Proceedings of the Edinburgh Mathematical Society (1984) 27, 91–101 ©

FRICKE'S OCTAVIC CURVE

by W. L. EDGE

(Received 31st May, 1983)

Summary

The paper is concerned with a curve F, the complete intersection of a quadric with a quartic surface, that admits a group of self-projectivities isomorphic to the symmetric group of degree 5. Every generator of the quadric is, as shown at the end of the paper, cut by F equianharmonically. F has 80 stalls, points where its osculating plane is stationary; they are of two kinds, 60 to be labelled Σ, the other 20 Ω. F also has inflections at 24 points which compose a figure encountered on earlier occasions. A search is made for tritangent planes of F of which, when reckoned according to proper multiplicity, there must be 2048. Among them are 60 all of whose three contacts are Σ while a further 120 each involve a single Σ among their contacts and 420 each involve a single Ω.

The existence of the group of self-projectivities is due to the presence of a certain basic pentahedron, so that the paper opens by describing this.

1. Preamble

Some account has been given [3, 4], using a system of supernumerary homogeneous coordinates x, y, z, t, u subject to the identity

$$S_1 \equiv x+y+z+t+u=0, \tag{1.1}$$

of the geometry of Bring's sextic curve B, the intersection of the quadric

$$Q: S_2 \equiv x^2+y^2+z^2+t^2+u^2=0 \tag{1.2}$$

and the "diagonal" cubic surface

$$D: S_3 \equiv x^3+y^3+z^3+t^3+u^3=0.$$

It is now proposed to study, analogously, the intersection F of Q with the quartic surface

$$\Phi: S_4 \equiv x^4+y^4+z^4+t^4+u^4=0; \tag{1.3}$$

F is an octavic curve, quadrisecant to every generator of Q. Since its plane projection from a point of itself has order 7 and has no multiple points other than two triple points [2] it has genus 9.

It seems proper to say, notwithstanding the fleeting reference [7, p. 202] to the curve $\beta=0$, *that F is Fricke's octavic. He not only explicitly mentions* [5, p. 368] *this* irreducible curve: he gives its genus and emphasises its invariance under a group of 120 self-transformations; but although he notes that invariant sets of 20, 24 points are intersections of F with certain special surfaces he does not give the particular geometrical attributes that serve to identify them and another set of 60, and which it is intended to give below (§§2, 3, 4).

The faces of the pentahedron of reference P meet by pairs in ten edges e_{ij}, by threes in ten vertices V_{ij}; V_{ij} is *opposite* to the edge common to those two faces *not* containing V_{ij} and is joined to it by the *diagonal plane* d_{ij}. Examples are

$$V_{12}:(1,-1,0,0,0); \quad e_{12}:x=y=0=z+t+u; \quad d_{12}:x+y=0=z+t+u.$$

Of course either, at choice, of the two equations given suffices to determine d_{12}, and any two of the three linearly dependent equations determine e_{12}.

The join of two vertices whose binary suffixes are disjoint is a *diagonal*; there are fifteen of them; the three through V_{ij} join it to the vertices on e_{ij}. One diagonal joins

$$V_{23}(0,1,-1,0,0) \quad \text{and} \quad V_{45}(0,0,0,1,-1),$$

its equations being $x=y+z=t+u=0$. All the diagonals are on D, which is therefore called the *diagonal surface*, as indeed they are on all the surfaces $S_{2n+1}=0$.

The polar plane π_{ij} of V_{ij} with respect to Q contains e_{ij}; that of V_{12} is $\pi_{12}:x=y$. The product of the ten π_{ij} is Fricke's A_{10}.

2. Twenty points Ω

Each d_{ij} meets Φ in a pair of conics with double contact: the points common to Φ and d_{12} satisfy

$$x^4+x^4+z^4+t^4+(z+t)^4=0,$$

$$x^4+(z^2+zt+t^2)^2=0,$$

$$(z^2+zt+t^2+ix^2)(z^2+zt+t^2-ix^2)=0,$$

a pair of conics in d_{12} both touching the lines $z=\omega t$, $z=\omega^2 t$ where they meet $x=0$, the two tangents meeting at V_{12}; here, as customary, ω is a complex cube root of 1.

