Bull. Aust. Math. Soc. **97** (2018), 43–46
doi:10.1017/S0004972717000715

THE $3k - 4$ THEOREM FOR ORDERED GROUPS

PREM PRAKASH PANDEY

(Received 9 March 2017; accepted 15 June 2017; first published online 28 September 2017)

Abstract

Recently, Freiman *et al.* ['Small doubling in ordered groups', *J. Aust. Math. Soc.* **96**(3) (2014), 316–325] proved two 'structure theorems' for ordered groups. We give elementary proofs of these two theorems.

2010 *Mathematics subject classification*: primary 11P70.
Keywords and phrases: $3k - 4$ theorem, ordered groups.

1. Introduction

The $3k - 4$ theorem is an inverse theorem in ordered groups recently proved by Freiman *et al.* in [1]. For any group G (written multiplicatively) and a subset S of G, we define $S^2 := \{ab : a, b \in S\}$. The main theorem of [1] is the following result.

THEOREM 1.1 [1, Theorem 1.3]. *Let G be an ordered group and S a finite subset of G. If $|S^2| \leq 3|S| - 3$, then the subgroup generated by S is an abelian subgroup of G.*

As a corollary to Theorem 1.1, Freiman *et al.* deduced a $3k - 4$ theorem for ordered groups.

THEOREM 1.2 [1, Corollary 1.4]. *Let G be an ordered group and S a finite subset of G with $|S| = k \geq 3$. If $|S^2| \leq 3|S| - 4$, then there exist two commuting elements x, y in G such that $S \subset \{yx^i : 0 \leq i \leq N\}$ for $N = |S^2| - |S|$.*

The study of the structure of such sets with small doubling is an important area of combinatorial group theory and there is a vast literature devoted to this theme (see, for example, [2–5]). Theorems 1.1 and 1.2 are important results. We give elementary proofs of Theorems 1.1 and 1.2.

2. Proofs

We shall always assume that G is an ordered group and S is a finite subset of G with k elements. We shall write $S = \{x_1, \ldots, x_k\}$ and assume that $x_1 < \cdots < x_k$. As in the case of integers, the following inequality holds:

$$|S^2| \geq 2|S| - 1. \tag{2.1}$$

In equation (2.1), equality holds only if S is a geometric progression, that is, S has the form $\{yx^i : 0 \le i \le k-1\}$ with two commuting elements $x, y \in G$. Analogous to the case of integers (see [6, Theorem 1.2]), we prove the following lemma.

Lemma 2.1. *If S is not a geometric progression, then $|S^2| \ge 2|S|$.*

Proof. Let $S = \{x_1, \ldots, x_k\}$ with $x_1 < \cdots < x_k$. Clearly,

$$x_1x_1 < x_1x_2 < \cdots < x_1x_k < x_2x_k < \cdots < x_kx_k$$

are $2|S| - 1$ distinct elements in S^2. If $|S^2| < 2|S|$, then

$$\{x_1x_1, x_1x_2, \ldots, x_1x_k, x_2x_k, \ldots, x_kx_k\} = S^2.$$

Now consider the elements $x_2x_1 < x_2x_2 < \cdots < x_2x_k$. All these elements are in S^2 and $x_1x_1 < x_2x_1, \ldots, x_2x_{k-1} < x_2x_k$. Thus,

$$x_2x_1 = x_1x_2, x_2x_2 = x_1x_3, x_2x_3 = x_1x_4, \ldots, x_2x_{k-1} = x_1x_k.$$

From these relations it follows that x_1 and x_2 commute and, for each $i > 2$, x_i is contained in the subgroup generated by $x_1, \ldots, x_{i-1}$. Consequently, each x_i commutes with each x_j for $i, j = 1, \ldots, k$. Put $y = x_1$ and $x = x_2x_1^{-1}$. Then x and y commute and $S = \{y, xy, x^2y, \ldots, x^{k-1}y\}$ is a geometric progression. Consequently, if S is not a geometric progression, we must have $|S^2| \ge 2|S|$. □

The proofs of Theorems 1.1 and 1.2 run along the same lines. We begin with the proof of Theorem 1.2.

