FUNCTION SPACES DETERMINED BY A LEVELLING LENGTH FUNCTION

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1. Introduction. In this paper we introduce the function spaces L^λ and $L^\lambda(B)$ which generalize the classical L^p and $L^p(B)$ spaces respectively. For those λ which possess what we call the *levelling* property we give a discussion of the conjugate space to $L^\lambda(B)$; our treatment applies, in particular, to the $L_{(w)}{}^p(B)$ and $M_{(w)}{}^q(B)$ spaces defined in [**7**, §2]. (*See also the note at end of this paper.*)

When specialized to apply to the classical $L^p(B)$ case, our methods are closely related to those used previously by other writers, including Bochner and Taylor [**3**], Pettis [**9**; **10**], Phillips [**11**; **12**], and Dieudonné [**4**]; but even for the classical case, our treatment is, we believe, simpler, more direct, and more complete than any given previously. (We are indebted to J. Dieudonné for pointing out that our own results could be strenghtened by applying Eberlein's Theorem and for a reference to the papers by Phillips.)

2. Terminology. Throughout this paper we shall use without comment the terminology given in [**7**, §2] as well as the following:

λ is called a *length function* if for every measurable function u with $0 \leqslant u(P) \leqslant \infty$ for almost all P, $\lambda(u)$ is defined with $0 \leqslant \lambda(u) \leqslant \infty$ and satisfies:

(L 1) $\lambda(u) = 0$ *whenever* $u(P) = 0$ *for almost all* P,

(L 2) $\lambda(u) \leqslant \lambda(u_1)$ *whenever* $u(P) \leqslant u_1(P)$ *for all* P,

(L 3) $\lambda(u_1 + u_2) \leqslant \lambda(u_1) + \lambda(u_2)$,

(L 4) $\lambda(ku) = k\lambda(u)$ *for all* $k > 0$,

(L 5) $u_1(P) \leqslant u_2(P) \leqslant \ldots$ *for all* P *implies* $\lambda(\sup u_n) = \sup \lambda(u_n)$.

We frequently write $\lambda(E)$ in place of $\lambda(1_E)$. S is called *coarse* if for some $A > 0$, $\gamma(e) < A$ implies $\gamma(e) = 0$; if S is not coarse we define $\lambda(+0)$ as inf $\lambda(e)$ for all e with $\gamma(e) > 0$. A set E is called λ-*null* if $\lambda(u) = 0$ whenever u vanishes outside E and E is called λ-*purely-infinite* if $\gamma(E) > 0$ and $\lambda(u) = \infty$ whenever $u(P) > 0$ on some subset of positive measure contained in E.

A length function λ is called *continuous* if

(L 6) $\lambda(u) = \sup_e \lambda(u_e)$.

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A continuous λ is called a *levelling* length function if it satisfies the following two conditions:

(L 7) *$\lambda(e) < \infty$ for some e with $\gamma(e) > 0$, and $\lambda(e) > 0$ for some e,*

(L 8) $$\lambda\Big(\sum_{i=1}^{m} k_{e,i}\Big) \geqslant \lambda\bigg(\Big(\frac{k_1\,\gamma(e_1) + k_2\,\gamma(e_2)}{\gamma(e_1+e_2)}\Big)_{e_1+e_2} + \sum_{i=3}^{m} k_{e,i}\bigg)$$

for all $m \geqslant 2$, disjoint $e_1, \ldots, e_m$ with all $\gamma(e_i) > 0$, and arbitrary $k_i \geqslant 0$.
Other possible conditions on λ are listed for reference as follows:

(L 9) *Either S is coarse or $\lambda(+0) = 0$.*

(L 10) *$\lambda(u) < \infty$ implies $u = u_E$ for some E which is a countable union of sets of finite measure.*

(L 11) *$\lambda(u) = \lambda(u_E) < \infty$ implies $u = u_E$.*

(L 12) *$\lambda(u) < \infty$, $\epsilon > 0$ imply $\lambda(u - u_e) < \epsilon$ for some e.*

(L 13) *$\lambda(u) < \infty$ implies $\lambda(u - u_N) \to 0$ as $N \to \infty$.*

For every length function λ we define the *conjugate length function* λ^* (sometimes denoted by μ) by the relation

$$\lambda^*(v) = \sup \int u(P)v(P)d\gamma(P),$$

for all u with $\lambda(u) \leqslant 1$. Notation such as (L 9)* refers to λ^* in place of λ.

For a given weight function w on S as described in [**7**, §2], $\lambda_{(w)}{}^p$ and $\mu_{(w)}{}^q$ shall denote the length functions:

$$\lambda_{(w)}{}^p(u) = \begin{cases} \left(\displaystyle\int_0^\gamma u^*(x)^p w^*(x)\,dx\right)^{1/p} & 1 \leqslant p < \infty, \\ u^*(0) = \text{ess. sup}\, u(P), & p = \infty; \end{cases}$$

$$\mu_{(w)}{}^q(v) = \begin{cases} \left(\displaystyle\int_0^\gamma \left(\frac{v^{*\circ}(x)}{w^*(x)}\right)^q w^*(x)\,dx\right)^{1/q} & 1 \leqslant q < \infty, \\ \sup\left(\dfrac{v^{*\circ}(x)}{w^*(x)}\right), & q = \infty. \end{cases}$$

If $f(P)$ has $|f(P)|$ measurable then $\lambda(f)$ will mean $\lambda(u)$ with $u(P) = |f(P)|$; L^λ will denote the space of numerical valued f which are measurable and have $\lambda(f)$ finite; $L^\lambda(B)$ will denote the space of *Bochner measurable* f, valued in B and with $\lambda(f)$ finite. L^λ, $L^\lambda(B)$ will sometimes be denoted by $L_{(w)}{}^p$, $L_{(w)}{}^p(B)$, respectively, when $\lambda = \lambda_{(w)}{}^p$ and by $M_{(w)}{}^q$, $M_{(w)}{}^q(B)$, respectively, when $\lambda = \mu_{(w)}{}^q$.

F (and similarly G) will always denote an additive set function defined for every e, with $F(e) = 0$ whenever $\gamma(e) = 0$; we define

$$\lambda(F) = \sup \lambda\left(\sum_{i=1}^{m} \left(\frac{|F(e_i)|}{\gamma(e_i)}\right)_{e_i}\right)$$

for all finite collections of disjoint e_i with all $\gamma(e_i) > 0$, and

$$|F|(e) = \sup (|F(e_1)| + \ldots + |F(e_m)|)$$

for all finite collections of disjoint $e_i \subset e$. For given G we let $V = V(G)$ denote the closed linear space determined by the values of the $G(e)$. G is said to be *majorized* on E (by the constant $M < \infty$) if $|G(e)| \leqslant M\,\gamma(e)$ for every $e \subset E$ (see Phillips [**12**, p. 133], Dieudonné [**4**, p. 130]).