So ten among the plane sections of Φ are pairs of conics with double contact. These contacts are on edges of P, those on e_{12} being $(0,0,1,\omega,\omega^2)$ and $(0,0,1,\omega^2,\omega)$ which are the intersections of e_{12} with Q: the *edges of P are bitangents of Φ and chords of F.* Their 20 intersections with F are geometrically significant: call them Ω-points.

https://doi.org/10.1017/S0013091500022161 Published online by Cambridge University Press

Since the tangent of F at $(0,0,1,\omega,\omega^2)$ is

$$z+\omega t+\omega^2 u=z+t+u=0$$

common to the tangent planes of Q and Φ it contains V_{12}. The passage of the tangents of F at both Ω on e_{ij} through V_{ij} involves *tritangent planes* of F: for example, $\omega u=\omega^2 t$ through e_{45} contains

the join of V_{12} on e_{45} to $(0,0,1,\omega,\omega^2)$ on e_{12},

the join of V_{23} on e_{45} to $(1,0,0,\omega,\omega^2)$ on e_{23},

the join of V_{31} on e_{45} to $(0,1,0,\omega,\omega^2)$ on e_{31}.

All of these joins are tangents of F, so that $u=\omega t$ is a tritangent plane (and meets F further in the Ω on e_{45}). So one finds 20 tritangent planes, two through each e_{ij}.

Note that d_{12} meets Q in the conic $x^2+z^2+zt+t^2=0$ which belongs to the pencil containing the two conics on Φ. Hence, as F is the common curve of Q and Φ, d_{12} has *four-point intersection with F at both points of the Ω pair.* All 20 Ω are stalls on F and d_{12} the stationary osculating plane at both.

3. Twenty-four points I

On p. 542 of [3] there were noticed 24 points of B that happen to be also on F; they are the intersections of B with Φ and compose the complete set of points common to the three surfaces Q, D, Φ. They are obtained from

$$a(1,\varepsilon,\varepsilon^2,\varepsilon^3,\varepsilon^4)$$

by imposing all 4! permutations on its y, z, t, u coordinates; here ε is $\exp(2\pi i/5)$. Certain pairs of these points are conjugate with respect to Q, so that their joins are generators. Any point

$$(1,\varepsilon^p,\varepsilon^q,\varepsilon^r,\varepsilon^s)$$

conjugate to a is in the tangent plane

$$x+\varepsilon y+\varepsilon^2 z+\varepsilon^3 t+\varepsilon^4 u=0$$

to Q and so

$$1+\varepsilon^{p+1}+\varepsilon^{q+2}+\varepsilon^{r+3}+\varepsilon^{s+4}=0$$

which will hold if (like p, q, r, s) $p+1$, $q+2$, $r+3$, $s+4$ are a permutation (mod 5) of 1, 2, 3, 4. Thus p, $q+1$, $r+2$, $s+3$ are, in some order, 0, 1, 2, 3 and, as $p\neq 0$, there are only three *disjoint* possibilities:

$$q=4, \qquad r=3, \qquad s=2.$$

Discussion is similar in all three instances. If, for example, $q=4$ then p, r, s are 1, 2, 3 as also are p, $r+2$, $s+3$; so consecutively, $s=3$, $r=1$, $p=2$ and a is conjugate to b $(1, \varepsilon^2, \varepsilon^4, \varepsilon, \varepsilon^3)$. If $r=3$ one returns to **a** which is of course conjugate to itself; if $s=2$ one arrives at $\bar{b}$ $(1, \varepsilon^3, \varepsilon, \varepsilon^4, \varepsilon^2)$. As b, $\bar{b}$ are transposed by complex conjugation they are both conjugate to $\bar{a}$ $(1, \varepsilon^4, \varepsilon^3, \varepsilon^2, \varepsilon)$. Thus there is on Q a quadrilateral whose vertices belong to both B and F and whose sides are generators, two in each regulus, and whose diagonals $a\bar{a}$, $b\bar{b}$ are polar lines.

