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OSCILLATION CRITERIA FOR SEMILINEAR EQUATIONS IN GENERAL DOMAINS

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Oscillation theory for nonlinear ordinary differential equations has been extensively developed in recent years by several authors. We refer the reader to the recent paper by Noussair and Swanson, [2], where an extensive bibliography may be found. The situation is somewhat different for the case of second order partial differential equations, an area which has recently been virtually untouched, except for the establishment of criteria which depend on a comparison with suitable linear equations and therefore are essentially linear in nature. A bibliography of such results may also be found in [2]. Of a more general nature have been the paper by the author [1], and the more recent work of Noussair and Swanson, [2]. Although the methods employed and the equations considered in [1] and [2] are different, both papers obtained results only for a class of exterior domains of R^n, of which the typical example is the complement of a bounded sphere. In this note we establish some oscillation criteria for more general unbounded domains and, therefore, answer at least partially one of the open questions mentioned by Noussair and Swanson at the end of their paper, [2].

Let $x=(x^1,\ldots,x^n)$ denote points of the n-dimensional Euclidean space R^n. We denote by D_i differentiation with respect to x^i and by $|x|$ the Euclidean length of x. The partial differential equation under consideration is the semilinear uniformly elliptic equation:

$$(1)\qquad \sum_{i,j=1}^{n} D_i(a_{ij}(x)D_ju)+c(x,u)=0, \qquad a_{ij}=a_{ji}$$

for $x\in G$, where G denotes an unbounded domain of R^n. We shall assume that $a_{ij}\in C^1(\bar{G})$ for $i,j=1,\ldots,n$ and that $c(x,t)$ always satisfies the following properties: (i) $c(x,t)\in C^0(\bar{G}\times R^1)$; (ii) $c(x,t)t^{-1}$ is nondecreasing in $t\in(0,\infty)$ for every $x\in G$; (iii) $-c(x,t)=c(x,-t)$ for all $(x,t)\in G\times R^1$; $c(x,t)\geq 0$ for all $(x,t)\in G\times(0,\infty)$. These global assumptions are made for convenience. It will be apparent from the sequel that the behaviour of the coefficients of (1) or of the domain G in any bounded subset of R^n is immaterial to our considerations.

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We observe that we do not require the a_{ij} to be radial functions, as was the case in [2]. Also, it will be clear by the presentation that by means of the results of [1] we could also treat the more general nonsymmetric equation:

$$\sum_{i,j=1}^{n} D_i(a_{ij}(x)D_ju)+2\sum_{j=1}^{n} b_j(x)D_ju+c(x,u)=0 \tag{2}$$

at the expense of complicating the presentation.

About G we shall only assume that there exists a family of bounded spheres $\{S_k\}_{k=1}^{\infty}$, where S_k has center at x_k and radius R_k, $k=1,\ldots,$ such that:

(i) $S_k \subset G$;

(ii) $x_k \in S_{k-1}$, $S_k \not\subset S_{k-1}$, $x_i \neq x_j$ if $i \neq j$;

(iii) for any integer N there exists an integer K such that if $k \geq K$ then $S_k \subset \{x \mid |x| > N\}$.

As a consequence of conditions (i), (ii), (iii) we can introduce with any such family $\{S_k\}_{k=1}^{\infty}$ another family of spheres $\{D_k\}_{k=1}^{\infty}$, where D_k also has center at x_k and radius d_k chosen so small that $\bar{D}_{k+1} \subset S_k$ and $\bar{D}_k \subset S_k - \bar{D}_{k+1}$ for $k=1,\ldots.$ It is sufficient, for example, to choose the d_k so that the following inequalities are satisfied:

$$d_1 < \min(R_1; |x_2 - x_1|)$$
$$d_{k+2} < \min(R_{k+1} - |x_{k+2} - x_{k+1}|; |x_{k+2} - x_{k+1}| - d_{k+1}, |x_{k+3} - x_{k+2}|)$$

for $k=0, 1.$

Clearly a sequence $\{D_k\}_{k=1}^{\infty}$ can be chosen explicitly in any specific example and in many ways. The possible values of the radii will depend on the geometry of the case in question, and not on the coefficients of equation (1). We illustrate these remarks by considering two very special examples where the calculations are relatively simple. Consider first the case where G is a cylindrical domain:

$$G=\left\{x \,\middle|\, \sum_{i=1}^{n-1} (x^i)^2 < R^2;\ x^n > -R\right\}. \tag{3}$$

