  $B$ . Then  $A$  and  $B$  are order-isomorphic by an order isomorphism taking  $A_i$  isomorphically to  $B_i$  for each  $i < \omega$ .

**Proof** Start with a ground model of  $V = L$ . Fix a diamond sequence

$$\langle (f_\alpha, g_\alpha, h_\alpha) : \alpha < \omega_2, \text{cof}(\alpha) = \omega_1 \rangle$$

for trapping triples  $(f, g, h)$  consisting of:

(1) A function  $f: \omega_2 \rightarrow ([\omega_2]^{\leq \omega})^\omega$ . The idea of  $f$  is that, with  $\omega_2$  identified with the ccc partial order we are about to build,  $[\omega_2]^{\leq \omega}$  contains the antichains. Thus,  $([\omega_2]^{\leq \omega})^\omega$  contains a name for each real number (construed as a subset of  $\omega$ ). Then for any nonmeager set  $X$  in the extension, we can find a ground model function  $f: \omega_2 \rightarrow ([\omega_2]^{\leq \omega})^\omega$  enumerating the names of the elements of  $X$ .

(2) Functions  $g, h: \omega_1 \rightarrow ([\omega_2]^{\leq \omega})^\omega$  intended to represent (enumerations of the names for the elements of) everywhere nonmeager sets of cardinality  $\omega_1$  with each of the sets  $\{g(\omega i + n) : n < \omega\}$  and  $\{h(\omega i + n) : n < \omega\}$ , for  $i < \omega$ , dense in  $\mathbb{R}$ .

So for each  $\alpha < \omega_2$  of cofinality  $\omega_1$ ,  $f_\alpha: \alpha \rightarrow ([\alpha]^{\leq \omega})^\omega$ , and  $g_\alpha, h_\alpha: \omega_1 \rightarrow ([\alpha]^{\leq \omega})^\omega$ . Also, for each  $(f, g, h)$  as in (1)–(2),  $\{\alpha < \omega_2 : \text{cof}(\alpha) = \omega_1, f \upharpoonright \alpha = f_\alpha, g \upharpoonright \alpha = g_\alpha \text{ and } h \upharpoonright \alpha = h_\alpha\}$  is stationary in  $\omega_2$ .

We will inductively define an  $\omega_2$ -stage finite support iteration

$$\langle \langle P_\alpha \rangle_{\alpha \leq \omega_2}, \langle \dot{Q}_\alpha \rangle_{\alpha < \omega_1} \rangle$$

as well as a  $P_\alpha$ -names  $\bar{M}_\alpha$  for oracles and one-to-one functions  $F_\alpha: P_\alpha \rightarrow \omega_2$  for  $\alpha < \omega_2$  such that the range of each  $F_\alpha$  is an initial segment of  $\omega_2$  which includes  $\alpha$  and for  $\beta < \alpha < \omega_2$ , we have  $F_\beta \subseteq F_\alpha$ . (At each stage,  $F_\alpha$  is any function satisfying these conditions.)

For  $\alpha < \omega_2$ , we will let  $\dot{X}_\alpha$  denote the  $P_\alpha$ -name for the set of real numbers whose elements have the names

$$\bigcup_{n < \omega} \{n\} \times F_\alpha^{-1}(f_\alpha(\xi)(n)), \quad \xi < \alpha.$$

Similarly, we will let  $\dot{A}_\alpha$  and  $\dot{B}_\alpha$  denote the  $\omega_1$ -sequences of  $P_\alpha$ -names for real numbers

$$\left\langle \bigcup_{n < \omega} \{n\} \times F_\alpha^{-1}(g_\alpha(\xi)(n)) : \xi < \omega_1 \right\rangle$$

and

$$\left\langle \bigcup_{n < \omega} \{n\} \times F_\alpha^{-1}(h_\alpha(\xi)(n)) : \xi < \omega_1 \right\rangle$$

respectively. At stage  $\alpha < \omega_2$  of the construction, if  $\text{cof}(\alpha) = \omega_1$  and if