$$\begin{array}{ll} a(1, \varepsilon, \varepsilon^2, \varepsilon^3, \varepsilon^4) & \bar{b}(1, \varepsilon^3, \varepsilon, \varepsilon^4, \varepsilon^2) \\ b(1, \varepsilon^2, \varepsilon^4, \varepsilon, \varepsilon^3) & \bar{a}(1, \varepsilon^4, \varepsilon^3, \varepsilon^2, \varepsilon). \end{array} \tag{3.1}$$

There are six such quadrilaterals: they arise from the one here displayed by keeping x and one of the other coordinates fixed while imposing the 3! permutations on the remaining three.

It was explained [**3**, p. 543] that the generators

$$ab, \ b\bar{a}, \ \bar{a}\bar{b}, \ \bar{b}a$$

are the tangents of B at

$$a, \ b, \ \bar{a}, \ \bar{b}.$$

They will now be seen to be the tangents of F at

$$b, \ \bar{a}, \ \bar{b}, \ a;$$

indeed not only do they touch F, they are *inflectional* tangents. For the coordinates of points on, say, ab occur on varying ρ, σ in

$$\rho+\sigma, \quad \rho\varepsilon+\sigma\varepsilon^2, \quad \rho\varepsilon^2+\sigma\varepsilon^4, \quad \rho\varepsilon^3+\sigma\varepsilon, \quad \rho\varepsilon^4+\sigma\varepsilon^3 \tag{3.2}$$

whose sum (of course) is zero, as the sum of their squares is too. But their cubes sum to $15\rho\sigma^2$, their fourth powers to $20\rho^3\sigma$ so that the three intersections of B and ab consist of one at b and two at a while the four intersections of F and ab consist of three at b and one at a.

Call these *inflections of* F the 24 points I. The same argument as was used [**3**, p. 543] to identify osculating planes of B shows, when applied to F, that the planes having 4-point intersection with F at a, b, $\bar{a}$, $\bar{b}$ are the respective tangent planes $\bar{b}ab$, $ab\bar{a}$, $b\bar{a}\bar{b}$, $\bar{a}\bar{b}a$ of Q. So far as their relation to B is concerned the six quadrilaterals were detected by Wiman [**10**, p. 19]; he also identified [**10**, p. 20] the osculating planes of B at their 24 vertices.

Take now the chord cd of F to be a side of any of these six quadrilaterals; every plane through cd has three-point intersection with F at either c or d. Those ten planes which join cd to lines λ in the opposite regulus belonging to those five quadrilaterals not involving cd each have three-point intersection with F also at one of the two intersections of λ with F; thus 60 tangent planes of Q have three-point intersections with F at two distinct places. These are not "ordinary" double osculating planes: they are specialised in the sense that their three coincident intersections at either contact are collinear.

If, however, λ belongs to the same quadrilateral as cd and so contains either c or d one is concerned with the intersections of F with the tangent plane of Q at a point I. Take a, with tangent plane $\bar{b}ab$; the points of this plane are obtained by varying ρ, σ, τ in $\rho+\sigma+\tau$, $\rho\varepsilon+\sigma\varepsilon^2+\tau\varepsilon^3$, $\rho\varepsilon^2+\sigma\varepsilon^4+\tau\varepsilon$, $\rho\varepsilon^3+\sigma\varepsilon+\tau\varepsilon^4$, $\rho\varepsilon^4+\sigma\varepsilon^3+\tau\varepsilon^2$ for which

$$S_2=10\sigma\tau, \quad S_4=10(2\rho^3\sigma+2\rho\tau^3+3\sigma^2\tau^2), \tag{3.3}$$

so that the eight points of F occur when

$$\sigma=0=\rho\tau^3 \quad \text{and} \quad \tau=0=\rho^3\sigma.$$

The former set is $\bar{b}+3a$, the latter is $a+3b$ so that the complete intersection is $\bar{b}+4a+3b$. This plane is clearly a tritangent plane two of whose three contacts coalesce at a; but it is further specialised in that its third contact is not an ordinary contact but a three-point contact, and this latter specialisation entitles it to be reckoned twice. So the tangent planes of Q at the $24I$ contribute 48 to the tally of tritangent planes of F.