We list a further possible condition on λ, namely:

(L 10)′ *$\lambda(F) < \infty$ implies $F = F_E$ for some E which is a countable union of sets of finite measure.*

In §6 we consider numerical valued $\dot{g} = g(c) = g(c, P)$ which, for fixed c, is defined for almost all P (the exceptional set depending on c) and is measurable and satisfies: $g(c_1, P) + g(c_2, P) = g(c_1 + c_2, P)$ for almost all P, for all c_1, c_2; $g(kc, P) = kg(c, P)$ for almost all P, for every scalar k and every c. We define

$$\lambda(\dot{g}) = \sup \lambda\left(\sum_{i=1}^{m} g(c_{e,i})\right)$$

for all finite collections of disjoint e_i and arbitrary c_i with $|c_i| = 1$. (We use the convention $g(c_{e,i}, P) = g(c_i, P)$ when P is in e_i, $= 0$ otherwise.) To be precise, $\dot{g}$ depends on the range of c; when not specified, c will range over B.

A subset H of B^* is called total on B if for every $v \in B$ with $v \neq 0$ there is some $c \in H$ with $cv \neq 0$. For such H a function $g(P)$, valued in B, is called *BH-integrable* if $g(P)c$ is measurable and integrable on every e for every c in the closed linear subspace $\tilde{H}$ determined by H; then G is called the *BH-integral* of g if $G(e)$ is valued in B and for every c in $\tilde{H}$ and every e,

$$G(e)c = \int_e g(P)c\,d\gamma(P).$$

$g(P)$, valued in B, is called *Bochner integrable* if it is Bochner measurable and $|g(P)|$ has finite integral on every e; then G, valued in B, is called the *Bochner integral* of g if for every e and every n there is a finitely valued $\sum_i c_{e,i}$ with disjoint $e_i \subset e$ such that

$$\gamma(e - \textstyle\sum_i e_i) < 1/n, \quad |g(P) - c_i| < 1/n$$

for all $P \in e_i$, and $|G(e) - \sum_i \gamma(e_i)c_i| < 1/n$.

We shall say that *B is separable-controlled by H* if H is a separable subset of B^* such that for every $v \in B$, $|v| = \sup|vc|$ for all $c \in H$ with $|c| = 1$ (cf. Pettis [**10**, p. 257]).

B is said to have *the RN property on S* if for every e of positive measure and every G valued in B and majorized on e, there exists, for every $\epsilon > 0$, a set e' of positive measure contained in e, and a c in B, such that

$$\left|\frac{G(e'')}{\gamma(e'')} - c\right| < \epsilon$$

for all e'' of positive measure contained in e'.

S is said to have a countable basis if there exists a sequence $e_1, e_2, \ldots$, such that for every e and every $\epsilon > 0$ there is a set e', the union of a subsequence of the e_n, such that $\gamma(e - ee') + \gamma(e' - ee') < \epsilon$.

3. Length functions. We refer to §2 of this paper and to [**7**, §2] for terminology.

It is easy to verify the following statements for an *arbitrary* length function λ: the conjugate λ^* is always continuous; $\lambda^{***} = \lambda^*$; if E is λ-null and $\gamma(E) > 0$ then E is λ^*-purely-infinite; if E is λ-purely-infinite then E is λ^*-null; $\lambda(u) = \lambda(u_1)$ whenever $u(P) = u_1(P)$ for almost all P; $\lambda(u) = 0$ if and only if $\{u(P) > 0\}$ is a λ-null set; $\lambda(u) < \infty$ implies $\{u(P) = \infty\}$ is a λ-null set; and $\lambda(u) \geqslant \lambda^{**}(u)$ for all u.

For an arbitrary length function λ the L^λ and, more generally, the $L^\lambda(B)$ are obviously linear, normed spaces with norm $\lambda(f)$, where f, f_1 are identified if and only if $\{f(P) \neq f_1(P)\}$ is a λ-null set. Completeness will be shown now by a variation of the von Neumann-Weyl argument [**13**, p. 111].

THEOREM 3.1. *$L^\lambda(B)$ is a Banach space.*

Proof. We need only show completeness. Given a sequence f_n with all $\lambda(f_n)$ finite and $\lambda(f_n - f_m) \to 0$ as $n, m \to \infty$, select an infinite subsequence g_i from the f_n such that

$$\sum_{i=1}^{\infty} \lambda(g_{i+1} - g_i)$$

is finite. For each P set

$$g_0(P) = |g_1(P)| + \sum_{i=1}^{\infty} |g_{i+1}(P) - g_i(P)|.$$

Then

$$\lambda(g_0) \leqslant \lambda(g_1) + \sum_{i=1}^{\infty} \lambda(g_{i+1} - g_i) < \infty$$

and hence $\{g_0(P) = \infty\}$ is a λ-null set E_0. Set $f(P) = 0$ for all P in E_0 and set

$$f(P) = g_1(P) + \sum_{i=1}^{\infty} (g_{i+1}(P) - g_i(P))$$

for all other P. Then $\lambda(f)$ is finite, $\lambda(f - g_i) \to 0$ as $i \to \infty$, and hence $\lambda(f - f_n) \leqslant \lambda(f - g_i) + \lambda(g_i - f_n) \to 0$ as n (and i) $\to \infty$.

Remark 1. Theorem 3.1 shows that the spaces $L_W{}^p(B)$, $L_{(w)}{}^p(B)$, and $M_{(w)}{}^q(B)$, in particular, are Banach spaces [cf. **7**, p. 276; **14**; **3a**, pp. 130, 131].

Remark 2. If f is in $L^\lambda(B)$ and E is λ-purely-infinite, then $f(P) = 0$ for almost all P in E.

4. Levelling length functions. We refer to §2 of this paper and to [**7**, §2] for terminology.

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By repetition, (L 8) implies, under the same conditions, the more general

$$\text{(L 8)}' \qquad \lambda\Big(\sum_{i=1}^{m} k_{e,i}\Big) \geqslant \lambda\bigg(\Big(\frac{k_1\gamma(e_1)+\ldots+k_r\gamma(e_r)}{\gamma(e_1+\ldots+e_r)}\Big)_{e_1+\ldots+e_r} + \sum_{i=r+1}^{m} k_{e,i}\bigg)$$

for all $2 \leqslant r \leqslant m$. For a continuous length function, (L 8) actually implies, as can be verified without difficulty:

(L 8)″ *For every finite or countable collection of disjoint e_i with all $\gamma(e_i) > 0$, $\lambda(u)$ is not increased if on each e_i, $u(P)$ is replaced by its average on e_i, namely*

$$\frac{1}{\gamma(e_i)}\int_{e_i} u(P)\,d\gamma(P).$$

The particular case of (L 8) with $m = 2$ and $k_1 = 1$, $k_2 = 0$, together with (L 4) and (L 2) gives

$$\text{(L 8)}''' \qquad \lambda(e_1) \geqslant \frac{\gamma(e_1)}{\gamma(e_1+e_2)}\,\lambda(e_1+e_2) \geqslant \frac{\gamma(e_1)}{\gamma(e_1+e_2)}\,\lambda(e_2).$$

It is now easy to verify the following statements for an arbitrary levelling length function; $0 < \lambda(e) < \infty$ whenever $0 < \gamma(e) < \infty$; E is λ-null if and only if $\gamma(E) = 0$; $\lambda(u) = 0$ if and only if $u(P) = 0$ for almost all P; there are no λ-purely-infinite sets; if $\gamma < \infty$ then $\lambda(S) < \infty$ but if $\gamma = \infty$ then $0 < \lambda(E) = \lambda(S) \leqslant \infty$ for all E with $\gamma(E) = \infty$; $\lambda(u) < \infty$ implies that u has finite integral on every e; and if S is not coarse then $\gamma(e) > 0$, $\gamma(e) \to 0$ implies $\lambda(e) \to \lambda(+0) \geqslant 0$.