$$\Vdash_{P_\alpha} \dot{X}_\alpha \text{ is not meager,}$$

then we use Lemma 4.8 to get a  $P_\alpha$ -name  $\bar{M}'_\alpha$  for an oracle so that if  $P$  is any forcing notion which satisfies the  $\bar{M}'_\alpha$ -cc, then  $X_\alpha$  remains nonmeager after forcing with  $P$ . Otherwise, in particular if  $\text{cof}(\alpha) \neq \omega_1$ , we let  $\bar{M}'_\alpha$  be any  $P_\alpha$ -name for an oracle.

For  $\beta < \alpha$ , let  $P_{\beta\alpha}$  be the usual  $P_\beta$ -name for a partial order such that  $P_\alpha$  is isomorphic to a dense subset of  $P_\beta * P_{\beta\alpha}$  (see [Ba]). Let  $\bar{M}_{\beta\alpha}$  be a  $P_\alpha$ -name for an oracle such that

- (1) If  $\Vdash_{P_\beta}$  “ $P_{\beta,\alpha}$  is  $\bar{M}_{\beta\alpha}$ -cc and  $\Vdash_{P_{\beta,\alpha}}$   $\dot{Q}_\alpha$  is  $\bar{M}_{\beta\alpha}$ -cc”, then  $\Vdash_{P_\beta}$  “ $P_{\beta,\alpha+1} = P_{\beta,\alpha} * \dot{Q}_\alpha$  is  $\bar{M}_{\beta\alpha}$ -cc”.

(There is such an  $\overline{M}_{\beta\alpha}$  by Proposition 4.4(2). In (1),  $\overline{M}_{\beta\alpha}$  is actually a  $P_\beta$ -name for a  $P_{\beta,\alpha}$ -name for an oracle. We denote the corresponding  $P_\alpha$ -name also by  $\overline{M}_{\beta\alpha}$ .)

Let  $\overline{M}_\alpha$  be a  $P_\alpha$ -name for an oracle such that

$$(2) \quad \Vdash_{P_\alpha} \text{“If } \dot{Q}_\alpha \text{ is } \overline{M}_\alpha\text{-cc, then } \dot{Q}_\alpha \text{ is } \overline{M}'_\alpha\text{-cc and } \overline{M}_{\beta\alpha}\text{-cc for all } \beta < \alpha\text{”}.$$

(Use Proposition 4.4(3).)

Now, if  $\text{cof}(\alpha) = \omega_1$  and if

$$\Vdash_{P_\alpha} \text{The ranges of } \dot{A}_\alpha, \dot{B}_\alpha \text{ are everywhere nonmeager and each of the sets } \{\dot{A}_\alpha(\omega i + n) : n < \omega\}, \{\dot{B}_\alpha(\omega i + n) : n < \omega\}, \text{ for } i < \omega, \text{ is dense in } \mathbb{R}.$$

then use Lemma 4.5 to get a  $P_\alpha$ -name  $\dot{Q}_\alpha$  for a partial order satisfying the  $\overline{M}_\alpha$ -cc and forcing an isomorphism between  $A_\alpha$  and  $B_\alpha$  as described in the statement of the lemma. In all other cases, take  $\dot{Q}_\alpha$  to name the partial order  $Q$  for adding one Cohen real. We have thus

$$(3) \quad \Vdash_{P_\alpha} \text{“}\dot{Q}_\alpha \text{ satisfies the } \overline{M}_\alpha\text{-cc”}.$$