Just as the intersection of F with the tangent plane of Q at a consists of $\bar{b}+4a+3b$ its intersection with the tangent plane of Q at $\bar{a}$ likewise consists of $b+4\bar{a}+3\bar{b}$; so the pair of planes cuts F in the set $4(a+b+\bar{a}+\bar{b})$. This same quadruple tetrad is also cut on F by the pair of planes tangent to Q at b and $\bar{b}$. Thus every quadric, other than Q itself, belonging to the pencil whose base is the skew quadrilateral $ab\bar{a}\bar{b}$ cuts this special set on F. This is the more significant in that the canonical sets on F happen to be its intersections with quadrics, so that while the canonical sets on any curve involve some every one of whose points is reckoned twice—in general there is a finite number $2^{p-1}(2^p-1)$ of these double sets on a curve of genus p—F is so specialised that six of its canonical sets consist of four points each reckoned four times.

4. Sixty points Σ

The chief feature of F, as of B, is invariance under the 120 operations of the group S_5 of permutations of the coordinates. The least complicated permutations are *transpositions*, leaving three coordinates unchanged while transposing the residual pair. Here there are ten of them; each is achieved by a harmonic inversion h_{ij} in a vertex V_{ij} of P and its polar plane π_{ij} with respect to Q. Take, for instance, $V_{12}(1,-1,0,0,0)$; its polar plane π_{12} is $x=y$ and

$$(x,y,z,t,u) \qquad (y,x,z,t,u)$$

https://doi.org/10.1017/S0013091500022161 Published online by Cambridge University Press

are transposed by h_{12} because the difference of these coordinate vectors gives V_{12}, their sum a point in π_{12}. Each point of F is joined to V_{12} by a chord of F, the two intersections with F being images in h_{12}. But should the point be among the eight intersections of F with π_{12} the tangent of F passes through V_{12} and an earlier discussion [**3**, p. 541] shows all these points to be *stalls* on F where its osculating plane is stationary, having four-point intersection. It also shows this stationary osculating plane to be the tangent plane of the cone of chords. Two of these eight stalls are the Ω on e_{12}; call the other six points Σ. In this way

$$20+6.10=80$$

stalls of F are accounted for, and there are no others. For the planes of space cut F in the octads of a g_8^3 and a standard formula [**9**, p. 188] gives $(3+1)(8+3.9-3)=128$ for the number of octads with a quadruple member. But these certainly include the 24 points I, and indeed twice over. This multiplicity is also a standard matter; an inflection, while counting *singly* in every octad cut by a plane through it but not containing the tangent, counts *trebly* among all those planes which do pass through the tangent, *quadruply* in a unique plane, so that the appropriate multiplicity is [**8**, p. 77; or, for a text book reference, **9**, p. 188]

$$1+3+4-\tfrac{1}{2}\cdot 3\cdot 4=2.$$

So $128-48=80$ is the tally of stalls on F. *Each plane π_{ij} meets F at eight stalls, two of them Ω, the other six Σ.*

5. Ten quartic cones of chords

The eight intersections of a plane through V_{12} with F consist of four pairs: each member of a pair is the image of the other in h_{12}; in other words, the chords of F through V_{12} generate a quartic cone q_{12}. Among its generators are e_{34}, e_{35}, e_{45} each joining a pair Ω, and the tangent plane of q_{12} along, say, e_{34} contains the tangents of F at both members of the Ω-pair; these tangents, as was implied in §2, meet at V_{34}. But the plane is d_{34}, having four-point intersection with F at both the Ω, so that the tangent plane of q_{12} along e_{34} meets q_{12} only in e_{34} reckoned four times. The section of q_{12} by a plane not containing V_{12} is a quartic curve, non-singular as will be seen, with undulations on e_{34}, e_{35}, e_{45}.

