Canad. Math. Bull. Vol. **60** (3), 2017 pp. 510–521
http://dx.doi.org/10.4153/CMB-2016-057-0

Convex-normal (Pairs of) Polytopes

Christian Haase and Jan Hofmann

Abstract. In 2012, Gubeladze (Adv. Math. 2012) introduced the notion of k-convex-normal polytopes to show that integral polytopes all of whose edges are longer than $4d(d+1)$ have the integer decomposition property. In the first part of this paper we show that for lattice polytopes there is no difference between k- and $(k+1)$-convex-normality (for $k \geq 3$) and improve the bound to $2d(d+1)$. In the second part we extend the definition to pairs of polytopes. Given two rational polytopes P and Q, where the normal fan of P is a refinement of the normal fan of Q, if every edge e_P of P is at least d times as long as the corresponding face (edge or vertex) e_Q of Q, then $(P+Q) \cap \mathbb{Z}^d = (P \cap \mathbb{Z}^d) + (Q \cap \mathbb{Z}^d)$.

1 Introduction

A polytope P has the integer decomposition property (IDP) if for every $k \in \mathbb{N}$ the dilation $kP = P + \cdots + P$ of P decomposes on the level of lattice points: $kP \cap \mathbb{Z}^d = (P \cap \mathbb{Z}^d) + \cdots + (P \cap \mathbb{Z}^d)$. Polytopes with the IDP turn up in many fields of mathematics. The name IDP comes from integer programming. In algebraic geometry these polytopes correspond to projectively normal embeddings of toric varieties. In commutative algebra they are called integrally closed.

So it is natural to ask which polytopes have the IDP. There has been a lot of research concerning this question in recent years. One way to prove the IDP for a given polytope is to cover it with simpler polytopes known to have the IDP. The first approach would be to use the easiest IDP polytopes, namely unimodular simplicies, and try to show that every polytope with the IDP can be triangulated into unimodular simplices. This does not work in general; in fact, it already fails in dimension 3 [KS03]. Relaxing triangulations to coverings with unimodular simplices, there is a famous 5-dimensional polytope with the IDP that does not have such a covering [BG99]. On the other hand, one very nice positive result is that given a lattice polytope P, if all edge lengths of P (with respect to the lattice) have a common factor $c \geq d-1$, then P has the IDP [EW91, LTZ93, BGT97].

The following conjecture, proposed during a workshop [HHM07], suggests that this is also true (maybe with a higher bound) in a more generalized setting, where the edge-lengths can be independent.

Conjecture *Simple lattice polytopes with long edges have the integer decomposition property, where long means some invariant, uniform in the dimension.*

This conjecture was then proved by Gubeladze in the following precise form.

Received by the editors May 19, 2016.
Published electronically November 15, 2016.
AMS subject classification: **52B20**, 14M25, 90C10.
Keywords: integer decomposition property, integrally closed, projectively normal, lattice polytopes.

Theorem ([Gub12]) *Let P be a lattice polytope of dimension d. If every edge of P has lattice length $\geq 4d(d+1)$, then P has the integer decomposition property.*

He proves this theorem by first introducing the notion of k-convex-normality and proving that a polytope is k-convex-normal if every edge has lattice length $\geq kd(d+1)$. Then he shows that 4-convex-normal lattice polytopes have the IDP.

In the first part of this paper we further examine k-convex-normal polytopes and show that if P is a lattice polytope and k-convex-normal for some $k \geq 3$, then P is also m-convex-normal for all $m \geq 2$ (Theorem 2.5). The lemma used to prove this theorem also allows us to improve Gubeladze's bound to $2d(d+1)$ (Corollary 2.7).

In the second part of the paper we extend the notion of convex-normal polytopes to pairs of polytopes. We show that given two polytopes P and Q, the map $(Q\cap\mathbb{Z}^d)\times(P\cap\mathbb{Z}^d)\to(Q+P)\cap\mathbb{Z}^d$ given by $(q,p)\mapsto q+p$ is surjective if the normal fan of P is a refinement of the normal fan of Q and every edge of P is at least d times as long as its corresponding face (edge or vertex) in Q (Theorem 4.2).

2 Convex-normality Revisited

Let $P\subseteq\mathbb{R}^d$ be a lattice polytope. Then P has the *integer decomposition property (IDP)*, if for all $k\in\mathbb{N}$ and all $z\in kP\cap\mathbb{Z}^d$, there exist $x_1,\dots,x_k\in P\cap\mathbb{Z}^d$ such that

$$z = x_1+\cdots+x_k.$$

Every one or two dimensional lattice polytope has the integer decomposition property. In dimension three, however, simplices do not need to posses the IDP.

For example $P=\operatorname{conv}\{(0,0,0),(1,1,0),(1,0,1),(0,1,1)\}$ does not have the IDP as $(1,1,1)\in 2P$ is not the sum of two lattice points in P.