If λ is a levelling length function, then

$$\begin{aligned} \lambda^*(e) &= \sup \int_e u(P)\,d\gamma(P), && \lambda(u) \leqslant 1,\\ &= \sup \int_e k\,d\gamma(P), && k \geqslant 0,\ k\lambda(e) \leqslant 1,\\ &= \gamma(e)\,\lambda(e)^{-1}, && \lambda(e) > 0. \end{aligned}$$

Thus we have the identity:

$$\lambda(e)\,\lambda^*(e) = \gamma(e).$$

Throughout the remainder of this paper λ will denote a levelling length function.

THEOREM 4.1. (i) $\lambda^{**} = \lambda$.
(ii) *λ^* is a levelling length function.*
(iii) *the $\lambda\lambda^*$-Hölder inequality*

$$\text{(4.1)} \qquad \int u(P)v(P)\,d\gamma(P) \leqslant \lambda(u)\,\lambda^*(v)$$

holds, as well as the converses: the supremum of the left-hand side of (4.1) *for all v with $\lambda^*(v) \leqslant 1$, is $\lambda(u)$; and the supremum of the left-hand side of* (4.1) *for all u with $\lambda(u) \leqslant 1$, is $\lambda^*(v)$. Moreover, the first of these converses holds even if v is further restricted to be constant on each of any finite or countable collection of e_i*

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on each of which the fixed u is constant and similarly the second converse holds even if u is further restricted to be constant on each of any finite or countable collection of e_i on which the fixed v is constant.

Proof of (i). For fixed m let $i = 1, \ldots, m$ and let e_i be fixed, disjoint sets of positive measure. The finitely valued functions in L^λ of the form $\alpha = \sum_i \alpha_{e,i}$ form an m-dimensional Banach space H. The conjugate H^* can clearly be represented as the set of finitely valued functions $\beta = \sum_i \beta_{e,i}$ where $\beta\alpha$ shall mean $\sum_i \alpha_i \gamma(e_i) \beta_i$; the norm of β in H^* has the value sup $\sum_i \alpha_i \gamma(e_i) \beta_i$ for all $\lambda(\alpha) \leqslant 1$, and this equals $\lambda^*(\beta)$ since λ is a levelling length function and (L 8)$''$ is valid. Hence for every such α there is a β with $\lambda^*(\beta) = 1$ and $\beta\alpha = \lambda(\alpha)$; it follows easily that $\lambda^{**}(u) \geqslant \lambda(u)$ for all finitely valued u, and hence for all u. Since $\lambda(u) \geqslant \lambda^{**}(u)$ holds for every length function, this proves (i).

Proof of (ii) *and* (iii). Since $\lambda^*(e) = \gamma(e)/\lambda(e)$ and $0 < \lambda(e) < \infty$ whenever $0 < \gamma(e) < \infty$ it follows that λ^* satisfies (L 7). Using (L 8)$''$ for λ we can write

$$(4.2) \qquad \lambda^*\left(\left(\frac{k_1 \gamma(e_1) + k_2 \gamma(e_2)}{\gamma(e_1 + e_2)}\right)_{e_1+e_2} + \sum_{i=3}^{m} k_{e,i}\right)$$
$$= \sup \left(\frac{k_1 \gamma(e_1) + k_2 \gamma(e_2)}{\gamma(e_1 + e_2)} \gamma(e_1 + e_2)\, h + \sum_{i=3}^{m} k_i \gamma(e_i)\, h_i\right)$$

for all non-negative h, h_i ($i \geqslant 3$) with $\lambda(\sum_{i=1}^{m} h_{e,i}) \leqslant 1$ (where $h_1 = h_2 = h$). Since this supremum is not decreased when the condition $h_1 = h_2$ is removed, we find that

$$\text{the expression (4.2)} \leqslant \sup \sum_{i=1}^{m} k_i \gamma(e_i)\, h_i$$

for all non-negative h_i with

$$\lambda\left(\sum_{i=1}^{m} h_{e,i}\right) \leqslant 1.$$

Hence

$$\text{the expression (4.2)} \leqslant \lambda^*\left(\sum_{i=1}^{m} k_{e,i}\right),$$

showing that λ^* satisfies (L 8). This proves (ii), and (iii) follows easily.

We note that $|F|(e)$ is a numerically valued additive set function for which $|F|(e) = 0$ whenever $\gamma(e) = 0$, and hence $\lambda(|F|)$ is defined.

THEOREM 4.2. *For every F valued in B,*

(i) $$|F(e)| \leqslant |F|(e) \leqslant \lambda(F_e)\, \lambda^*(e) \leqslant \lambda(F)\, \lambda^*(e)$$

for every e,

(ii) $$\lambda(F) = \lambda(|F|).$$

Proof. Let m be fixed, $i = 1, \ldots, m$ and suppose $e = e_1 + \ldots + e_m$ with e_i disjoint sets of positive measure. Then

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$$\lambda(F_e) \geqslant \lambda \sum_i \left(\frac{|F(e_i)|}{\gamma(e_i)}\right)_{e_i} \geqslant \lambda\left(\frac{\sum_i |F(e_i)|}{\gamma(e)}\right)_e$$
$$= \frac{\sum_i |F(e_i)|}{\gamma(e)} \lambda(e),$$

showing that $\sum_i |F(e_i)| \leqslant \lambda(F_e)\, \lambda^*(e)$. This implies (i).