Now suppose that for some  $P_{\omega_2}$ -name  $\dot{X}$  we have

$$\Vdash_{P_{\omega_2}} \dot{X} \text{ is not meager.}$$

(Every nonmeager set in any extension has a name forced by the weakest condition to be nonmeager since there is always a nonmeager set.) Fix a name  $\dot{f}$  such that

$$\Vdash_{P_{\omega_2}} \dot{f}: \omega_2 \rightarrow \dot{X} \text{ is onto.}$$

Then define  $f: \omega_2 \rightarrow ([\omega_2]^{< \omega})^\omega$  so that if

$$\tau_\xi = \bigcup_{n < \omega} \{n\} \times F^{-1}(f(\xi)(n)), \quad \xi < \omega_2,$$

then for each  $\xi < \omega_2$ ,

$$\Vdash_{P_{\omega_2}} \dot{f}(\xi) = \tau_\xi.$$

There is a closed unbounded set  $C \subseteq \omega_2$  such that for each  $\alpha \in C$  of cofinality  $\omega_1$  we have:

- (i)  $f \upharpoonright \alpha: \alpha \rightarrow ([\alpha]^{< \omega})^\omega$ .
- (ii)  $\forall \xi < \alpha, \tau_\xi$  is a  $P_\alpha$ -name.
- (iii)  $\Vdash_{P_\alpha} \{\tau_\xi : \xi < \alpha\}$  is not meager.

(For (iii), note that when  $\alpha$  has cofinality  $\omega_1$ , each  $P_\alpha$ -name for a meager set is a  $P_\beta$ -name for some  $\beta < \alpha$ . Thus, if  $M$  is an elementary submodel of  $H_\theta$  for a suitably large  $\theta$  such that  $|M| = \omega_1, M^\omega \subseteq M, \langle \tau_\xi : \xi < \omega_2 \rangle \in M$  and  $\alpha = M \cap \omega_2 \in \omega_2$  has cofinality  $\omega_1$ , then for each (nice)  $P_\alpha$ -name  $\sigma$  for a meager Borel set, we have  $\sigma \in M$  and hence  $M$  knows about a maximal antichain of conditions each deciding a  $\xi$  for which  $\tau_\xi$  is forced not to be in  $\sigma$ . The antichain is countable and hence contained in

$M$ . For each condition in the antichain, the least  $\xi$  which it decides is in  $M$  and hence below  $\alpha$ . Hence  $\Vdash_{P_\alpha} \{ \tau_\xi : \xi < \alpha \}$  is not contained in  $\sigma$ .)

Choose such an  $\alpha$  of cofinality  $\omega_1$  for which  $f \upharpoonright \alpha = f_\alpha$ . By (i) and (ii), the definition of  $\tau_\xi$  would not change if we used  $f_\alpha$  instead of  $f$  and  $F_\alpha$  instead of  $F$ . Then from the definition of  $\dot{X}_\alpha$  we get

$$\Vdash_{P_\alpha} \dot{X}_\alpha = \{ \tau_\xi : \xi < \alpha \}.$$

So at stage  $\alpha$  we chose a  $P_\alpha$ -name  $\overline{M}_\alpha$  and we arranged that

$$\Vdash_{P_\alpha} \text{“} P_{\alpha,\gamma} \text{ is } \overline{M}_\alpha\text{-cc”}.$$

(This follows easily by induction on  $\gamma \geq \alpha$  and Propositions 4.4(1,2). (Recall that  $P_{\alpha,\gamma}$  can be viewed in canonical way as an iteration: see [Ba]. At limits  $\gamma$  use Propositions 4.4(1). At stages  $\gamma + 1$ , use (3) to get  $\Vdash_{P_\gamma}$  “ $\dot{Q}_\gamma$  satisfies the  $\overline{M}_\gamma$ -cc” and then use (2) and (1) with  $(\beta, \alpha)$  replaced by  $(\alpha, \gamma)$ .)