Given a rational polytope Q with vertex set $\operatorname{vert}(Q)$ we set

$$G(Q) := \bigcup_{v\in\operatorname{vert}(Q)} (v+\mathbb{Z}^d)\cap Q;$$

that is, we base the lattice in one vertex after the other and take the union of those shifted lattices inside Q. Note that if Q is a lattice polytope, then $G(Q)=Q\cap\mathbb{Z}^d$.

Following Gubeladze, we call a rational polytope $P\subseteq\mathbb{R}^d$ *k-convex-normal* for some $k\in\mathbb{Q}$, if for all rational $c\in[2,k]$:

$$cP = G((c-1)P)+P. \tag{2.1}$$

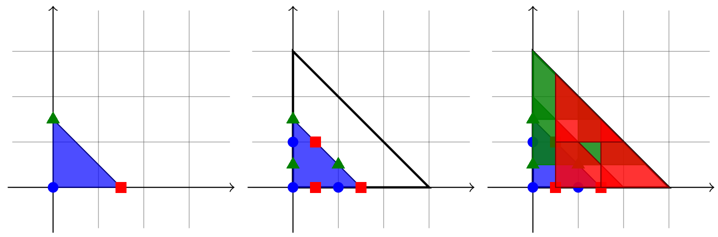
Observe that the inclusion $\supseteq$ is always true.

Example 2.1 In Figure 1, where the polytope Q is $\operatorname{conv}\{(0,0),(\frac{3}{2},0),(0,\frac{3}{2})\}$ we get

$$G(Q)=\Big\{(0,0),(1,0),(0,1),\ \big(\tfrac{3}{2},0\big),\big(\tfrac{1}{2},0\big),\big(\tfrac{1}{2},1\big),\ \big(0,\tfrac{3}{2}\big),\big(0,\tfrac{1}{2}\big),\big(1,\tfrac{1}{2}\big)\Big\}.$$

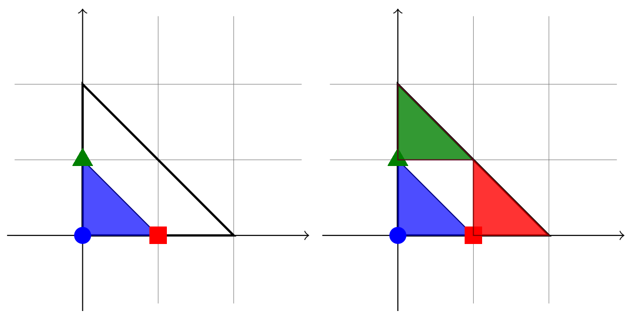
The shapes in the figures encode which vertex produced the base point for the corresponding copy of P and we can see that Q is 2-convex-normal.

https://doi.org/10.4153/CMB-2016-057-0 Published online by Cambridge University Press



(A) Q (B) Q with $G(Q)$ (C) $G(Q)+Q=2Q$

Figure 1. A 2-convex-normal polytope.



(A) P with $G(P)$ (B) $G(P)+P\neq 2P$

Figure 2. A polytope that is not 2-convex-normal.

Example 2.2 An easy example of a polytope which is not even 2-convex-normal is the 2-dimensional standard simplex $P=\operatorname{conv}\{(0,0),(1,0),(0,1)\}$ as shown in Figure 2.

Our first lemma highlights a special behavior of $G(rP)$ when P is a lattice polytope.

Lemma 2.3 *Let P be a lattice polytope and $r\in\mathbb{Q}_{>0}$. Then*

$$G(rP)+G(P)\subseteq G((r+1)P).$$

Proof Let $x=rv+u\in G(rP)$ and $y=w+u'\in G(P)$ with $v,w\in\operatorname{vert}(P)$ and $u,u',v,w\in\mathbb{Z}^d$. As $x\in rP$ and $y\in P$ it follows that $z=x+y\in(r+1)P$ and also

$$z=x+y=rv+u+w+u'=(r+1)v+(w-v+u+u')\in\operatorname{vert}((r+1)P)+\mathbb{Z}^d,$$

so $z\in G((r+1)P)$. ∎

The other inclusion "⊇" does not hold. In fact, $G(rP)+G(P)\supseteq G((r+1)P)$ holds for all integral r if and only if P has the integer decomposition property. Now we can prove the main lemma of this section, concerning equation (2.1), which P must satisfy to be k-convex-normal.