To prove (ii), since $\lambda(|F|) \geqslant \lambda(F)$ is clear (in fact, $|F|(e) \geqslant |F(e)|$ for every e), we need only prove $\lambda(|F|) \leqslant \lambda(F)$. Hence we may suppose $\lambda(F)$, and by (i), all $|F|(e)$, to be finite. Now let m be fixed, $i = 1, \ldots, m$, and e_i disjoint sets of positive measure; then for every $\epsilon > 0$ there exist disjoint decompositions

$$e_i = \sum_j e_{ij} \qquad (j = 1, \ldots, m_i),$$

with $\gamma(e_{ij}) > 0$ for all i, j and

$$|F|(e_i) = \sum_j |F(e_{ij})| + \epsilon_i$$

with $0 \leqslant \epsilon_i < \epsilon\gamma(e_i)$ for all i. Then

$$\lambda \sum_i \left(\frac{|F|(e_i)}{\gamma(e_i)}\right)_{e_i} \leqslant \inf_{\epsilon>0} \left(\lambda \sum_i \left(\sum_j \frac{|F(e_{ij})|}{\gamma(e_i)}\right)_{e_i} + \epsilon \sum_i \lambda(e_i)\right)$$
$$\leqslant \inf_{\epsilon>0} \left(\lambda \sum_i \sum_j \left(\frac{|F(e_{ij})|}{\gamma(e_{ij})}\right)_{e_{ij}} + \epsilon \sum_i \lambda(e_i)\right)$$
$$\leqslant \lambda(F).$$

This implies $\lambda(|F|) \leqslant \lambda(F)$ and completes the proof of (ii).

We note that for a levelling length function: each of (L 11) and (L 12) implies (L 10) but (L 10) need not imply either of (L 11) or (L 12); and (L 12) need not imply (L 11). One of (L 9), (L 9)* can fail to hold but at least one of them holds since $\lambda(e)\, \lambda^*(e) = \gamma(e)$.

THEOREM 4.3. *If* (L 9)* *holds and* $\lambda(F) < \infty$ *then*:

(i) $$F\left(\sum_{i=1}^m e_i\right) \to F(e) \text{ and } |F|\left(\sum_{i=1}^m e_i\right) \to |F|(e)$$

whenever $$e = \sum_{i=1}^\infty e_i.$$

(ii) *For every* $E = e_1 + e_2 + \ldots$ *there is a point function* $u(E) = u(E, P) \geqslant 0$ *for all* P *for which* $\lambda(F_E) = \lambda(|F_E|) = \lambda(u(E))$ *and*

$$|F|(e) = \int_e u(E, P)\, d\gamma(P)$$

for all $e \subset E$.

(iii) $\lambda(F_{E(n)}) \to \lambda(F_E)$ *whenever*

$$E = \sum_{i=1}^\infty E_i$$

where $E(n)$ *denotes* $E_1 + \ldots + E_n$.

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Proof. By Theorem 4.2 and the additivity of F and $|F|$, setting

$$e' = e - \sum_{i=1}^{m} e_i,$$

we obtain

$$|F(e) - F\left(\sum_{i=1}^{m} e_i\right)| = |F(e')| \leqslant \lambda(F)\,\lambda^*(e'),$$

and the same relation holds with $|F|$ in place of F. When $m \to \infty$, $\gamma(e') \to 0$ and (L 9)* implies that $\lambda^*(e') \to 0$. This proves (i).

The Radon-Nikodym theorem then shows that $u(E)$ exists with

$$|F|(e) = \int_e u(E, P)\, d\gamma(P)$$

for all $e \subset E$. By (L 8)″, $\lambda(F_E) \leqslant \lambda(u(E))$. To complete the proof of (ii) we need only show that for $e \subset E$ with $u(E, P)$ bounded on e, say $u(E, P) \leqslant K$ on e, $\lambda((u(E))_e) \leqslant \lambda(F_E)$. For this purpose, with fixed m and $i = 1, \ldots, m$, let e_i be the subset of e on which $K(i-1)/m \leqslant u(E, P) \leqslant Ki/m$. Then for almost all P,

$$(u(E))_e(P) \leqslant \sum_{i=1}^{m}{}' \left(\frac{|F|(e_i)}{\gamma(e_i)}\right)_{e_i}(P) + \frac{K}{m},$$

where $\sum'$ indicates that the i for which $\gamma(e_i) = 0$ are to be omitted. Hence

$$\lambda((u(E))_e) \leqslant \lambda(F_E) + \frac{K}{m}\lambda(e)$$

for all m, implying $\lambda((u(E))_e) \leqslant \lambda(F_E)$ and proving (ii).

To prove (iii) we need only show that $\lambda(F_E) \leqslant \sup \lambda(F_{E(n)})$ for all n since $\lambda(F_{E(n)})$ is non-decreasing with increasing n and does not exceed $\lambda(F_E)$. We need therefore only show that for fixed m $(i = 1, \ldots, m)$, and disjoint e_i of positive measure contained in E,

$$\lambda \sum_i \left(\frac{|F(e_i)|}{\gamma(e_i)}\right)_{e_i} \leqslant \sup \lambda(F_{E(n)})$$

for all n. Now by (i), for each i, $|F(e_{i,r})| \to |F(e_i)|$ as $r \to \infty$, where $e_{i,r}$ denotes $e_i(E_1 + \ldots + E_r)$. Hence it is sufficient to show that for every r,

$$\lambda \sum_i \left(\frac{F(e_{i,r})}{\gamma(e_{i,r})}\right)_{e_{i,r}} \leqslant \lambda(F_{E(n)})$$

for some n; and this is clearly so for $n \geqslant r$.

COROLLARY 1. *If* (L 9)* *holds and* $\lambda(F_e)$ *is finite then* e *can be decomposed into a finite or countable number of disjoint* e_i *on each of which* F *is majorized.*

COROLLARY 2. *If* (L 9)* *holds and* $\lambda(F)$ *is finite and* (L 10)′ *holds then there exists a decomposition* $S = E_0 + e_1 + e_2 + \ldots$ *with disjoint* E_0, e_i *such that* F *is majorized on each* e_i *and vanishes on* E_0.

COROLLARY 3. (L 11) *and* (L 9)* *together imply* (L 10)′ *if* λ *is a levelling length function.*

5. $L^\lambda(B)^*$ in terms of set functions.

THEOREM 5.1. *If* (L 12) *and* (L 13) *hold, the relation*

$$G(e)c = \Phi(c_e) \tag{5.1}$$

for all c *in* B *and all* e, *sets up a* (1, 1) *linear correspondence between all* Φ *in* $L^\lambda(B)^*$ *and all* G, *valued in* B^*, *for which* $\lambda^*(G)$ *is finite; for corresponding elements,* $|\Phi| = \lambda^*(G)$.

Proof. For given Φ in $L^\lambda(B)^*$, $|\Phi(c_e)| \leqslant |\Phi||c|\,\lambda(e)$ and hence (5.1) defines $G(e)$, valued in B^*, as an additive set function with $G(e) = 0$ whenever $\gamma(e) = 0$. To see that $\lambda^*(G) \leqslant |\Phi|$, let $i = 1, \ldots, m$ for fixed m, and choose arbitrary disjoint e_i of positive measure, arbitrary c_i in B with $|c_i| = 1$ and arbitrary scalars α_i; set $f(P) = \sum_i \alpha_i c_{e,i}(P)$. Then f is in $L^\lambda(B)$, $\lambda(f) = \lambda(\sum_i \alpha_{e,i})$ and $|\Phi(f)| \leqslant |\Phi|\lambda(f)$; hence

$$\left|\sum_i \alpha_i \gamma(e_i)\beta_i\right| \leqslant |\Phi|\lambda\left(\sum_i \alpha_{e,i}\right), \quad \beta_i = \frac{G(e_i)c_i}{\gamma(e_i)};$$

the converse of the $\lambda\lambda^*$-Hölder inequality now shows that $\lambda^*(\sum_i \beta_{e,i}) \leqslant |\Phi|$. Since the c_i can be chosen to make $G(e_i)c_i$ arbitrarily near to $|G(e_i)|$ for each i,

$$\lambda^*\left(\sum_i \left(G(e_i)/\gamma(e_i)\right)_{e_i}\right) \leqslant |\Phi|$$

implying $\lambda^*(G) \leqslant |\Phi|$.