Hence, by the choice of  $\overline{M}_\alpha$ ,

$$(4) \quad \Vdash_{P_\alpha} \Vdash_{P_{\alpha,\gamma}} \dot{X}_\alpha \text{ is not meager}$$

from which it follows that

$$\Vdash_{P_\alpha} \Vdash_{P_{\alpha,\omega_2}} \dot{X}_\alpha \text{ is not meager}$$

since if this failed then we would have

$$p \Vdash_{P_\alpha} q \Vdash_{P_{\alpha,\omega_2}} \dot{X}_\alpha \subseteq \dot{B}$$

for some conditions  $p \in P_\alpha, q \in P_{\alpha,\omega_2}$  and some name  $\dot{B}$  for a meager Borel set. But then for some  $\gamma$ , we have  $\alpha < \gamma < \omega_2, q \in P_{\alpha,\gamma}$  and  $\dot{B}$  is a  $P_\gamma$ -name and this contradicts (4).

By what we have established, there are guaranteed to be sets of cardinality  $\omega_1$  which are not meager in any extension by  $P_{\omega_2}$ . Hence there are guaranteed to be everywhere nonmeager sets of cardinality  $\omega_1$ . Suppose that for some  $P_{\omega_2}$ -names  $\dot{A}$  and  $\dot{B}$  for  $\omega_1$ -sequences we have

$$\Vdash_{P_{\omega_2}} \text{The ranges of } \dot{A}, \dot{B} \text{ are everywhere nonmeager and each of the sets } \{ \dot{A}(\omega i + n) : n < \omega \}, \{ \dot{B}(\omega i + n) : n < \omega \}, \text{ for } i < \omega, \text{ is dense in } \mathbb{R}.$$

(By what we just said, every pair of everywhere nonmeager sets  $A$  and  $B$  of cardinality  $\omega_1$ , together with choices of countably many disjoint countable dense subsets of each one, has a name such that the weakest condition forces the desired properties.)

Define  $g, h: \omega_1 \rightarrow ([\omega_2]^{< \omega})^\omega$  so that if

$$\sigma_\xi = \bigcup_{n < \omega} \{n\} \times F^{-1}(g(\xi)(n)), \quad \xi < \omega_1$$

and

$$\tau_\xi = \bigcup_{n < \omega} \{n\} \times F^{-1}(h(\xi)(n)), \quad \xi < \omega_1$$

then for each  $\xi < \omega_1$ ,

$$\Vdash_{P_{\omega_2}} \dot{A}(\xi) = \sigma_\xi$$

and

$$\Vdash_{P_{\omega_2}} \dot{B}(\xi) = \tau_\xi.$$

For all large enough  $\alpha < \omega_2$ , we have:

- (i)  $g, h: \omega_1 \rightarrow ([\alpha]^{< \omega})^\omega$ .
- (ii)  $\forall \xi < \alpha, \sigma_\xi$  and  $\tau_\xi$  are  $P_\alpha$ -names.

Choose any such  $\alpha$  of cofinality  $\omega_1$ . By (i) and (ii), the definitions of  $\sigma_\xi$  and  $\tau_\xi$  would not change if we used  $g_\alpha$  instead of  $g$ ,  $h_\alpha$  instead of  $h$ , and  $F_\alpha$  instead of  $F$ . Then from the definitions of  $\dot{A}_\alpha$  and  $\dot{B}_\alpha$  we get

$$\Vdash_{P_\alpha} \text{The ranges of } \dot{A}_\alpha, \dot{B}_\alpha \text{ are everywhere nonmeager and each of the sets } \{\dot{A}_\alpha(\omega i + n) : n < \omega\}, \{\dot{B}_\alpha(\omega i + n) : n < \omega\}, \text{ for } i < \omega, \text{ is dense in } \mathbb{R}.$$

(Being everywhere nonmeager is trivially downward absolute.) Then  $\dot{Q}_\alpha$  was chosen to add an order isomorphism between  $A_\alpha$  and  $B_\alpha$  of the desired type.