Lemma 2.4 *Let P be a 2-convex-normal lattice polytope and $c > 2$. Then*

$$G((c-2)P) + P = (c-1)P \text{ implies } G((c-1)P) + P = cP.$$

Proof $G((c-1)P) + P \subseteq cP$ is always true, hence we only have to show the other direction $cP \subseteq G((c-1)P) + P$:

$$cP = (c-1)P + P = \Big(G((c-2)P) + P \Big) + P = G((c-2)P) + 2P,$$

but P is 2-convex-normal so that $2P = G(P) + P$ and hence:

$$cP = G((c-2)P) + 2P = G((c-2)P) + G(P) + P \subseteq G((c-1)P) + P,$$

where the inclusion follows from Lemma 2.3. ∎

Now, given a lattice polytope P, if P satisfies equation (2.1) for $c = s-1$, it will also satisfy the equation for s. In particular, if P satisfies the equation for all rational c in the interval $[2,3]$, then P satisfies it for all rational $c \geq 2$. This proves the following theorem.

Theorem 2.5 *Let P be a lattice polytope. If P is 3-convex-normal, then P is also k-convex-normal for all $k \geq 2$.*

Let e be the edge of a rational polytope P connecting vertices v and w. By $\ell(e)$ we denote the lattice length of e; *i.e.,* let u be the smallest integer vector on the line spanned by $w - v$; then $e = ku$ for some $k \in \mathbb{Q}$ and $\ell(e) := |k|$. We also consider degenerate edges with $v = w$; in this case we set $\ell(e) = 0$. The previous theorem together with the lemma that 4-convex-normal polytopes have the integer decomposition property ([Gub12, Lemma 6.2]), implies that a lower bound of $\ell(e) \geq 3d(d+1)$ for every edge e of P would be enough. But using Lemma 2.4 directly, we can do better.

Corollary 2.6 *Let P be a lattice polytope. If P is 2-convex-normal, then P has the integer decompositions property.*

Proof As P is 2-convex-normal, using Lemma 2.4 repeatedly we know that $kP = G((k-1)P) + P$ for all $k \in \mathbb{N}$. Now given $z \in kP \cap Z^d$ for some $k \in \mathbb{N}$, we know that $z = x + y$ with $y \in P$, $x \in G((k-1)P) = (k-1)P \cap \mathbb{Z}^d$ and therefore $y \in P \cap \mathbb{Z}^d$. By induction we can find $x_1, \ldots, x_{k-1} \in P \cap \mathbb{Z}^d$ such that $x = x_1 + \cdots + x_{k-1}$. ∎

Gubeladze [Gub12, Theorem 1.2] proves that given a polytope P, if every edge of P has at least lattice length $kd(d+1)$, then P is k-convex-normal. Combining this with the previous corollary yields the improved bound we promised in the introduction.

Corollary 2.7 *Let P be a lattice polytope. If for every edge e of P the lattice length $\ell(e) \geq 2d(d+1)$, then P has the integer decomposition property.*

3 Convex-normality for Pairs of Polytopes

In this section we extend the above definitions and results to pairs of polytopes.

Definition 3.1 A pair of rational polytopes (Q, P) is called *convex-normal* if

$$Q + P = G(Q) + P.$$

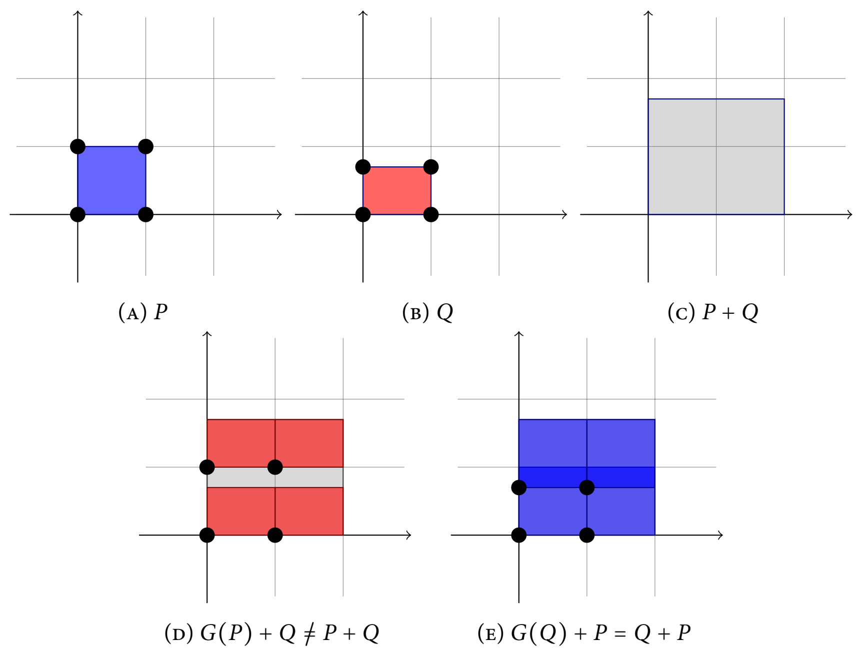
Note that we only have to show $Q + P \subseteq G(Q) + P$ as the other inclusion is always true, since $G(Q) \subset Q$. Furthermore, this notion is invariant under independent translations of P and Q by rational vectors. A small calculation shows that $G(Q - w) = G(Q) - w$. Hence, we can set two vertices $v \in \operatorname{vert}(P)$ and $w \in \operatorname{vert}(Q)$ to $\mathbf{0}$. In these terms a single polytope P is k-convex-normal if for all rational $c \in [2, k]$ the pairs $((c-1)P, P)$ are convex-normal.