On the other hand, the $\lambda\lambda^*$-Hölder inequality shows that $|\Phi(f)| \leqslant \lambda^*(G)\,\lambda(f)$ for finitely valued f and (L 12), (L 13) imply that these f are dense in $L^\lambda(B)$. This means that $|\Phi| \leqslant \lambda^*(G)$ and so $|\Phi| = \lambda^*(G)$.

Conversely, for given G with $\lambda^*(G)$ finite, (5.1) defines $\Phi(c_e)$ for all c in B and all e; for finitely valued $f = \sum_i c_{e,i}$ we set $\Phi(f) = \sum_i \Phi(c_{e,i})$. Then $\Phi(f)$ is defined and

$$|\Phi(f)| \leqslant \lambda^*(G)\,\lambda(f)$$

for a set of f dense in $L^\lambda(B)$. It follows that $\Phi(f)$ extends uniquely to all f in $L^\lambda(B)$ so that the extension Φ is in $L^\lambda(B)^*$ and $|\Phi| \leqslant \lambda^*(G)$. The preceding two paragraphs now show that $|\Phi| = \lambda^*(G)$.

6. Set functions in terms of integrals.

We refer to §2 for terminology.

If $\dot{g}$ is such that $g(c, P)$ is identical with $g(P)c$ for some point function $g(P)$ valued in B^*, for all c in B, and if $|g(P)|$ is measurable, then $\lambda^*(\dot{g}) \leqslant \lambda^*(g)$ and inequality may actually occur. However, if the point function $g(P)$ is Bochner measurable, it is not difficult to verify that equality holds.

LEMMA 6.1. *Suppose* G, *valued in* B^*, *corresponds by the relation*

$$G(e)c = \int_e g(c, P)\,d\gamma(P) \tag{6.1}$$

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for all c in B and all e, to a $\dot{g} = g(c, P)$ *for which*

$$\int_e |g(c, P)| \, d\gamma(P)$$

is finite for every c and every e. Then

$$\lambda^*(G) = \lambda^*(\dot{g}) \leqslant \infty. \tag{6.2}$$

Proof. When c varies over all elements in B with $|c| = 1$,

$$|G(e)| = \sup \left| \int_e g(c, P) \, d\gamma(P) \right| \leqslant \sup \int_e |g(c, P)| \, d\gamma(P).$$

(L 8)″ then implies that $\leqslant$ holds in (6.2). To prove that $\geqslant$ holds in (6.2), we need only show that for disjoint e_i and $|c_i| = 1$,

$$\lambda^* \sum_i g(c_{e,i}) \leqslant \lambda^*(G).$$

We need only consider the case that for every i, $k \leqslant |g(c_i, P)| \leqslant K$ for all P in e_i for some $0 < k < K < \infty$. Now suppose $0 < \eta < 1$ and decompose each e_i into disjoint subsets $e_i = \sum_j e_{ij}$ $(j = 1, \ldots, m_i)$, so that for each e_{ij}, $\gamma(e_{ij}) > 0$ and there is scalar h_{ij} such that for all P in e_{ij},

$$|g(c_i, P)| \geqslant |h_{ij}|, \quad |g(c_i, P) - h_{ij}| \leqslant \eta |h_{ij}|,$$

(this can clearly be done, using a partition of the complex numbers $k \leqslant |z| \leqslant K$ by a finite number of parallels to the axes of reals and pure imaginaries). Then for all P in e_{ij},

$$\frac{|G(e_{ij})|}{\gamma(e_{ij})} \geqslant \frac{1}{\gamma(e_{ij})} \left(\int_{e_{ij}} |h_{ij}| \, d\gamma(P) - \int_{e_{ij}} |g(c_i, P) - h_{ij}| \, d\gamma(P) \right)$$
$$\geqslant |h_{ij}|(1 - \eta) \geqslant (1 - \eta)(1 + \eta)^{-1} |g(c_i, P)|.$$

This implies $\lambda^*(G) \geqslant (1 - \eta)(1 + \eta)^{-1} \lambda^* \sum_i g(c_{e,i})$ for all $0 < \eta < 1$ and hence $\geqslant$ holds in (6.2). This establishes (6.2).

Remark. If $|g(c, P)|$ has *finite* integral on every e for every c then there always exists a G, valued in B^* which corresponds to $\dot{g}$ by (6.1). This is shown by the argument of [**5**, p. 308].

THEOREM 6.1. *If* (L 10)′* *holds or if S has property* (*R*) *then the relation* (6.1) *sets up a* (1, 1) *linear correspondence between the* $\dot{g}$ *with* $\lambda^*(\dot{g})$ *finite and the* G, *valued in* B^* *with* $\lambda^*(G)$ *finite; for corresponding elements* $\lambda^*(G) = \lambda^*(\dot{g})$.

Proof. For given $\dot{g}$, if $\lambda^*(\dot{g})$ is finite, the $\lambda\lambda^*$-Hölder inequality shows that

$$\int_e |g(c, P)| \, d\gamma(P) \leqslant \lambda(e) \lambda^*(\dot{g}) |c|$$

for all c in B and all e; hence (6.1) defines G, valued in B^*, and Lemma 6.1 shows that $\lambda^*(G) = \lambda^*(\dot{g})$.

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Conversely, if G, valued in B^*, has $\lambda^*(G)$ finite, then

$$|G(e)c| \leqslant \lambda^*(G)\,\lambda(e)|c|$$

so that $G(e)c$ is, for fixed c, a numerically valued absolutely continuous set function. By the Radon-Nikodym theorem, for given $E = e_1 + e_2 + \ldots$ there exists an integrable function $g(E)(c, P)$ satisfying (6.1) for all $e \subset E$. If (L 10)′* holds or if S has property (R) then $\dot{g} = g(c, P)$ may be defined by "combining" the $g(E)(c, P)$ so that $\dot{g}$ will satisfy the requirements of the theorem.

THEOREM 6.2. *If* (L 10)′* *holds or if S has property* (R), *and if also* (L 12) *and* (L 13) *hold, then the relation*

$$\Phi(c_e) = \int_e g(c, P)\, d\gamma(P) \tag{6.3}$$

sets up a (1, 1) *linear correspondence between the Φ in $L^\lambda(B)^*$ and the $\dot{g}$ with $\lambda^*(\dot{g})$ finite; for corresponding elements $|\Phi| = \lambda^*(\dot{g})$.*

Proof. This theorem is an immediate consequence of Theorems 6.1 and 5.1.