In this case all the spheres can be chosen of fixed radius and with centers on the x_n axis. For example, we can choose the families $\{S_k\}_{k=1}^{\infty}$; $\{D_k\}_{k=1}^{\infty}$ by the formulas

$$S_k=\left\{x \,\middle|\, \sum_{i=1}^{n-1} (x^i)^2 + (x^n - (k-1)\,\delta R)^2 < R^2\right\} \tag{4}$$

$$D_k=\left\{x \,\middle|\, \sum_{i=1}^{n-1} (x^i)^2 + (x^n - (k-1)\,\delta R)^2 < \mu^2 R^2\right\} \tag{5}$$

for $k=1,2,\ldots,$ where δ, μ denote any numbers satisfying: $0 < 2\mu < \delta < \delta + \mu < 1$. Examples of noncylindrical domains can also be given where the spheres can be chosen of fixed radii. As an example of a domain where the radii must depend on k we consider the domain G given by:

$$G=\left\{x \mid 0<x^n<\left(\sum_{i=1}^{n-1}(x^i)^2\right)^{-1}\right\}\cap\left\{x \,\middle|\, \sum_{i=1}^{n-1}(x^i)^2<1\right\}. \tag{6}$$

In this case we can, for example, choose $\{S_k\}_{k=2}^{\infty}$, $\{D_k\}_{k=2}^{\infty}$ according to the formulas:

$$S_k=\left\{x \,\middle|\, \sum_{i=1}^{n-1}(x^i)^2+\left(x^n-1+\frac{1}{2k}-\sum_{i=1}^{k-1}\frac{1}{2i}\right)^2<\frac{1}{k^2}\right\} \tag{7}$$

$$D_k=\left\{x \,\middle|\, \sum_{i=1}^{n-1}(x^i)^2+\left(x^n-1+\frac{1}{2k}-\sum_{i=1}^{k-1}\frac{1}{2i}\right)^2<\frac{1}{16k^2}\right\}. \tag{8}$$

With S_1, D_1 chosen conveniently, simple calculations show that these families satisfy the conditions required.

The following extension of the classical three sphere theorem will be needed in the sequel:

LEMMA 0. *Let $u>0$ satisfy the differential inequality*:

$$\sum_{i,j=1}^{n} D_i(a_{ij}D_ju)\le 0$$

in the annular domain S_p-D_p, that is: $d_p\le|x-x_p|<R_p$. Then for all x in S_p-D_p the following estimate holds:

$$u(x)\ge h_p(|x-x_p|)\inf_{|x-x_p|=d_p}[u(x)],$$

where, for $d_p\le r\le R_p$,

$$h_p(r)=\frac{\displaystyle\int_r^{R_p}\exp\left[-\int_{d_p}^{s}c_p(\rho)\rho^{-1}\,d\rho\right]ds}{\displaystyle\int_{d_p}^{R_p}\exp\left[-\int_{d_p}^{s}c_p(\rho)\rho^{-1}\,d\rho\right]ds},$$

and c_p is any function in $C^0([d_p, R_p])$ such that for all $x\in S_p-D_p$,

$$\frac{\sum_{i=1}^n a_{ii}-\sum_{i,j=1}^n a_{ij}\dfrac{(x^i-x_p^i)(x^j-x_p^j)}{|x-x_p|^2}+\sum_{i,j=1}^n D_j(a_{ij})(x^j-x_p^i)}{\sum_{i,j=1}^n a_{ij}\dfrac{(x^i-x_p^i)(x^j-x_p^j)}{|x-x_p|^2}}\le c_p(|x-x_p|).$$

Proof. Consider the annulus $d_p\le|x-x_p|\le R_\alpha$ where $R_\alpha\uparrow R_p$. Since u is positive, by an extension of the three sphere theorem [4, p. 132], it follows that in this annulus we have:

$$u(x) \geq \inf_{|x-x_p|=d_p}[u(x)] \cdot \frac{\int_{|x-x_p|}^{R_\alpha} \exp\left[-\int_{d_p}^{s} c_p(\rho)\rho^{-1}\, d\rho\right] ds}{\int_{d_p}^{R_\alpha} \exp\left[-\int_{d_p}^{s} c_p(\rho)\rho^{-1}\, d\rho\right] ds}$$

The conclusion now follows by letting $R_\alpha \to R_p$.