Example 3.2 As seen in Example 2.1, the pair $(1.5\cdot\Delta_2, 1.5\cdot\Delta_2)$ is convex-normal and the pair (Δ_2, Δ_2) is not. More generally, P is 2-convex-normal if and only if (P, P) is convex-normal.

Example 3.3 Convex-normality is not symmetric. When we set

$$P = \operatorname{conv}\begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \quad \text{and} \quad Q = \operatorname{conv}\begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0.7 & 0.7 \end{pmatrix},$$

Figure 3 illustrates that $G(Q) + P = Q + P$ but $G(P) + Q \neq P + Q$.



(A) P (B) Q (C) $P + Q$

(D) $G(P) + Q \neq P + Q$ (E) $G(Q) + P = Q + P$

Figure 3. Convex-normality of pairs is not symmetric.

The second definition we need is an extension of the integer decomposition property to pairs of polytopes.

https://doi.org/10.4153/CMB-2016-057-0 Published online by Cambridge University Press

Definition 3.4 A pair of polytopes (Q, P) has the *integer decomposition property (IDP)*, if the map

$$\begin{array}{ccccc} (Q \cap \mathbb{Z}^d) & \times & (P \cap \mathbb{Z}^d) & \longrightarrow & (Q+P) \cap \mathbb{Z}^d \\ (q & , & p) & \longmapsto & q+p \end{array}$$

is surjective, that is, if $(P+Q) \cap \mathbb{Z}^d = (P \cap \mathbb{Z}^d) + (Q \cap \mathbb{Z}^d)$.

If the pairs (P, nP) have the integer decomposition property for all $n \in \mathbb{N}$, then P is a lattice polytope and has it as well.

The pair (Δ_2, Δ_2) from the example above has the integer decomposition property, so we see that pairs of polytopes with the IDP are not always convex-normal. But the converse implication is true.

Lemma 3.5 *Let P be a polytope and let Q be a lattice polytope such that (Q, P) is convex-normal. Then (Q, P) has the integer decomposition property.*

Proof As (Q, P) is convex-normal, we know that $Q + P = G(Q) + P$.

As Q is a lattice polytope, we have $G(Q) = Q \cap \mathbb{Z}^d$, and hence

$$\begin{aligned} (Q+P) \cap \mathbb{Z}^d &= (G(Q)+P) \cap \mathbb{Z}^d = ((Q \cap \mathbb{Z}^d) + P) \cap \mathbb{Z}^d \\ &= (Q \cap \mathbb{Z}^d) + (P \cap \mathbb{Z}^d). \end{aligned}$$ ∎

In the remainder of this paper we will prove a sufficient condition, based on edge lengths, for a pair (Q, P) to be convex-normal.

Given a polytope P, if F is a face of P we write $F \prec P$. For every nonempty face F of P there exists a linear functional c_F, such that $c_F^t x$ is maximal over P if and only if $x \in F$. We also say that c_F defines the face F. The set

$$C_F = \Big\{ c : \{ z : \max_{x \in P} c^t x = c^t z \} \supseteq F \Big\}$$

is a polyhedral cone. The *normal fan* $\mathcal{N}(P)$ of P is the collection of these cones over all nonempty faces of P. The correspondence $F \longleftrightarrow C_F$ is an inclusion reversing bijection; *i.e.*, given two faces $F, F' \prec P$, then $F \subseteq F'$ if and only if $C_{F'} \subseteq C_F$.

In the above examples P and Q had the same normal fan. If we drop this condition, there are pairs of polytopes with arbitrarily long edges lacking the integer decomposition property and not being convex-normal.

Example 3.6 Set

$$Q = \operatorname{conv}\begin{pmatrix} 0 & 1 & 0 \\ 0 & k & 1 \end{pmatrix} \quad \text{and} \quad P = \operatorname{conv}\begin{pmatrix} 0 & -l & -(l-1) \\ 0 & 1 & 1 \end{pmatrix},$$

see Figure 4.