7. $L^\lambda(B)^*$ in terms of point functions. We refer to §2 for terminology; discussions of measurability, integrability and integral for vector-valued functions (and additional relevant references) are given in [**2**], [**5**], [**8**], and [**9**]. The BH-integral of $g(P)$, if it exists at all, will be unique since H is required to be total on B but the BH-integral may depend on B and H for a given g; however, it can be shown that every H^*H-integrable g (this is the special case of $B = H^*$) does have an H^*H-integral G [**5**, p. 308].

The function g is Bochner measurable if and only if g is locally almost-separable-valued, that is, for every e there is an e_0 of zero measure contained in e such that the set of values of $g(P)$, for all P in $e - e_0$, is a separable set and, for every c in this separable set, $|g(P) - c|$ is measurable; thus the Bochner measurability of g, though dependent on h is *not* dependent on B provided that for every e, almost all values of $g(P)$ for P in e are contained in B. If g is Bochner integrable, then the Bochner integral always exists, it is uniquely determined and it depends on g but not on B. Bochner integrability implies that the BH-integral exists, and coincides with the Bochner integral for every $H \subset B^*$ with H total on B.

If B is separable-controlled by H then B can be considered as part of $\tilde{H}^*$ which will also be separable-controlled by H; thus B is separable-controlled if and only if B can be imbedded isometrically in the conjugate to a separable Banach space. Every separable B is, of course, separable-controlled. If B is separable-controlled by H then B is separable-controlled by a suitable countable subset of elements of norm 1 contained in H.

LEMMA 7.1. *Suppose* (L 9) *and either* (L 10)′* *or property* (R) *for S hold. If $\lambda^*(G)$ is finite and $V(G)$ is separable-controlled by a Banach space H then G is the H^*H-integral of some g, valued in H^*, such that $|g(P) - h|$ is measurable for*

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every h in H^* and $\lambda^*(g) = \lambda^*(\dot{g}) = \lambda^*(G)$ where $\dot{g} = g(c) = g(c, P) = g(P)c$ and c varies over H.

LEMMA 7.2. *Suppose* (L 9) *and either* (L 10)′* *or property* (R) *for* S *hold. If $\lambda^*(G)$ is finite and $V(G) \subset B^*$ for some separable B^*, then there is a $g(P)$, valued in B^*, such that G is the Bochner integral of g and $\lambda^*(g) = \lambda^*(G)$.*

LEMMA 7.3. *Suppose* (L 9) *and either* (L 10)′* *or property* (R) *for* S. *hold. If $\lambda^*(G)$ is finite and, for every e, $V(G_e)$ is separable and locally weakly compact, then there is a $g(P)$, valued in $V(G)$ such that G is the Bochner integral of g and $\lambda^*(g) = \lambda^*(G)$.*

Remark. If (L 9) holds and $\lambda^*(G)$ is finite and e has a countable basis, then $V(G_e)$ is necessarily separable.

Proof of Lemma 7.1. Suppose G majorized by M on some e. Since G may be considered as valued in H^*, Theorem 6.1 implies the existence of a $\dot{g}$ such that *for every c* in H,

$$\left| \int_{e_1} g(c, P)\, d\gamma(P) \right| = |G(e_1)c| \leqslant M|c|\, \gamma(e_1)$$

for every $e_1 \subset e$. It follows that $|g(c, P)| \leqslant M|c|$ outside of a set of zero measure (depending on c) and we may suppose $g(c, P) = 0$ for P outside e.

Since H is separable there exists a countable set (c_k) of elements dense in H and including all finite rational linear combinations of its members. Then for every P outside a single set N of zero measure, $g(c_k, P)$ is rational linear in c_k and $|g(c_k, P)| \leqslant M|c_k|$ for all k. For fixed P set $g_1(c_k, P) = g(c_k, P)$ outside N, $= 0$ in N and for the remaining c in H, define $g_1(c, P)$ to be $\lim g_1(c^m, P)$ where c^m is any subsequence of the c_k with $c^m \to c$ as $m \to \infty$. It is easily verified that $g_1(c, P)$ is now uniquely defined, and that $|g_1(c, P)| \leqslant M|c|$ for all c and all P; therefore $g_1(c, P)$ can be expressed in the form $g(P)c$ with $g(P)$ valued in H^* and uniquely defined and with $|g(P)| \leqslant M$ for every P. Now for each c in H, $g(P)c = g_1(c, P)$ is measurable and

$$\begin{aligned}\int_e |g(P)c - g(c, P)|\, d\gamma(P) \leqslant \int_e |g(P)(c - c_k)|\, d\gamma(P) \\ + \int_e |g(P)c_k - g(c_k, P)|\, d\gamma(P) \\ + \int_e |g(c - c_k, P)|\, d\gamma(P) \\ \leqslant 2M\gamma(e)|c - c_k|\end{aligned}$$

for all k. This implies that for every fixed c in H, $g(P)c$ and $g(c, P)$ can be identified.

Corollaries 1 and 2 to Theorem 4.3 enable us to decompose S into sets e on which G is majorized, and we can do this in such a way, that by combining the $g(P)$ corresponding to these e, we get a single $g(P)$ which is H^*H-integrable and has G as its H^*H-integral and for which $\lambda^*(\dot{g}) = \lambda^*(G)$ where $\dot{g}$ denotes $g(c, P) = g(P)c$.

Now let (c_j), with each $|c_j| = 1$, be a countable subset of H by which H^* is separable-controlled. Since $g(P)c$ is measurable for every c it follows that for every h in H^*, $(g(P) - h)c_j$ and $\sup |(g(P) - h)c_j|$ for all j, are measurable, and hence $|g(P) - h|$ is measurable. Furthermore $|g(P)| = \sup |g(c_j, P)|$ for all j, from which it follows that

$$\lambda^*(g) = \lambda^*(\dot{g}) = \lambda^*(G).$$

Proof of Lemma 7.2. Since B^* is separable-controlled by B, Lemma 7.1 applies (with B in place of H) and gives a $g(P)$ which is Bochner measurable since B^* is separable. Furthermore, $\lambda^*(g) < \infty$ implies that $|g(P)|$ has finite integral on each e and hence that g has a Bochner integral, necessarily coinciding with its B^*B-integral G. This proves the lemma.

Proof of Lemma 7.3. This follows from Lemma 7.2 since a separable and locally weakly compact space is reflexive [**1**, Théorème 13, p. 189] and hence $V(G_e)$ can be taken as B^* with $B = V(G_e)^*$.

THEOREM 7.1. *Suppose the following hold:* (L 9), *either* (L 10)'* *or property* (R) *for* S, (L 12), *and* (L 13). *If* B *is separable then the relation, for every* f *in* $L^\lambda(B)$,

$$\Phi(f) = \int f(P)g(P)\, d\gamma(P) \tag{7.1}$$

sets up a (1, 1) *linear correspondence between the* Φ *in* $L^\lambda(B)^*$ *and the* B^*B-*integrable* $g(P)$ *with finite* $\lambda^*(g)$; *for corresponding elements* $\lambda^*(g) = |\Phi|$.