We recall that equation (1) is oscillatory iff any nontrivial solution of (1) has a zero in $\{x \mid x \in G \text{ and } |x| > R\}$ for any given $R > 0$.

LEMMA 1. *Let u denote a solution of* (1) *which is positive in $\bigcup_{i=l}^{\infty} S_i$. Then for all $x \in S_i$ $(i \geq l+2)$ the following estimate holds*:

$$u(x) \geq \tau g_i(x),$$

where:

$$g_i(x) = \begin{cases} f_{i-1}(x) & \text{if } \quad x \in D_i, \\ f_i(x) & \text{if } \quad x \in S_i - D_i, \end{cases}$$

τ is a constant depending on the value of u on S_l and $\{S_i\}_{i=1}^{l}$, and f_i is given by:

$$f_i(x) = \left\{\prod_{j=2}^{i} h_{j-1}[|x_j - x_{j-1}| + d_j]\right\} h_i[|x - x_i|] \qquad i \geq l+1,$$

where h_p is given in Lemma 0.

Proof. Since $u > 0$ in $\bigcup_{i=l}^{\infty} S_l$, it follows that u satisfies the inequality

$$\sum D_i(a_{ij} D_j u) \leq 0.$$

By Lemma 0 we conclude that for $x \in S_l - D_l$,

$$u(x) \geq h_l(|x - x_l|) \cdot \inf_{|x-x_l|=d_l}[u(x)].$$

Repeating the procedure for S_{l+1} gives for $x \in S_{l+1} - D_{l+1}$,

$$u(x) \geq h_{l+1}(|x - x_{l+1}|) \inf_{|x-x_{l+1}|=d_{l+1}}[u(x)]. \tag{9}$$

But by the previous result,

$$\inf_{|x-x_{l+1}|=d_{l+1}}[u(x)] \geq \inf_{|x-x_l|=d_l}\{[u(x)]\} h_l(|x_{l+1} - x_l| + d_{l+1})$$

Substituting into (9) gives:

$$u(x) \geq \tau\left\{\prod_{j=1}^{l} h_j(|x_{j+1} - x_j| + d_{j+1})\right\} h_{l+1}(|x - x_{l+1}|)$$

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where

$$\tau = \inf_{|x-x_l|=d_l} (u(x)) \cdot \left[\prod_{j=1}^{l-1} h_j(|x_{j+1}-x_j|+d_{j+1}) \right]^{-1}.$$

The conclusion of the Lemma now follows by induction.

THEOREM 1. *Let the above structure and notation hold. Assume further that* (a_{ij}) *is bounded as a form and that, for any constant* $\tau > 0$,

$$\lim_{k\to\infty} \left\{ R_k^{2-n} \int_{\{x \mid |x-x_k|<(R_k/2)\}} \left[\frac{c(x, \tau g_k(x))}{\tau g_k(x)} \right] dx \right\} = +\infty, \tag{10}$$

then all solutions of (1) *oscillate.*

Proof. By assumption (iii) on $c(x, t)$, if u is a solution of (1) which does not oscillate, then we may assume that u is positive in $\bigcup_{i=l}^{\infty} S_i$ for some integer l. By Lemma 1 it follows that if $x \in S_k$ and $k > l+1$ then $u(x) \geq \tau g_k(x)$, for some constant τ. We next appeal to the Swanson-Picone identity as was done in [1] to conclude that for any $\phi \in C_0^\infty(S_k)$,

$$0 \leq \int_{S_k} \left\{ \sum_{i,j=1}^{n} a_{ij} D_i\phi D_j\phi - \frac{c(x, u)}{u} \phi^2 \right\} dx$$

By the monotonicity of $c(x, t)t^{-1}$ this implies:

$$0 \leq \int_{S_k} \left\{ \sum a_{ij} D_i\phi D_j\phi - \frac{c(x, \tau g_k)}{\tau g_k} \phi^2 \right\} dx.$$

To obtain a contradiction we merely have to find a function ϕ which makes the integral on the right hand side negative. Condition (10) is sufficient for the existence of such a ϕ as an immediate consequence of Theorem 2 of [5].