If we look at (nQ, nP), then the edges of both nQ and nP all have lattice length n and there are $O(n^4)$ lattice points in $(nP \cap \mathbb{Z}^2) + (nQ \cap \mathbb{Z}^2)$, but $k \cdot l \cdot O(n^2)$ lattice points in $nP + nQ$. Hence for $k, l \gg n$, the pair (nQ, nP) neither has the integer decomposition property nor is it convex-normal.

https://doi.org/10.4153/CMB-2016-057-0 Published online by Cambridge University Press

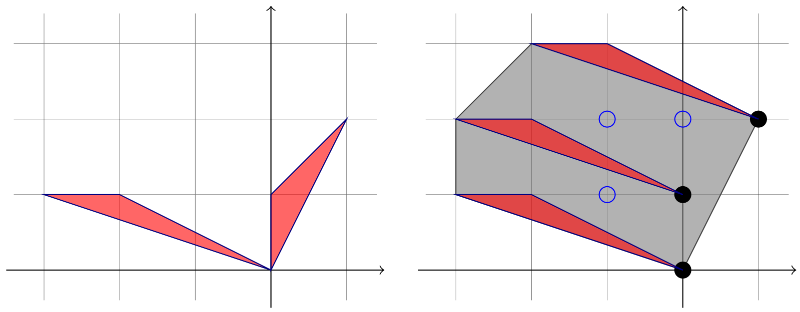


Figure 4. $Q + P$ and $G(Q) + P$ for $n = 1$, $k = 2$ and $l = 3$.

For a pair (Q, P) of polytopes to be convex-normal, it is not enough if both polytopes have the integer decomposition property, are k-convex-normal, or have long edges. The examples suggest that we need a condition on the normal fans of P and Q, and in fact that is what we need.

Given two d-polytopes Q and P, if $\mathcal{N}(P)$ is a refinement of $\mathcal{N}(Q)$, then for every cone $C \in \mathcal{N}(P)$ there exists a cone $D \in \mathcal{N}(Q)$ s.t. $C \subseteq D$. In this case we can define a map $\Phi' : \mathcal{N}(P) \to \mathcal{N}(Q)$, where $\Phi'(C)$ is defined as the smallest cone in $\mathcal{N}(Q)$ containing C. This map preserves inclusions and has a corresponding map $\Phi : \mathcal{L}(P) \to \mathcal{L}(Q)$ on the face lattices of P and Q, taking a face $F < P$ with corresponding cone C_F to the face $G < Q$ with corresponding cone $C_G = \Phi'(C_F)$.

Example 3.7 In Figure 5 we illustrate the map with

$$P = \operatorname{conv}\begin{pmatrix} 0 & 3 & 3 & 2 & -1 & -1 \\ 0 & 0 & -2 & -3 & -3 & -1 \end{pmatrix} \quad \text{and} \quad Q = \operatorname{conv}\begin{pmatrix} 0 & 2 & 2 & 0 \\ 0 & 0 & -2 & -2 \end{pmatrix}.$$

For example, the edge e from $(-1, -1)$ to $(0, 0)$ in P corresponds to the vertex $(0, 0)$ in Q, *i.e.*, $\Phi(e) = (0, 0)$, because e corresponds to cone $\binom{-1}{1} \in \mathcal{N}(P)$ and the smallest cone of $\mathcal{N}(Q)$ containing it is $\operatorname{cone}\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, which is the normal cone belonging to $(0, 0)$ in Q.

4 A Sufficient Condition for Convex-normality of (Q, P)

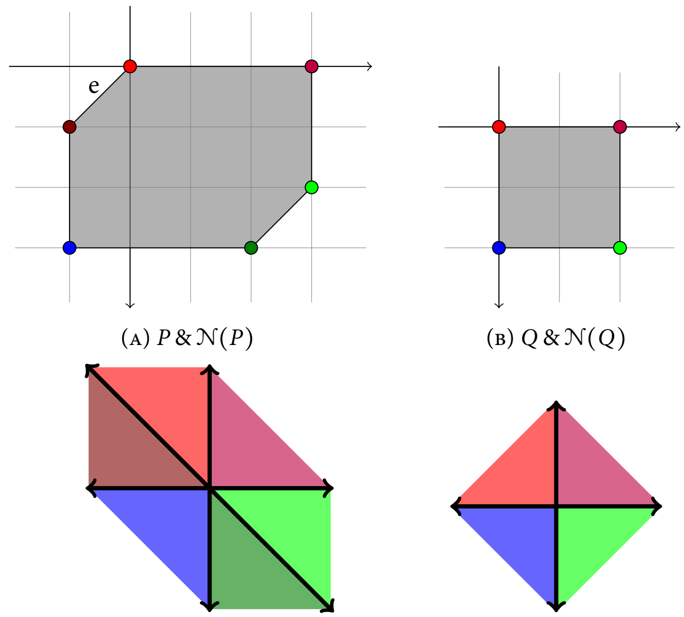
Now that we have all the tools lined up, we can start the proof with the following lemma, which is the base case for our induction.

Lemma 4.1 *Let $P = [0, q]$ and $Q = [0, m]$ be intervals with $q \geq \min\{1, m\}$; then (Q, P) is convex-normal.*

Proof Set $l := \lfloor m \rfloor$. If $l \geq 1$, then $q \geq 1$ and

$$Q + P = [0, q + m] = \Big(\bigcup_{i=0}^{l} i + [0, q] \Big) \cup m + [0, q] \subseteq G(Q) + P.$$

https://doi.org/10.4153/CMB-2016-057-0 Published online by Cambridge University Press



(A) P & $\mathcal{N}(P)$ (B) Q & $\mathcal{N}(Q)$

Figure 5. Each face of P corresponds to a face of Q.