Proof. To any Φ in $L^\lambda(B)^*$ corresponds a G, valued in B^* with $\lambda^*(G) = |\Phi|$ by Theorem 5.1. Since each $V(G)$ is separable-controlled by B, Lemma 7.1 gives the existence of a unique function $g(P)$ having G as B^*B-integral and with $\lambda^*(g) = \lambda^*(G) = |\Phi|$. Relation (7.1) follows at once for all finitely valued f. Since the finitely valued f are dense in $L^\lambda(B)$ and since, for such f,

$$\int |f(P)g(P)|\, d\gamma(P) \leqslant \lambda(f)\, \lambda^*(g),$$

it follows that the numerical valued function $f(P)g(P)$ is measurable and (7.1) holds for every f in $L^\lambda(B)$.

Conversely, suppose $g(P)$ is B^*B-integrable and $\lambda^*(g) < \infty$. Then for every finitely valued f, (7.1) defines $\Phi(f)$ so that $|\Phi(f)| \leqslant \lambda^*(g)\lambda(f)$ and Φ has a unique extension which satisfies (7.1) for all f in $L^\lambda(B)$.

THEOREM 7.2. *If* B^* *is separable and* (L 9), *either* (L 10)'* *or property* (R) *for* S, (L 12), *and* (L 13) *hold, then* $L^\lambda(B)^* = L^\mu(B^*)$ *where* $\mu = \lambda^*$. *If* B^{**} *is separable and* (L 9), (L 9)*, *either* (L 10)' *and* (L 10)'* *or property* (R) *for* S, (L 12), (L 12)*, (L 13), *and* (L 13)* *all hold then* $L^\lambda(B)^{**} = L^\lambda(B^{**})$.

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Proof. For every G, valued in B^*, with $\lambda^*(G)$ finite, $V(G)$ is separable and separable-controlled by B [**1**, Théorème 12, p. 189]. Lemma 7.2 shows that $L^\lambda(B)^* \subset L^\mu(B^*)$. Since $L^\lambda(B)^* \supset L^\mu(B^*)$ is easily verified with the help of the $\lambda\lambda^*$-Hölder inequality, $L^\lambda(B)^* = L^\mu(B^*)$. The other parts of the theorem follow immediately from this.

THEOREM 7.3. *Suppose B is reflexive. If B is separable or if every e has a countable basis, and if* (L 9), *either* (L 10)′* *or property* (R) *for S,* (L 12), *and* (L 13) *hold, then $L^\lambda(B)^* = L^\mu(B^*)$; on the other hand, if* (L 9), (L 9)*, *either* (L 10)′ *and* (L 10)′* *or property* (R) *for S,* (L 12), (L 12)*, (L 13), *and* (L 13)* *all hold then $L^\lambda(B)^* = L^\mu(B^*)$ and $L^\lambda(B)$ is reflexive.*

Proof. The first part of the theorem follows from Theorem 5.1, Lemma 7.3, and the remark following Lemma 7.3, since for a Banach space, reflexitivity is equivalent to locally weak compactness [**6**].

To prove the second part of the theorem we need only show that if B is locally weakly compact a_ed (L 9), (L 9)*, either (L 10)′ and (L 10)′* or property (R) for S, (L 12), (L 12)*, (L 13), (L 13)* all hold, then $L^\lambda(B)$ is locally weakly compact; indeed it is sufficient to show, under these conditions, that any sequence of finitely valued functions $f_n = \sum_i c_{n,e,i}$ with all $\lambda(f_n) \leqslant 1$ has a subsequence which converges weakly. But these f_n may be considered as valued in B_1 where B_1 is a suitable separable subspace of B. Since the first part of this theorem shows that $L^\lambda B_1)$ is reflexive, hence locally weakly compact, the theorem follows.

Theorems 7.1, 7.2, and 7.3 were developed in the classical $L^p(B)$ case, by Bochner and Taylor [**3**], Pettis [**10**], Dunford [**5**], Phillips [**11**; **12**], and Dieudonné [**4**].

8. The RN property.

We refer to §2 for terminology.

THEOREM 8.1. *For given B and S let λ vary over all levelling length functions satisfying* (L 9), (L 12), (L 13), *and, if S does not have the property* (R), (L 10)′*. *Then the relation $L^\lambda(B)^* = L^\mu(B^*)$, with $\mu = \lambda^*$, holds for one of these λ if and only if it holds for all of them and if and only if B^* has the RN property on S. If B is reflexive then B has the RN property on every S.*

Proof of necessity. If G, valued in B^*, is majorized on e then G_e is related by (5.1) to a Φ in $L^\lambda(B)^*$, hence to a Bochner measurable $g(P)$ which has G_e as its Bochner integral. Since g can be approximated by finitely valued functions the RN property can be established.

Proof of sufficiency. Any Φ in $L^\lambda(B)^*$ corresponds by (5.1) to some G, valued in B^*, with $\lambda^*(G)$ finite. It is sufficient to show that if G is majorized by M on some e, then there is a g such that G_e is the Bochner integral of g. Now by the RN property, for fixed n, e can be decomposed into a finite or countable number of disjoint e_{ni} of positive measure such that, for some c_{ni},

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$$\left|\frac{G(e')}{\gamma(e')} - c_{ni}\right| < \frac{1}{n}$$

for all $e' \subset e_{ni}$ with $\gamma(e') > 0$. Define $g_n(P)$ as the countably valued function with value c_{ni} on e_{ni} for all n and i. Let N be union of all intersections of any finite collection of the e_{ni} which have zero measure. Then N is also a set of zero measure, and for any P not in N and any n, m there is a set of positive measure $e' = e_{ni}\, e_{mj}$ containing P; hence

$$|g_n(P) - g_m(P)| \leqslant \left|\frac{G(e')}{\gamma(e')} - g_n(P)\right| + \left|\frac{G(e')}{\gamma(e')} - g_m(P)\right| \leqslant \frac{1}{n} + \frac{1}{m}$$

so that $g_n(P)$ converges uniformly, for all P outside N, to a limit $g(P)$. Clearly g is Bochner measurable and Bochner integrable.

Now for any e' and any n, let

$$e_{nk}' = e' \sum_{i=1}^{k} e_{ni}.$$

Then for every n,

$$\begin{aligned} \left|G(e') - \int_{e'} g(P)\, d\gamma(P)\right| &\leqslant \sup_k \left|G(e_{nk}') - \int_{e_{nk}'} g_n(P)\, d\gamma(P)\right| \\ &\quad + \int_{e'} |g_n(P) - g(P)|\, d\gamma(P) \\ &\leqslant \frac{2}{n}\gamma(e). \end{aligned}$$

This shows that G_e is the Bochner integral of g as required and proves the sufficiency part of the theorem.