The equation of q_{12} is the outcome of eliminating x and y from (1.1), (1.2) and (1.3), and the identity

$$(x+y)^4+2(x^4+y^4)\equiv(x^2+y^2)\{x^2+y^2+2(x+y)^2\}$$

shows the outcome to be

$$(z+t+u)^4-2(z^4+t^4+u^4)=(z^2+t^2+u^2)\{z^2+t^2+u^2-2(z+t+u)^2\} \tag{5.1}$$

wherein fourth powers cancel in accordance with e_{34}, e_{35}, e_{45} being on q_{12}. The undulations are apparent on writing (5.1) as

$$(t+u)(u+z)(z+t)(z+t+u)=(tu+uz+zt)^2 \tag{5.2}$$

showing $t+u=0$ to have all its four intersections with this plane quartic coincident at $t=u=0$. The bitangency of $z+t+u=0$ (the section of d_{12}) and its points Ω of contact are also apparent, and the curve is easily shown to be free of multiple points.

So F is, in ten different ways, in (2, 1) correspondence with a curve of genus 3. And this curve is in itself special being, as the symmetry of (5.2) shows, invariant under a dihedral group of six self-projectivities. It has three undulations, each accounting for two of its 24 inflections because while its Hessian sextic cuts it at its inflections it touches it at any undulation. The equation (5.2) shows that its three undulation tangents touch a conic at their contacts, and that the Hessian duad on the conic of these three contacts are contacts of the quartic with a bitangent.

Apart from the three edges of P through V_{ij}, undulation generators of q_{ij}, the cone has $24-2.3=18$ inflectional generators; its tangent planes along these are *double osculating planes* of F, the two contacts of each being paired in h_{ij}. So 180 such planes are recognised.

6.

The 28 bitangent planes each touching a quartic cone along two of its generators include, for a cone q_{ij}, diagonal planes with special properties. For example: in d_{34}, d_{35}, d_{45} the two generators of contact with q_{12} coincide on an undulation generator, so that 25 bitangent planes remain; these include d_{12} which does touch q_{12} along distinct generators, but is special in that the two contacts of F with d_{12} on either coincide at an Ω on e_{12}.

There are, apart from diagonal planes, 24 bitangent planes of each q_{ij}; they are quadritangent to F so that each is to be counted as four tritangent planes and one obtains a contribution of 960 to the full number of tritangent planes of F.

7. Tritangent planes whose three contacts are Σ

Tritangent planes all of whose three contacts with F are stalls occur just as with B [**4**, p. 216], though of course with B there were no additional intersections. If $(1, 1, \xi, \eta, \zeta)$ is a Σ in π_{12}

$$\xi+\eta+\zeta=\xi^2+\eta^2+\zeta^2=\xi^4+\eta^4+\zeta^4=-2$$

so that ξ, η, ζ are the zeros of

$$\theta^3+2\theta^2+3\theta+\tfrac{3}{2}. \tag{7.1}$$

The six Σ in π_{12} occur on permuting ξ, η, ζ, no two of them being equal; all 60 Σ occur on permuting 1, 1, ξ, η, ζ in all $5!/2!=60$ possible different ways. The generator

$$z/\xi=t/\eta=u/\zeta \tag{7.2}$$

https://doi.org/10.1017/S0013091500022161 Published online by Cambridge University Press

of q_{12} touches F at $(1, 1, \xi, \eta, \zeta)$ and the plane $t/\eta = u/\zeta$ joining (7.2) to e_{45} is tritangent to F at

$$(\xi, 1, 1, \eta, \zeta), \quad (1, \xi, 1, \eta, \zeta), \quad (1, 1, \xi, \eta, \zeta).$$

In π_{12}, correspondingly, the join $t/\eta = u/\zeta$ of (ξ, η, ζ) on (5.2) to the undulation $(1, 0, 0)$ touches (5.2) at $(1, \eta, \zeta)$; this can be directly verified, using the fact that (7.1) is zero when $\theta = \xi, \eta, \zeta$.

So each q_{ij} has six generators, each belonging to three tritangent planes whose contacts are all Σ, and each such plane so arises from three q_{ij}. Thus 60 tritangent planes of F are identified.