If $l < 1$, then $q \geq l$ and:

$$Q + P = (0 + P) \cup (m + P). \qquad \blacksquare$$

Now we can prove the main result.

Theorem 4.2 *Let P and Q be rational d-polytopes such that $\mathcal{N}(P)$ is a refinement of $\mathcal{N}(Q)$ and such that $\ell(e_P) \geq d \cdot \ell(e_Q)$ for every edge $e_P \prec P$ and corresponding face (edge or vertex) $e_Q = \Phi(e_P) \prec Q$. Then (Q, P) is convex-normal.*

Proof Lemma 4.1 took care of the base case, hence let P and Q be d-polytopes with $d \geq 2$.

Step 1: Subdividing $Q + P$:

Without loss of generality we assume $\mathbf{0} \in \operatorname{vert}(P)$ and $\mathbf{0} = \Phi(0) \in \operatorname{vert}(Q)$ and start by subdividing $Q + P$ by assigning weights/heights to the vertices of P and Q. Vertices of Q and the vertex $\mathbf{0}$ of P get height 0 and all the other vertices of P get height 1. We use those heights to define new polytopes P' and Q' in $\mathbb{R}^{d+1}$ as follows:

$$Q' := \operatorname{conv}\{(w, 0) : w \in \operatorname{vert}(Q)\},$$
$$P' := \operatorname{conv}\big((\mathbf{0}, 0) \cup \{(u, 1) : u \in \operatorname{vert}(P) \backslash \{v\}\}\big).$$

Then the projection of $P' + Q'$ onto the first d coordinates is $P + Q$ and the lower boundary of $P' + Q'$ induces a subdivision of $P + Q$ into the following pieces:

$$0 + Q \quad \text{and} \quad F_Q + (\operatorname{conv}(0, F_P))$$

https://doi.org/10.4153/CMB-2016-057-0 Published online by Cambridge University Press

for faces $F_Q \prec Q$ and faces $F_P \prec P$, with $0 \notin F_P$ and $\Phi(F_P) = F_Q$. Compare Figure 6.

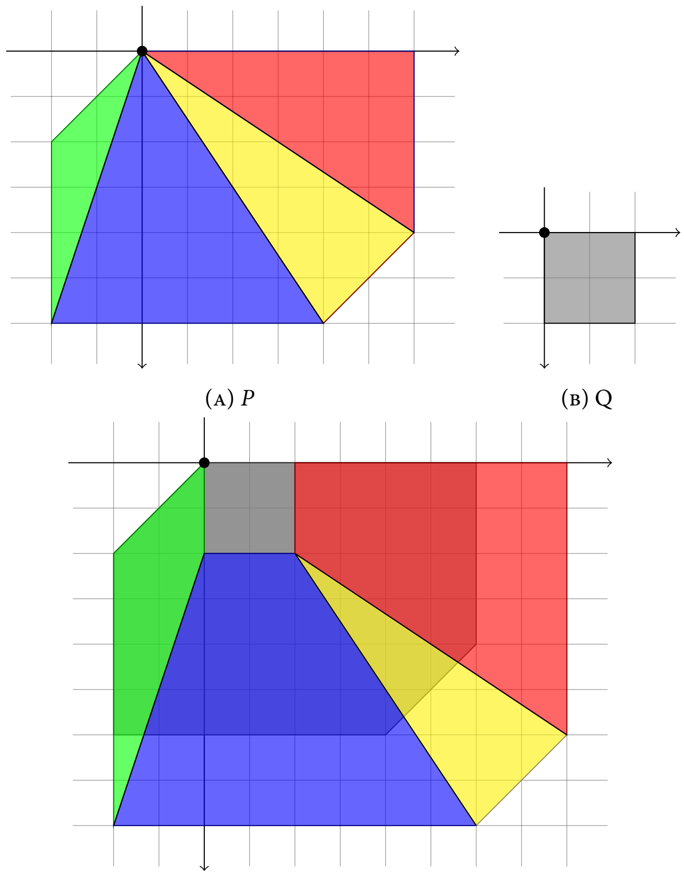


Figure 6. $Q + P$ subdivided into $0 + Q$ and $F_Q + (\operatorname{conv}(0, F_P))$.

Another decomposition of $P + Q$ we will be using is the following:

$$I := \Big(\frac{d-1}{d}\Big)P + Q \quad \text{ and } \quad B := \overline{(P+Q) \setminus I},$$

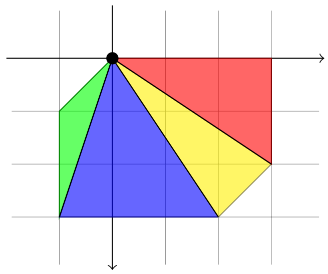
where I stands for the "inner" part of $P + Q$ and B stands for the "boundary" part of $P + Q$; see Figure 7.