9. The $L_{(w)}{}^p(B)$ and $M_{(w)}{}^q(B)$ spaces.

We refer to [**7**, §2] and to §2 of this paper for terminology.

Theorem 5.1 of [**7**] states that $\lambda_{(w)}{}^{p*} = \mu_{(w)}{}^q$ and it is not difficult to verify, with the help of the remark following Theorem 5.1 of [**7**], that $\lambda_{(w)}{}^p$ satisfies (L 8). It follows that $\lambda_{(w)}{}^p$ and $\mu_{(w)}{}^q$ are conjugate levelling length functions. They have the following properties:

(L 9) holds for $\lambda_{(w)}{}^p$ except when S fails to be coarse with $p = \infty$ (the proof is easy).

(L 9) holds for $\mu_{(w)}{}^q$ except when S fails to be coarse with $q = \infty$ and with $w^*(x)$ bounded (from the relation $\mu(e) = \gamma(e)/\lambda(e)$).

(L 10) holds for $\lambda_{(w)}{}^p$ for all p in Case (C_1) and for $1 \leqslant p < \infty$ in Case (C_2) [**7**, Corollary to Theorem 5.6]. (L 10)′ holds in these cases also.

(L 10) holds for $\mu_{(w)}{}^q$ except when $q = \infty$ in Case (C_2) [**7**, Corollary to Theorem 5.7]. (L 10)′ holds in these cases also.

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(L 11) holds for $\lambda_{(w)}^p$ only in Case (C_1) with $1 \leqslant p < \infty$ and $w^*(x) > 0$ for all $0 < x < \gamma$ and in Case (C_2) with $1 \leqslant p < \infty$ [**7**, Theorem 5.3(i)].

(L 11) holds for $\mu_{(w)}^q$ except when $q = \infty$ [**7**, Theorem 5.3(ii)].

(L 12) holds for $\lambda_{(w)}^p$ only in Case (C_1) with $1 \leqslant p \leqslant \infty$ and in Case (C_2) with $1 \leqslant p < \infty$ [**7**, Theorem 5.6].

(L 12) holds for $\mu_{(w)}^q$ except when $q = \infty$ in Case (C_2) [**7**, Theorem 5.7].

(L 13) holds for $\lambda_{(w)}^p$ in all cases [**7**, Theorem 5.4(i)].

(L 13) holds for $\mu_{(w)}^q$ except when $q = \infty$ with $w^*(x)$ unbounded [**7**, Theorem 5.5(i)].

By specializing Theorems 7.1, 7.2, and 7.3 to these particular spaces we obtain:

(i) Suppose B is separable and specialize Theorem 7.1. Then: (α) $L_{(w)}^p(B)^*$ can be identified with the Banach space of $g(P)$ valued in B^* which are B^*B-integrable and have finite norm $\mu_{(w)}^q(g)$ in the following cases: Case (C_1) with $1 \leqslant p < \infty$, Case (C_1) with $p = \infty$ and S coarse, Case (C_2) with $1 \leqslant p < \infty$ (assuming, if $p = 1$, that S has property (R)); (β) $M_{(w)}^q(B)^*$ can be identified with the Banach space of $f(P)$ valued in B^* which are B^*B-integrable and have finite norm equal to $\lambda_{(w)}^p(f)$ in the following cases: Case (C_1) with either $1 \leqslant q < \infty$ or $q = \infty$ and S coarse, Case (C_2) with $1 \leqslant q < \infty$ (assuming, if $q = 1$, that S has property (R)), Case (C_3) with $1 \leqslant q < \infty$ (assuming for all $1 \leqslant q < \infty$, that S has property (R)), and Case (C_3) with $q = \infty$ and S coarse (assuming that S has property (R)).

(ii) Suppose B^* is separable and specialize Theorem 7.2. Then $L_{(w)}^p(B)^* = M_{(w)}^q(B^*)$ and $M_{(w)}^q(B)^* = L_{(w)}^p(B^*)$ in the cases detailed in (α), (β) respectively of the (i) preceding.

(iii) Suppose B^{**} is separable and specialize Theorem 7.2. Then $L_{(w)}^p(B)^{**} = L_{(w)}^p(B^{**})$ and $M_{(w)}^q(B)^{**} = M_{(w)}^q(B^{**})$ in the cases: Case (C_1) with $1 < p < \infty$, Case (C_2) with $1 < p < \infty$, and Case (C_1) with S coarse and either $p = 1$ or $p = \infty$.

(iv) Suppose either every e has a countable basis or B is separable. If B is reflexive then, specializing Theorem 7.3, $L_{(w)}^p(B)^* = M_{(w)}^q(B^*)$ in the cases listed in (α) of (i) preceding and $M_{(w)}^q(B)^* = L_{(w)}^p(B^*)$ in the cases listed in (β) of (i) preceding.

(v) Specialize Theorem 7.3. If B is reflexive then $L_{(w)}^p(B)$ and $M_{(w)}^q(B)$ are reflexive conjugate spaces in the cases listed in (iii) preceding.

A more detailed study of Cases (C_1) and (C_2) with $p = 1$ or $p = \infty$ and of Case (C_3) with $1 \leqslant p \leqslant \infty$ will be given in a subsequent publication.

Added Sept. 15, 1953. The discussion given above actually applies to *arbitrary* length functions if the following changes are made:

(i) If G is an additive set function, valued in B^*, defined for all e with $\lambda(e)$ finite and such that $G(e) = 0$ whenever $\lambda(e) = 0$, let $\lambda^*(G)$ be defined as

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sup $\sum_i |G(e_i)| \alpha_i$ for all finite collections of disjoint e_i and non-negative α_i for which $\lambda(\sum_i \alpha_{e,i}) \leqslant 1$.

(ii) (L 9) shall read: either S is coarse or for every E with $\lambda(E) < \infty$, $\lambda(e) \to 0$ whenever $\gamma(e) \to 0$ with $e \leqslant E$.

(iii) (L 12) shall read: $\lambda(u) < \infty$, $\epsilon > 0$ imply $\lambda(u - u_e) < \epsilon$ for some e with $\lambda(e) < \infty$.

Then $\lambda^*(G) = \lambda^*(|G|)$ and $|G|(e) \leqslant \lambda^*(G)\lambda(e)$. If $\lambda^*(G) < \infty$ then G is absolutely continuous on every e with $\lambda(e) < \infty$; Corollaries 1 and 2 to Theorem 4.3 continue to hold if a λ-purely-infinite set E_∞ may be added to the decompositions.

Now Theorems 5.1, 6.1, 6.2, Lemmas 7.1, 7.2, 7.3 and Theorems 7.1, 7.2, 7.3 and 8.1 continue to hold for arbitrary length functions (not required to be levelling). Note that for Theorems 7.2 and 7.3 we use the additional assumption:

(L 14) Every non-negative measurable function $u(P)$ can be expressed as $u = u_1 + u_2$ with $\lambda^{**}(u) = \lambda^{**}(u_1) = \lambda(u_1)$ and $\lambda^{**}(u_2) = 0$.

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