CORRESPONDENCE.

EXPONENTIAL AND LOGARITHMIC FUNCTIONS.

To the Editor of the *Mathematical Gazette.*

SIR,—Mr. Tuckey's Note 1805 (*Mathematical Gazette*, February, 1945, p. 23) is to the point; I would, however, suggest that in this case a double entry table is more useful than the graphical representation. We require the proof of three things:

(i) that a^x is continuous, in other words, $a^h \to 1$ as $h \to 0$;

(ii) that $(a^h - 1)/h$ tends to a limit which depends on a;

(iii) that when this limit is 1, a is about 2·71828.

I have found the following sufficient.

Table I gives values of a^h for $a = 1$, 2, 3, 10, 100, 1000, 1000000, and for $h = 0{\cdot}1$, 0·01, 0·001, 0·0001. The values should be obtained by seven-figure logarithms and the convergence to 1 is shown clearly even when a is one million.

Tables II giving values of $(a^h - 1)/h$ for the above values of a and h can then be written down at sight, and shows the convergence to a limit increasing with a; for example, 0·69 when $a = 2$, and 1·10 when $a = 3$. This shows that e will lie between 2 and 3.

The derivatives of e^x and $\log x$ follow, and we can calculate $\log_e 2$ from the integral $\int_1^2 dx/x$ by Simpson's rule to 6 places, and having $\log_{10} 2$ from the tables we get $\log_{10} e$ easily to 5 places and so e itself. Yours, etc.,

Melbourne. R. J. A. BARNARD.

To the Editor of the *Mathematical Gazette.*

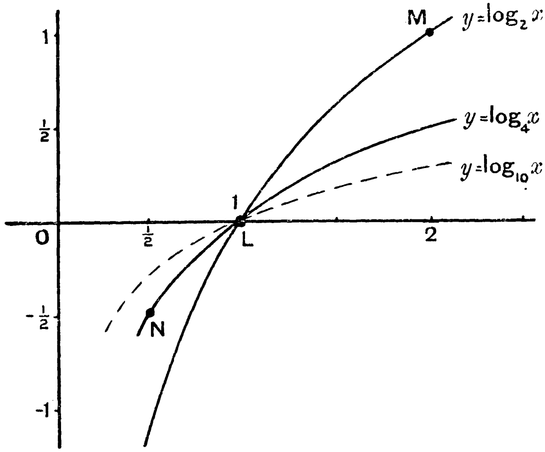
SIR,—Most teachers will agree with Messrs. Durell and Robson that the method of introducing the logarithm by investigating $\text{hyp}\,(t) \equiv \int_1^t dx/x$ is exciting, has outlook value and should be employed with mathematical specialists. On the other hand, it is difficult. It involves transforming a definite integral, concluding that if $\text{hyp}\,(e^n) = n$, then $\text{hyp}\,(x) \equiv \log_e x$, *i.e.* that there is only one function f which has the property $f(e^x) = x$, and finally proving a special case of

$$\frac{d}{dt}\int_c^t g(x)\;dx = g(t).$$

Although it is more natural to think of integrating $1/x$ than of differentiating a^x or $\log_a x$, the latter process is much easier, and need not really be much of a " bolt from the blue " After investigating algebraic and trigonometric functions and finding that the latter are sometimes required for integrating the former, it is quite natural to consider the logarithmic function as another function met with in elementary work. Nearly every boy, as Mr. Tuckey has pointed out, will know in advance that $D \log_e x = 1/x$. If he hasn't looked ahead in the textbook he will have seen it at the end of his tables, on the back of his slide-rule or in his diary. For the incurious innocent the calculation of $\log\,(a + h)$ in terms of $\log a$ and h or the fact that the gradient of $y = \log x$ appears to be inversely proportional to x could be made starting-points for the investigation.

https://doi.org/10.1017/S0025557200215295 Published online by Cambridge University Press

I prefer to differentiate $\log_a x$ rather than a^x, using the following method, for which no originality is claimed. Plot $y = \log_a x$ for $a = 10$, 2 and 4. Since $\log_a x = \log_a b \times \log_b x$, the curves $y = \log_a x$ for variable a are such that any one of them is a sideways stretch from the x-axis of any other.



On the curve $y = \log_a x$ consider two points P, Q whose abscissae are x_1 and $x_1(1+h)$. The gradient of the line PQ is

$$\frac{1}{x_1} \frac{\log_a (1+h)}{h}.$$

The gradient of the tangent at P is μ_a/x_1, where

$$\mu_a = \lim_{h \to 0} \frac{\log_a (1+h)}{h}.$$

Thus (i) the gradient of $y = \log_a x$ is proportional to $1/x$;

(ii) μ_a is the gradient at the point $L(1, 0)$ on the curve and μ_a decreases steadily as a increases.

Consider now in particular the curves $y = \log_2 x$ on which the point $M(2, 1)$ lies and $y = \log_4 x$ on which the point $N(\frac{1}{2}, -\frac{1}{2})$ lies. Since the gradient of these curves decreases as x increases they are concave downwards and

$$\mu_2 > \text{gradient of } LM = 1;$$

$$\mu_4 < \text{gradient of } LN = 1.$$

Thus there is a number e between 2 and 4 for which $\mu_e = 1$, and the gradient of $y = \log_e x$ is $1/x$. Yours, etc.,

R. C. Lyness.

TECHNICAL MATHEMATICS.

To the Editor of the *Mathematical Gazette*.

Sir,—As a mathematics teacher who has just returned to school after four and a half years doing research work in industry and helping with part-time technical classes, I was very interested to read the report in the October

https://doi.org/10.1017/S0025557200215295 Published online by Cambridge University Press