In the next step we will be using our first subdivision to cover the boundary part. We will then show that covering I is easy because it lies in $0 + P$.

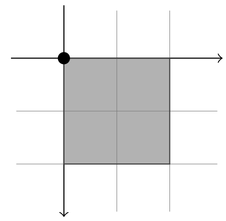
Step 2.1: Covering B:

Let $x \in B$; then $x \notin Q$, and hence we can find facets $F_P \prec P$ and $F_Q \prec Q$ such that $x \in F_Q + (\operatorname{conv}(0, F_P))$ coming from our subdivision in Step 1. Hence, x can

https://doi.org/10.4153/CMB-2016-057-0 Published online by Cambridge University Press



(A) $\left(\frac{d-1}{d}\right)P$



(B) Q

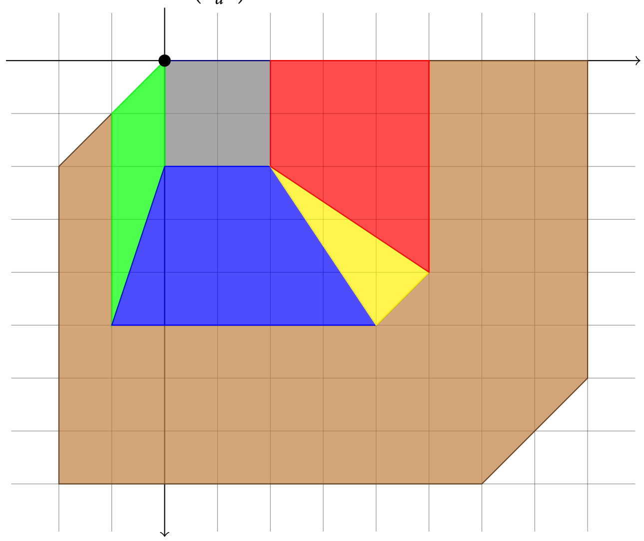


Figure 7. $Q+P$ divided into I and B.

be written as $x = q + \mu p$, with $q \in F_Q \prec Q$, $p \in F_P \prec P$ and $0 \le \frac{d-1}{d} \le \mu \le 1$. Then $z := q + \frac{d-1}{d} p$ is contained in $\frac{d-1}{d} F_P + F_Q$. Furthermore, $(F_Q, \frac{d-1}{d} F_P)$ is convex-normal by induction, as $\mathcal{N}(\frac{d-1}{d} F_P)$ is a refinement of $\mathcal{N}(F_Q)$, and given edges $e_{F_Q} \prec F_Q$ and $\frac{d-1}{d} e_{F_P} \prec \frac{d-1}{d} F_P$ ($\Leftrightarrow e_{F_P} \prec F_P$), we have

$$\ell\Big(\frac{d-1}{d} e_{F_P}\Big) = \Big(\frac{d-1}{d}\Big)\ell(e_{F_P}) \ge \Big(\frac{d-1}{d}\Big) \cdot d\ell(e_{F_Q}) = (d-1)\ell(e_{F_Q}).$$

Hence we can find a point $g \in G(F_Q)$ such that $z \in g + \frac{d-1}{d} F_P$, and since $p \in F_P \subseteq \operatorname{conv}(0, F_P)$, we get $x \in g + \operatorname{conv}(0, F_P) \subseteq g + P$, as illustrated in Figure 8.

Step 2.2: Covering I:

Now we are left with covering the points in the inner part I of $P+Q$. We claim that $I \subseteq P$, which implies $I \subseteq 0 + P \subseteq G(Q) + P$. First we reformulate the problem by using that $I = (\frac{d-1}{d})P + Q \subseteq P$ is equivalent to $Q \subseteq \frac{1}{d}P$.

To show the latter, suppose $Q \notin \frac{1}{d}P$; then there exists a vertex u of Q that does not lie in $\frac{1}{d}P$. This implies that there exists a functional c such that $c^t u = b$ and $c^t x < b$ for

https://doi.org/10.4153/CMB-2016-057-0 Published online by Cambridge University Press

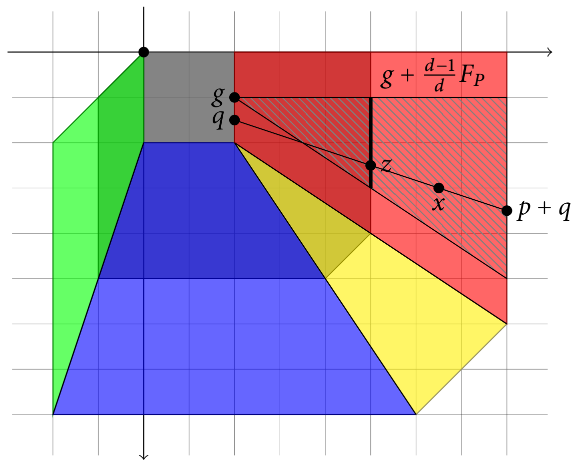


Figure 8. Covering B using induction.