This completes the proof of the theorem. ■

### 5 A Measure-Theoretic Analog of perfect( $2^\omega$ )

A measure theoretic version of Laczkovich’s question is not completely obvious because perfect sets carry many measures. We consider the following measures on  $2^\omega$  which we will call canonical. Given  $P \subseteq 2^\omega$  a perfect set, define

$$T_P = \{s \in 2^{< \omega} : P \cap U(s) \neq \emptyset\},$$

where  $U(s) = \{x \in 2^\omega : s \subseteq x\}$ . We say that  $s \in T_P$  splits iff both  $s0$  and  $s1$  are in  $T_P$ . The canonical measure  $\mu_P$  is the one supported by  $P$  and determined by declaring  $\mu_P(U(s)) = 1/2^n$  iff  $s \in T_P$  and  $|\{i < |s| : s \upharpoonright i \text{ splits}\}| = n$ . An equivalent view is to take the natural map from  $2^{< \omega}$  to the splitting nodes of  $T_P$  and the homeomorphism  $h: 2^\omega \rightarrow P \subseteq 2^\omega$  induced by it and then  $\mu_P$  is the measure corresponding to the product measure  $\mu$  on  $2^\omega$ , i.e.,  $\mu_P(A) = \mu(h^{-1}(A))$ .

**Theorem 5.1** *It is relatively consistent with ZFC that for any set  $B \subseteq 2^\omega$  which is not of measure zero, there exists a perfect set  $P$  of measure zero such that  $B \cap P$  does not have measure zero in the canonical measure  $\mu_P$  on  $P$ .*

**Proof** The model is the one used by Roslanowski and Shelah in the proof of [RS, Theorem 3.2]. It is obtained by forcing over a model of CH with an  $\omega_2$ -stage countable support iteration  $\langle \langle P_\alpha \rangle_{\alpha < \omega_2}, \langle \dot{Q}_\alpha \rangle_{\alpha < \omega_2} \rangle$  of the measured creature forcing  $\mathbb{Q} = \mathbb{Q}_4^{\text{mt}}(K^*, \Sigma^*, \mathbf{F}^*)$  defined in [RS, Section 2]. We use the notation of [RS] concerning

this partial order. The definition involves in particular a rapidly growing sequence of powers of 2,  $\langle N_i = 2^{M_i} : i < \omega \rangle$ . Forcing with  $\mathbb{Q}$  gives rise to a continuous function  $h: \prod_{i<\omega} N_i \rightarrow 2^\omega$ . We will make use of the following result concerning this function. The measure on  $\prod_{i<\omega} N_i$  in this proposition is the product of the uniform probability measures on the factors and the measure on  $2^\omega$  is the usual product measure. In the remainder of this proof, we denote both of these measures, as well as their product, by  $\mu$ , letting the context distinguish them.

[RS, Proposition 2.6] Suppose that  $A \subseteq \prod_{i<\omega} N_i \times 2^\omega$  is a set of outer measure one. Then, in  $V^{\mathbb{Q}}$ , the set

$$\left\{ x \in \prod_{i<\omega} N_i : (x, h(x)) \in A \right\}$$

has outer measure one.

We shall also need to know that  $\mathbb{Q}$  is proper and that countable support iterations of  $\mathbb{Q}$  preserve Lebesgue outer measure. The former is [RS, Corollary 1.14]. The latter is explained in the proof of [RS, Theorem 3.2]. (The explanation refers the reader to some very general preservation theorems for iterated forcing. For the reader who wants to verify this without learning these general theorems, we indicate that it also follows from the special case of these theorems, preservation of  $\square^{\text{random}}$ , given in [Go] by imitating the proof in [Pa] that Laver forcing satisfies what is called there  $\star$  and by noting that  $\star$  implies preservation of  $\square^{\text{random}}$ .)