8. Further tritangent planes

Now, again as in [**4**, p. 216], there are, through these 60 generators of the q_{ij} tritangent planes only one of whose contacts is a stall. For through any point other than an inflection on a non-singular plane quartic there pass ten lines touching the curve elsewhere, so that through a join of V_{ij} to a point Σ in π_{ij} there pass ten planes touching q_{ij} along other generators g. But among these are three of the tritangent planes of F just noticed; seven other planes remain which are tritangent to F, their other two contacts being images on g in h_{ij}. Each q_{ij} thus affords 42 such planes, so that 420 more of the tritangent planes of F are found.

There are also ten planes through the join of V_{ij} to either Ω on e_{ij} that touch q_{ij} along other generators; d_{ij} is one of these, and they also include three of those noted in §2. For example: $(0, 0, 1, \omega, \omega^2)$ is joined to V_{12} by the line $z = \omega^2 t = \omega u$ lying in the three planes

$$\omega^2 t = \omega u, \quad \omega^2 u = \omega z, \quad \omega^2 z = \omega t$$

meeting π_{12} in the tangents of (5.2) at

$$(0, \omega, \omega^2), \quad (\omega^2, 0, \omega), \quad (\omega, \omega^2, 0).$$

There remain, apart from $z + t + u = 0$, six further tangents of (5.2) through $(0, 0, 1, \omega, \omega^2)$; their joins to V_{12} are tritangent planes of F, and the companion Ω on e_{12} provides another six. So, in all, 120 more tritangent planes of F are found.

9.

Apart from the ten planes d_{ij} the number of tritangent planes accounted for is

$$20 + 48 + 60 + 960 + 420 + 120 = 1628,$$

and to these must be added the d_{ij} counted with their proper multiplicity. Any tritangent plane not fixed by any of the 120 self-projectivities of F will belong to an orbit of 120.

https://doi.org/10.1017/S0013091500022161 Published online by Cambridge University Press

10.

Among the many equations, obtained by Cayley [**1**, p. 77], involving the characters of an algebraic curve C in [3] is the pair, both self-dual,

$$t+t'=\tfrac{1}{3}(r-4)(r-5)(r-6)-4p(r-10)-i(r-6)-2\tau(r-8),$$

$$t-t'=(r-6)(n-m)$$

where t is the number of tritangent planes, t' the number of triads of concurrent tangents. Here n is the order of C, m the class (number of osculating planes through a point), r the rank (number of tangents meeting a line), p the genus, i the number of inflections, τ the number of double tangents. For F

$$n=8, \quad m=48, \quad r=32, \quad p=9, \quad i=24, \quad \tau=0,$$

so that $t=2048$, $t'=3088$. If it could convincingly be shown that each d_{ij} accounts for *six* tritangent planes one could observe that

$$2048=1688+360,$$

allowing for three orbits of 120.

11. The equianharmonic property

Since the four points a, b, $\bar{b}$, $\bar{a}$ of (3.1) are linearly independent any point in the space has, for some p, q, r, s coordinates given by

$$(x, y, z, t, u)=(p, q, r, s)\begin{pmatrix} 1 & \varepsilon & \varepsilon^2 & \varepsilon^3 & \varepsilon^4 \\ 1 & \varepsilon^2 & \varepsilon^4 & \varepsilon & \varepsilon^3 \\ 1 & \varepsilon^3 & \varepsilon & \varepsilon^4 & \varepsilon^2 \\ 1 & \varepsilon^4 & \varepsilon^3 & \varepsilon^2 & \varepsilon \end{pmatrix};$$

this is to supersede the supernumerary coordinates by homogeneous coordinates with $ab\bar{b}\bar{a}$ as tetrahedron of reference. Then [**7**, p. 187]

$$\tfrac{1}{10}\Sigma x^2=ps+qr, \quad \tfrac{1}{10}\Sigma x^4=2(p^3q+q^3s+s^3r+r^3p)+3(p^2s^2+q^2r^2)+12pqrs.$$