To illustrate the above theorem we consider as a special example the n-dimensional Emden-Fowler equation:

$$\Delta u + d(x)u^\gamma = 0$$

for two different types of subdomains $G \subset R^n$, $n \geq 3$. Here we assume that d is a nonnegative continuous function and that γ is the ratio of two odd positive integers, $\gamma \geq 1$. Assume first that G is the cylinder given by (3) with $\{S_k\}_{k=1}^\infty$; $\{D_k\}_{k=1}^\infty$ as defined by (4), (5). In this case, $h_p(r) = (R^{2-n} - r^{2-n})/(R^{2-n} - d^{2-n})$, and consequently:

$$\begin{aligned} f_i(x) &= \prod_{j=2}^{i} \left(\frac{R^{2-n} - (\delta+\mu)^{2-n} R^{2-n}}{R^{2-n} - \mu^{2-n} R^{2-n}} \right) \left(\frac{R^{2-n} - |x-x_i|^{2-n}}{R^{2-n} - \mu^{2-n} R^{2-n}} \right) \\ &= \left[\frac{1-(\delta+\mu)^{2-n}}{1-\mu^{2-n}} \right]^{i-1} \frac{R^{2-n} - |x-x_i|^{2-n}}{R^{2-n}(1-\mu^{2-n})}. \end{aligned}$$

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Since $D_k \subset \{x \mid |x-x_k|<R/2\}$, then

$$\int_{\{x\,||x-x_k|<R/2\}} \left[\frac{c(x,\tau g_k(x))}{\tau g_k(x)}\right] dx \geq \int_{\{x\,||x-x_k|<\mu R\}} \tau^{\gamma-1}\, d(x) g_k^{\gamma-1}(x)\, dx$$

$$\geq \tau_0 \left\{\left[\frac{(\delta+\mu)^{2-n}-1}{\mu^{2-n}-1}\right]^{k(\gamma-1)} \int_{\{x\,||x-x_k|<\mu R\}} d(x)\, dx\right\}$$

for some positive constant τ_0. Condition (10) of Theorem 1 is thus implied by the simple criterion:

$$\lim_{k\to\infty}\left\{\left[\frac{(\delta+\mu)^{2-n}-1}{\mu^{2-n}-1}\right]^{k(\gamma-1)} \int_{\{x\,||x-x_k|<\mu k\}} d(x)\, dx\right\} = +\infty.$$

Consider next the case where G, $\{S_k\}_{k=1}^{\infty}$, $\{B_k\}_{k=1}^{\infty}$ are as given by (6), (7), (8) respectively. Repeating the procedure of the previous example, we find:

$$f_i(x) = \left\{\prod_{j=2}^{i}\left(\frac{1-(4j)^{n-2}(3j+1)^{2-n}}{1-4^{n-2}}\right)\right\} \frac{(i)^{n-2}-|x-x_i|^{2-n}}{(i)^{n-2}-(4i)^{n-2}}.$$

Since once again $D_k \subset \{x \mid |x-x_k|<R_k/2\}$ condition (10) is guaranteed by the criterion:

$$\lim_{k\to\infty}\left\{k^{n-2}\left(\prod_{j=2}^{k-1}\frac{1-(4j)^{n-2}(3j+1)^{2-n}}{1-4^{n-2}}\right)^{\gamma-1}\right.$$

$$\left.\times \int_{D_k} d(x)\{(|x-x_{k-1}|(k-1))^{2-n}-1\}^{\gamma-1}\cdot dx\right\} = +\infty.$$

Other examples can be given for domains in the shape of spirals and for the more general operator (1). Although the calculations for such cases are theoretically straightforward, they can become very lengthy and complicated.