Recall that  $N_i = 2^{M_i}$ . We identify  $N_i$  with the set of binary sequences of length  $M_i$ . The map  $h: \prod_{i<\omega} N_i \rightarrow 2^\omega$  is determined from a generic sequence of finite maps  $(W(i): N_i \rightarrow 2 : i < \omega)$  added by  $\mathbb{Q}$ .  $h$  is defined by  $h(x)(i) = W(i)(x(i))$  for each  $i$ . We use the  $W(i)$ 's to define a perfect set  $P \subseteq 2^\omega$  by the condition that  $x \in P$  if and only if there exists  $\gamma \in 2^\omega$  such that  $x$  is the concatenation of the sequence  $s_0, i_0, s_1, i_1, \dots$ , where  $\gamma$  is the concatenation of  $s_0, s_1, s_2, \dots$  and where each  $s_k$  has length  $M_k$  and  $i_k = W(k)(s_k) \in \{0, 1\}$ .  $P$  is essentially the same as the graph of  $h$  but we spell out the details to be sure the canonical measure is the one we want. Another way to define  $P$  is as follows:

- (i) Let  $l_i = M_i + \sum_{j<i} (M_j + 1)$ . Let  $l_{-1} = -1$ . The  $l_i, i < \omega$ , are the nonsplitting levels of the tree  $T_P$  which can be determined by the next two conditions.
- (ii) If  $s \in T_P$  and  $l_{i-1} < |s| < l_i$  then both  $s0$  and  $s1$  are in  $T_P$ .
- (iii) If  $s \in T_P$  and  $|s| = l_i$ , then only  $sj$  in  $T_P$  where  $W(i)(t) = j$  and  $s = rt$  is the concatenation of  $r$  and  $t$  where  $|t| = M_i$  and  $r$  has the appropriate length.
- (iv) Define

$$P = [T_P] \stackrel{\text{def}}{=} \{x \in 2^\omega : \forall n(x \upharpoonright n \in T_P)\}$$

Every time we pass a nonsplitting level  $l_i$  we lose half the measure and so  $P$  is a perfect set of measure zero for the usual measure on  $2^\omega$ .

Let  $\rho: \prod_{i<\omega} N_i \times 2^\omega \rightarrow 2^\omega$  be the natural homeomorphism given by  $\rho(x, z)$  is the concatenation of the sequence  $x_0, z_0, x_1, z_1, \dots$ , where we are identifying  $N_i$  with the set of binary sequences of length  $M_i$ .

**Claim 5.2**  $\rho$  is measure-preserving.

**Proof** By a standard uniqueness theorem for the extension of a measure from an algebra to the  $\sigma$ -algebra it generates, it suffices to verify that  $\rho^{-1}[C]$  has the same measure as  $C$  for every clopen set  $C \subseteq 2^\omega$ . As above, for  $k < \omega$  and  $s \in 2^k$ , let us write  $U(s) = \{x \in 2^\omega : s \subseteq x\}$ . Similarly, for  $s \in \prod_{i < k} N_i$  we write  $V(s) = \{x \in \prod_{i < \omega} N_i : s \subseteq x\}$ . Every clopen set  $C \subseteq 2^\omega$  can be partitioned into clopen sets of the form  $U(r)$ , where for some  $k < \omega$ ,  $s^r \in \prod_{i < k} N_i$  and  $t^r \in 2^k$ ,  $r$  is the concatenation of  $s^r_0, t^r_0, \dots, s^r_{k-1}, t^r_{k-1}$ . (These are simply the basic open sets  $U(r)$  for which  $r$  has length  $\sum_{i < k} (M_i + 1)$  for some  $k < \omega$ .) Hence it suffices to verify  $\mu(\rho^{-1}[U(r)]) = \mu(U(r))$  for  $r$  of this form. We have

$$\mu(\rho^{-1}[U(r)]) = \mu(V(s^r) \times U(t^r)) = \left(\prod_{i < k} 2^{-M_i}\right) 2^{-k} = 2^{-\sum_{i < k} (M_i + 1)} = \mu(U(r)).$$

This proves the claim. ■

Let  $g: \prod_{i < \omega} N_i \rightarrow \prod_{i < \omega} N_i \times 2^\omega$  be the homeomorphism of  $\prod_{i < \omega} N_i$  onto the graph of  $h$  given by  $g(x) = (x, h(x))$ . We have  $\rho[h] = P$  (i.e., the graph of  $h$  corresponds to  $P$  under  $\rho$ ).