Proof of Theorem 1.2. We shall use induction on k. For $k = 3$, we have $|S^2| \le 5$. We have five distinct elements $x_1^2 < x_1x_2 < x_2^2 < x_2x_3 < x_3^2$ in S^2. Since $x_1x_3 \in S^2$, then x_1x_3 must equal one of these five elements. By comparing elements in pairs, we find $x_1x_3 = x_2^2$. Similarly, $x_1x_2 = x_2x_1$. Let $y = x_1$ and $x = x_2x_1^{-1}$. Then x and y commute and $S = \{y, yx, yx^2\}$.

Now we assume that $k \ge 4$ and that the theorem is true for any subset T of G with $3 \le |T| \le k-1$. Take $T = \{x_1, \ldots, x_{k-1}\}$.

Case 1. $|T^2| \le 3|T| - 4$.

By the induction hypothesis, there are commuting elements x, y in G such that $T \subset \{yx^j : j = 0, \ldots, M\}$ and $M = |T^2| - |T|$. If $x_kT \cap T^2 = \emptyset$, then, taking account of x_k^2, we see that $|S^2| \ge |T^2| + (|T| + 1)$. Since $|T^2| \ge 2|T| - 1$, we immediately obtain $|S^2| \ge 3|S| - 3$, which contradicts the hypothesis. Thus, $x_kT \cap T^2 \ne \emptyset$. Consequently, there are $yx^i, yx^u, yx^v \in T$ such that $x_kyx^i = yx^uyx^v$. This gives $x_k = yx^{(u+v-i)}$ and $S \subset \{yx^j : j = 0, \ldots, M'\}$ with $M' = \max\{M, u+v-i\}$. The map $yx^j \mapsto j$ gives a 2-isomorphism of S with a subset of $\mathbb{Z}$. From Freiman's $3k-4$ theorem for the integers (see [6, Theorem 1.16]), it follows that $M' \le N$ and the theorem is proved.

https://doi.org/10.1017/S0004972717000715 Published online by Cambridge University Press

Case 2. $|T^2| \geq 3|T| - 3 = 3|S| - 6$.

Using the order relation of G, we see that the elements x_k^2 and $x_k x_{k-1}$ of S^2 are not in T^2. Consider the element $x_{k-1}x_k$ of S^2. If $x_{k-1}x_k \neq x_k x_{k-1}$, then $|S^2| \geq |T^2| + 3$, which contradicts the hypothesis. So, we obtain $x_{k-1}x_k = x_k x_{k-1}$. Next, we consider the element $x_{k-2}x_k$ of S^2. If $x_{k-2}x_k \neq x_{k-1}^2$, then again $|S^2| \geq |T^2| + 3$, leading to a contradiction. Thus, $x_{k-2}x_k = x_{k-1}^2$. Similarly, $x_k x_{k-2} = x_{k-1}^2$ and so

$$x_{k-1}x_k = x_k x_{k-1}, x_{k-2}x_k = x_k x_{k-2} = x_{k-1}^2.$$

Put $y = x_k$ and $x = x_{k-1}x_k^{-1}$. Then x and y commute and $x_k = y, x_{k-1} = yx, x_{k-2} = yx^2$. Considering the elements $x_{k-3}x_k, x_{k-4}x_k, \ldots, x_1 x_k$ successively, we see that each of the x_i is of the form yx^{t_i}. Clearly, S is 2-isomorphic to the subset $\{t_i : 1 \leq i \leq k\}$ of $\mathbb{Z}$ and again the theorem follows from Freiman's $3k - 4$ theorem for the integers. □

Proof of Theorem 1.1. We shall use induction on k. For $k = 2$, the theorem holds trivially. Now let $k \geq 3$ and assume that the theorem is true for any set T with $|T| \leq k - 1$. Put $T = \{x_1, \ldots, x_{k-1}\}$.