We conclude by observing that the procedure used in Theorem 1 can be replaced in special cases by simpler considerations which take advantage of the specific global structure considered. As an example, we state:

LEMMA 2. *Assume G contains the infinite rectangle:*

$$E = \{x \mid x^1 > 0, \quad 0 < x^i < \alpha \quad \text{for} \quad i = 2, \ldots, n\}.$$

If for all constants $\tau > 0$ the ordinary differential equation:

$$y''(t) + y(t)\{C(t) - (\pi^2/\alpha^2)(n-1)\} = 0 \tag{11}$$

is oscillatory then all solutions of:

$$\Delta u + c(x, u) = 0 \tag{12}$$

oscillate in G or, for all k sufficiently large, satisfy $\inf_{\{x \mid x\in E, x^1=k\}}(|u(x)|) = 0$,

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where:

$$C(t)=\frac{2^{n-1}}{\alpha^{n-1}}\int_{\{x\mid x\in E, x^1=t\}}\left[\frac{c(x,\tau\omega)}{\tau\omega}\right]\prod_{i=2}^{n}\sin^2\left(\frac{\pi}{\alpha}x^i\right)dx^2\cdots dx^n,$$

$$\omega=\exp(-(\pi/\alpha)x^1(n-1)^{1/2})\prod_{i=2}^{n}\sin\left(\frac{\pi}{\alpha}x^i\right).$$

Proof. Assume to the contrary that there exists a solution u of (12) which is positive in E for $x^1\geq k$ and whose infimum on the cross section is not eventually always zero. We observe that, by the Hopf maximum principle, if $x\in E\cap\{x\mid k<x^1<k+m;\ \alpha-\beta<x^i<\beta\ \text{ for }\ i=2,\ldots,n\}$ then $u(x)\geq v_{m,\beta}(x)$ where: $0<\beta<\alpha$,

$$v_{m,\beta}(x)=\phi_{m,\beta}(x^1)\prod_{i=2}^{n}\sin\left(\frac{\pi}{2\beta-\alpha}(x^i-\alpha+\beta)\right)$$

and $\phi_{m,\beta}$ solves the boundary value problem:

$$\phi''-\left(\frac{\pi}{2\beta-\alpha}\right)^2(n-1)\phi=0;\qquad \phi(k)=\inf_{\{x\mid x\in E;\, x^1=k\}}[u(x)];\qquad \phi(k+m)=0.$$

Letting $\beta\to\alpha$ and then $m\to+\infty$ it follows that for some constant τ_0 depending only on k and the values of u on $x^1=k$, $u(x)\geq\tau_0\omega(x)$ for all $x\in E\cap\{x\mid x^1\geq k\}$. Again a contradiction will be obtained if for some $m>0$ and function $\phi\in C_0^\infty(E\cap\{x\mid k<x^1<k+m\})$ the following inequality holds:

$$\int_{E\cap\{x\mid k<x^1<k+m\}}\sum_{i=1}^{n}(D_i\phi)^2-\frac{c(x,\tau_0\omega)}{\tau_0\omega}\phi^2\,dx\leq 0. \tag{13}$$

A direct calculation shows that the function $\psi(x^1)\prod_{i=2}^{n}\sin((\pi/\alpha)x^i)$, where ψ denotes a solution of (11), with $\tau=\tau_0$ and $\psi(k)=\psi(k+m)=0$, satisfies (13) and the result follows.

Corollary 1. *Let the conditions of Lemma* 2 *hold, except assume now that* $\bar{E}\subset G$. *Then equation* (12) *is oscillatory.*

Any of the results for ordinary linear differential equations can now be employed to obtain an oscillation criteria for (12) in G much as was done in [1] for the case of an exterior domain. As an example we state:

Corollary 2. *Let the conditions and notation of Lemma* 2 *and Corollary* 1 *hold. Assume further that, for all constants* $\tau>0$,

$$\lim_{k\to\infty}\left[\tau\frac{2^{n-1}}{\alpha^{n-1}}\left\{\int_{E\cap\{x\mid 0<x^1<k\}}d(x)\omega^{\gamma-1}\prod_{i=2}^{n}\sin^2\left(\frac{\pi}{\alpha}x^i\right)dx\right\}-\frac{\pi^2}{\alpha^2}(n-1)k\right]=+\infty. \tag{14}$$

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Then all solutions of the previously considered Emden-Fowler equation oscillate in G.

Proof. Condition (14) may be rewritten as:

$$\lim_{k\to\infty}\left[\int^{k}\left\{C(t)-\frac{\pi^2}{\alpha^2}(n-1)\right\}dt\right]=+\infty.$$

Equation (11) is then oscillatory by the well known Leighton-Wintner test, [3].

Since the particular type of equation as well as the type of domain considered were of key importance in the proof of Lemma 2, the problem of establishing an analogue of Lemma 2 for the general equations (1), (2) remains open.

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