**Claim 5.3** For any Borel set  $B \subseteq 2^\omega$

$$\mu_P(B) = \mu(g^{-1}[\rho^{-1}[B]])$$

and similarly for outer measure.

**Proof** Since the range of  $g$  is the graph of  $h$ , we have

$$\mu(g^{-1}[\rho^{-1}[B]]) = \mu(g^{-1}[\rho^{-1}[B] \cap h]) = \mu(g^{-1}[\rho^{-1}[B \cap P]]).$$

Similarly, since  $\mu_P$  concentrates on  $P$ ,  $\mu_P(B) = \mu_P(B \cap P)$ . Hence, it suffices to prove the claim for Borel subsets of  $P$ .

Given  $s \in \prod_{i < k} N_i$ , define  $t^s \in 2^k$  by  $t^s_i = W(i)(s(i))$  for all  $i < k$ , and write  $r^s$  for the concatenation of  $s_0, t^s_0, \dots, s_{k-1}, t^s_{k-1}$ . Using the notation for basic open sets from the proof of the previous claim, we have

$$\mu_P(U(r^s)) = 2^{-\sum_{i < k} M_i} = \prod_{i < k} 2^{-M_i}.$$

Also,  $\rho^{-1}[U(r^s)] = V(s) \times U(t^s)$  and  $g^{-1}[\rho^{-1}[U(r^s)]] = g^{-1}[V(s) \times U(t^s)] = V(s)$ , so

$$\mu(g^{-1}[\rho^{-1}[U(r^s)]]) = \mu(V(s)) = \prod_{i < k} 2^{-M_i}.$$

Thus, the claim holds for basic open sets of the form  $U(r^s)$ . Every clopen subset of  $P$  is partitioned by such sets, so the claim holds for all clopen sets, and hence, as in the proof of Claim 5.2, for all Borel sets. This proves the claim. ■

Now we prove Theorem 5.1 in the case that  $B \subseteq 2^\omega$  has outer measure one. It follows from the usual Lowenheim–Skolem arguments that if we let  $B_\alpha = V^{\mathbb{P}_\alpha} \cap B$  then there will exist  $\alpha < \omega_2$  such that  $B_\alpha \in V^{\mathbb{P}_\alpha}$  and

$$V^{\mathbb{P}_\alpha} \models B_\alpha \text{ has outer measure one.}$$

Letting  $A = \rho^{-1}[B_\alpha]$  (which has outer measure one by Claim 5.2) in [RS, Proposition 2.6] cited above (applied in  $V^{\mathbb{P}_\alpha}$ ) and using Claim 5.3, we have that

$$V^{\mathbb{P}_{\alpha+1}} \models B_\alpha \text{ has } \mu_P \text{ outer measure one.}$$

Because  $\mathbb{Q}$  is proper, the remainder  $\mathbb{P}_{\omega_2}/\mathbb{P}_{\alpha+1}$  of the forcing is isomorphic in  $V^{\mathbb{P}_{\alpha+1}}$  to a countable support iteration of  $\mathbb{Q}$  and hence preserves outer measure. It follows that in the final model  $V^{\mathbb{P}_{\omega_2}}$ ,  $B$  has  $\mu_P$  outer measure one.

Now in the case that  $B$  has outer measure less than one, replace it by  $B' = Q + B$  where  $Q$  is a countable dense subset of  $2^\omega$ . Then  $B'$  has outer measure one, and so we know there exists a measure zero perfect  $P$  such that  $B'$  has positive  $\mu_P$  outer measure. Hence for some  $q \in Q$  we have that  $q + B$  has positive  $\mu_P$  outer measure. But then  $B$  has positive  $\mu_{q+P}$  outer measure.

This completes the proof of the theorem. ■

**Problem 5.4** Is it relatively consistent with ZFC to have simultaneously both the category property, perfect( $2^\omega$ ), and its measure theory analogue, Theorem 5.1?

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