all $x \in \frac{1}{d}P$. When we use the simplex method to maximize c over $\frac{1}{d}P$ starting in 0, we get a monotone edge path from 0 to an optimal vertex u'. As $\mathcal{N}(\frac{1}{d}P)$ is a refinement of $\mathcal{N}(Q)$, we have an inclusion-preserving map $\mathcal{L}(\frac{1}{d}P) \to \mathcal{L}(Q)$ between the two face lattices. Using this map, we get a corresponding edge path in Q, which also ends in an optimal vertex u'', as $c \in C_{u'} \subseteq C_{u''}$. But as every edge in $\frac{1}{d}P$ is at least as long as the corresponding face (edge or vertex) in Q, we have

$$c^t u' \geq c^t u'' = c^t u \quad \lightning.$$

Hence no vertex of Q is lying outside of $\frac{1}{d}P$, so that $Q \subseteq \frac{1}{d}P$, which finishes our proof. ∎

Theorem 4.2 requires Q to be a lot smaller than P. But in conjunction with the following lemma, it can be used in certain cases where Q is allowed to be big.

Lemma 4.3 *Let P be a rational polytope and Q be a lattice polytope, with*

$$Q = Q_1 + \cdots + Q_s$$

where the Q_i are lattice polytopes such that the pairs (Q_i, P) are convex-normal for all i. (For example, they could satisfy the conditions of the previous theorem.) Then (Q, P) is convex-normal.

Proof As (Q_i, P) are convex-normal, we get

$$\begin{aligned} Q + P &= (Q_1 + \cdots + Q_s) + P \\ &= G(Q_1) + \cdots + G(Q_s) + P \\ &\subseteq G(Q_1 + \cdots + Q_s) + P \\ &= G(Q) + P, \end{aligned}$$

https://doi.org/10.4153/CMB-2016-057-0 Published online by Cambridge University Press

where the second equality is true because the Minkowski sum is commutative and associative, and the inclusion is true because the Q_i are lattice polytopes. ∎

In particular, if Q is a lattice polytope and (Q, P) is convex-normal, then (kQ, P) is convex-normal for all $k \in \mathbb{N}$. Putting together Lemma 3.5, Theorem 4.2, and Lemma 4.3 we get the following corollary.

Corollary 4.4 *Let P and Q be rational polytopes, where $\mathcal{N}(P)$ is a refinement of $\mathcal{N}(Q)$. If Q has a decomposition into lattice polytopes $Q = Q_1 + \cdots + Q_s$ and every edge of P is at least d times as long as the corresponding edge in Q_i for all i, then (Q, P) has the integer decomposition property.*

Acknowledgements We would like to thank Petra Meyer. The first part of the paper grew out of her master's thesis.

References

[BG99] W. Bruns and J. Gubeladze, *Normality and covering properties of affine semigroups*. J. Reine Angew. Math. **510**(1999), 161–178. http://dx.doi.org/10.1515/crll.1999.044

[BGT97] W. Bruns, J. Gubeladze, and N. G. Trung, *Normal polytopes, triangulations, and Koszul algebras*. J. Reine Angew. Math. **485**(1997), 123–160.

[EW91] G. Ewald and U. Wessels, *On the ampleness of invertible sheaves in complete projective toric varieties*. Results Math. **19**(1991), no. 3–4, 275–278. http://dx.doi.org/10.1007/BF03323286

[Gub12] J. Gubeladze, *Convex normality of rational polytopes with long edges*. Adv. Math. **230**(2012), no. 1, 372–389. http://dx.doi.org/10.1016/j.aim.2011.12.003

[HHM07] C. Haase, T. Hibi, and D. Maclagan, *Mini-workshop: Projective normality of smooth toric varieties, abstracts from the mini-workshop: held August 12-18, 2007*. Oberwolfach Reports **4**(2007), 2283–2319.

[KS03] J.-M. Kantor and K. S. Sarkaria, *On primitive subdivisions of an elementary tetrahedron*. Pacific J. Math. **211**(2013), 123–155. http://dx.doi.org/10.2140/pjm.2003.211.123

[LTZ93] J. Y. Liu, L. E. Trotter, Jr., and G. M. Ziegler, *On the height of the minimal Hilbert basis*. Results Math. **23**(1993), no. 3–4, 374–376. http://dx.doi.org/10.1007/BF03322309

Mathematik, FU Berlin, 14195 Berlin, Germany
e-mail: haase@math.fu-berlin.de janhofmann@math.fu-berlin.de