Now parametrise Q, with equation $ps+qr=0$, by

$$p{:}q{:}r{:}s=\lambda_1\mu_1{:}\lambda_2\mu_1{:}\lambda_1\mu_2{:}-\lambda_2\mu_2$$

so that [**7**, pp. 179, 180] λ_1/λ_2 and μ_1/μ_2 are parameters in its two reguli. This point satisfies $\Sigma x^4=0$, and so lies on F if, and only if [**6**, p. 381; **7**, p. 196; **5**, p. 368]

$$\lambda_1^4\mu_1\mu_2^3+\lambda_1^3\lambda_2\mu_1^4-3\lambda_1^2\lambda_2^2\mu_1^2\mu_2^2-\lambda_1\lambda_2^3\mu_2^4-\lambda_2^4\mu_1^3\mu_2=0; \tag{11.1}$$

https://doi.org/10.1017/S0013091500022161 Published online by Cambridge University Press

the double binary quartic accords with F being quadrisecant to every line on Q.

A binary quartic

$$aX^4+4bX^3Y+6cX^2Y^2+4dXY^3+eY^4$$

has two invariants

$$I\equiv ae-4bd+3c^2, \qquad J\equiv\begin{vmatrix} a & b & c \\ b & c & d \\ c & d & e \end{vmatrix},$$

and discriminant $\Delta\equiv I^3-27J^2$. But, regarding (11.1) as a binary quartic in λ_1, λ_2 so that

$$a=\mu_1\mu_2^3, \quad b=\tfrac{1}{4}\mu_1^4, \quad c=-\tfrac{1}{2}\mu_1^2\mu_2^2, \quad d=-\tfrac{1}{4}\mu_2^4, \quad e=-\mu_1^3\mu_2, \tag{11.2}$$

I is identically zero, as it is also, similarly, for (11.1) regarded as a binary quartic in μ_1, μ_2. Thus *every line on Q is cut equianharmonically by F.*

Incidentally: whenever $I=0$ Δ is a multiple of J^2; this accords with F having not an ordinary contact but an inflection whenever it touches a line on Q. A last point of interest is that (11.2) make

$$J=\tfrac{1}{16}\mu_1\mu_2(\mu_1^{10}+11\mu_1^5\mu_2^5-\mu_2^{10});$$

the parameters of the twelve lines in the "μ" regulus that are inflectional tangents of F are zeros of this [7, p. 56] icosahedral duodecimic; and the same is true of the "λ" regulus.

REFERENCES

1. A. Cayley, On the theory of the curve and torse, *Quarterly J. of Pure and Applied Maths.* **11** (1871), 294–317: *Mathematical Papers* **8** (Cambridge Univ. Press, 1895), 72–91.

2. A. Cayley, On the curves situate on a surface of the second order, *Phil. Mag.* **22** (1861), 35–38: *Mathematical Papers* **5** (Cambridge Univ. Press, 1892), 70–72.

3. W. L. Edge, Bring's curve, *J. London Math. Soc.* (2) **18** (1978), 539–545.

4. W. L. Edge, Tritangent planes of Bring's curve, *J. London Math. Soc.* (2) **23** (1981), 215–222.

5. R. Fricke, Entwicklungen zur Transformationen fünfter und siebenter Ordnung einiger spezieller automorpher Funktionen, *Acta Mathematica* **17** (1893).

6. P. Gordan, Ueber die Auflösung der Gleichungen von fünften Grade, *Math. Annalen* **13** (1878), 375–404.

7. F. Klein, *Vorlesungen über das Ikosaeder* (Teubner, Leipzig, 1884).

8. C. Segre, Introduzione alla geometria sopra un ente algebrico semplicemente infinito, *Annali di mat.* (2) **22** (1894), 41–142: *Opere I* (Rome, 1957), 198–304.

9. F. Severi, *Vorlesungen über algebraische Geometrie* (Teubner, Leipzig, 1921).

10. A. Wiman, Ueber die algebraischen Curven von den Geschlechtern $p=4$, 5, 6 welche eindeutige Transformationen in sich besitzen, *Svenska Vet. Akad. Handlingar*: Bihang till Handlingar **21** (1895), afd. 1, No. 3, 41 pp.

Inveresk House
Musselburgh
EH21 7UA