Case 1. $|T^2| \leq 3|T| - 3$.

By the induction hypothesis, T generates a commutative subgroup. If $x_k T \cap T^2 \neq \emptyset$ or $T x_k \cap T^2 \neq \emptyset$, then x_k lies in the subgroup generated by T. Consequently, S generates a commutative subgroup. So, we can assume that $x_k T \cap T^2 = \emptyset$ and $T x_k \cap T^2 = \emptyset$.

Using the order relation in G, we see that $x_k^2 \notin T^2 \cup x_k T$ and so

$$|S^2| \geq |T^2| + |T| + 1. \tag{2.2}$$

If T is not a geometric progression, then, using Lemma 2.1 and (2.2), we see that $|S^2| \geq 3|S| - 2$, which contradicts the hypothesis. Thus, T must be a geometric progression.

Next, observe that if $x_k T \neq T x_k$, then we have an element in $T x_k$ which is not in $T^2 \cup x_k T \cup \{x_k^2\}$. This leads to

$$|S^2| \geq |T^2| + |T| + 1 + 1$$

and so $|S^2| \geq 3|S| - 2$, which contradicts the hypothesis. Therefore, we must have $x_k T = T x_k$. Using the order relation, we see that x_k commutes with all the elements of T and consequently S generates an abelian group.

Case 2. $|T^2| > 3|T| - 3$.

As in the proof of Theorem 1.2 (following the arguments used in Case 2), we see that either $|S^2| \geq |T^2| + 3$ or $S = \{yx^{t_i} : 1 \leq i \leq k\}$ with commuting elements x and y. The first alternative leads to a contradiction. Consequently, $S = \{yx^{t_i} : 1 \leq i \leq k\}$ with commuting elements x and y and the theorem is proved. □

Remark 2.2. From the proof of Theorem 1.2, it is clear that the subgroup generated by S (with $|S| > 2$) is, in fact, generated by $|S| - 1$ or fewer elements.

Acknowledgement

We thank the referees for comments which improved the presentation of the paper.

References

[1] G. A. Freiman, M. Herzog, P. Longobardi and M. Maj, 'Small doubling in ordered groups', *J. Aust. Math. Soc.* **96**(3) (2014), 316–325.

[2] G. A. Freiman, M. Herzog, P. Longobardi, M. Maj, A. Plagne, D. J. S. Robinson and Y. V. Stanchescu, 'On the structure of subsets of an orderable group, with some small doubling properties', *J. Algebra* **445** (2016), 307–326.

[3] G. A. Freiman, M. Herzog, P. Longobardi, M. Maj, A. Plagne and Y. V. Stanchescu, 'Small doubling in ordered groups: generators and structures', *Groups Geom. Dyn.* **11**(2) (2017), 585–612.

[4] G. A. Freiman, M. Herzog, P. Longobardi, M. Maj and Y. V. Stanchescu, 'Direct and inverse problems in additive number theory and in non-abelian group theory', *European J. Combin.* **40** (2014), 42–54.

[5] G. A. Freiman, M. Herzog, P. Longobardi, M. Maj and Y. V. Stanchescu, 'A small doubling structure theorem in a Baumslag–Solitar group', *European J. Combin.* **44** (2015), 106–124.

[6] M. Nathanson, *Additive Number Theory: Inverse Problems and the Geometry of Sumsets*, Graduate Texts in Mathematics, 165 (Springer, New York, 1996).

PREM PRAKASH PANDEY, Department of Mathematics, IISER Berhampur, Government ITI Berhampur, Khodasingi, Berhampur-760010, India
e-mail: premshivaganga@gmail.com

https://doi.org/10.1017/S0004972717000715 Published online by Cambridge University Press