COMPOSITIO MATHEMATICA

Upper semi-continuity of the Hilbert–Kunz multiplicity

Ilya Smirnov

Compositio Math. **152** (2016), 477–488.

doi:10.1112/S0010437X15007800

FOUNDATION COMPOSITIO MATHEMATICA

LONDON MATHEMATICAL SOCIETY
EST. 1865

Compositio Math. 152 (2016) 477–488
doi:10.1112/S0010437X15007800

Upper semi-continuity of the Hilbert–Kunz multiplicity

Ilya Smirnov

Abstract

We prove that the Hilbert–Kunz multiplicity is upper semi-continuous in F-finite rings and algebras of essentially finite type over an excellent local ring.

1. Introduction

Let R be a commutative Noetherian ring of characteristic $p > 0$. For a prime ideal $\mathfrak{p}$ the (normalized) Hilbert–Kunz function of $\mathfrak{p}$ is defined to be

$$e \mapsto \frac{\ell_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}^{[p^e]}R_\mathfrak{p})}{p^{ed}},$$

where e is a positive integer and $\mathfrak{p}^{[p^e]}$ is the ideal generated by the p^eth powers of elements of $\mathfrak{p}$. The Hilbert–Kunz multiplicity of $\mathfrak{p}$, $\mathrm{e}_{HK}(\mathfrak{p})$, is defined as the limit of this sequence. The Hilbert–Kunz theory originates in the work of Kunz [Kun69, Kun76], and Hilbert–Kunz multiplicity was introduced by Monsky [Mon83] in 1983.

From the beginning, there was a perception that the Hilbert–Kunz theory could be used to measure singularities. In fact, in 1969 Kunz showed that the Hilbert–Kunz function detects singularity, and later, in 2000, Watanabe and Yoshida generalized this for the Hilbert–Kunz multiplicity. In [WY00] they showed that an unmixed local ring $(R, \mathfrak{m})$ is regular if and only if $\mathrm{e}_{HK}(\mathfrak{m}) = 1$.

Since it follows from the work of Kunz that $\mathrm{e}_{HK}(\mathfrak{m}) \geqslant 1$, one may expect that, when $\mathrm{e}_{HK}(\mathfrak{m})$ gets close to 1, the singularity of R should be better. A notable example of this was given by Blickle and Enescu in [BE04] and then improved by Aberbach and Enescu in [AE08]. They show that if $\mathrm{e}_{HK}(\mathfrak{m})$ is sufficiently close to 1, then R has to be Gorenstein and F-regular.

This work is devoted to a global property of the Hilbert–Kunz multiplicity: upper semi-continuity. A formal definition of upper semi-continuity is given in Definition 1, but via Nagata's criterion (Proposition 8) this question can be considered as a distribution property of singularities of the local rings of R. Namely, given a prime ideal $\mathfrak{p}$ of R, we want to know if the Hilbert–Kunz multiplicity of a generic prime containing $\mathfrak{p}$ is close to the Hilbert–Kunz multiplicity of $R_\mathfrak{p}$; see Proposition 9. So, we may say that we are trying to understand if Hilbert–Kunz multiplicity sees that the singularity of a general prime containing $\mathfrak{p}$ is close to that of $\mathfrak{p}$.

We also want to draw reader's attention to Example 7, where we show that the Hilbert–Kunz multiplicity of a general prime containing $\mathfrak{p}$ need not to be equal to $\mathrm{e}_{HK}(\mathfrak{p})$. This illustrates the subtlety of the problem, and since the corresponding statement for the Hilbert–Samuel multiplicity is true, demonstrates a difference between two theories.

Received 10 November 2014, accepted in final form 24 August 2015, published online 2 February 2016.
2010 Mathematics Subject Classification 13D40, 13A35, 13F40 (primary).
Keywords: Hilbert–Kunz multiplicity, semi-continuity, F-finite rings.

In 1976 [Kun76] Kunz proved that for a fixed e the Hilbert–Kunz function $\ell(R_{\mathfrak{p}}/\mathfrak{p}^{[p^e]}R_{\mathfrak{p}})/p^{ed}$ is upper semi-continuous. Shepherd-Barron [She79] supplied a more elementary proof and showed the necessity of an equidimensionality assumption. Twenty years later, in [ES05], Enescu and Shimomoto revisited this work and asked whether the Hilbert–Kunz multiplicity is upper semi-continuous. They also proved that the Hilbert–Kunz multiplicity is dense upper semi-continuous on the maximum spectrum, which is a weaker condition. In 2011 there was a group working on this question in the AIM workshop 'Relating Test Ideals and Multiplier Ideals'.

We prove that the Hilbert–Kunz multiplicity is upper semi-continuous in locally equidimensional F-finite rings and locally equidimensional rings of essentially finite type over an excellent local ring, a mild restriction that is satisfied by complete local domains and domains finitely generated over a field. To achieve this, we want to control the convergence rate of the Hilbert–Kunz function and, building on Tucker's estimates from [Tuc12], we show that it can be controlled generically for F-finite domains (Theorem 19) and for algebras of essentially finite type over a complete local domain (Theorem 22). This method allows us to reduce upper semi-continuity of the limit of a sequence (the Hilbert–Kunz multiplicity) to upper semi-continuity of a term of the sequence (a fixed Hilbert–Kunz function), and the latter is known from the work of Kunz. This strategy should be useful for other numerical invariants in positive characteristic, for example, for F-signature. However, an application of this approach seems to require a better understanding of F-signature; see Remark 26.

The structure of the proof is as follows. We start with general preliminaries in § 2. Then we develop the machinery of global convergence estimates in §§ 3 and 4: § 3 treats the F-finite case; § 4 obtains the same result for algebras of essentially finite type over a complete local domain, so may be skipped by a reader interested only in the F-finite case. Then the estimates are used in § 5 to prove the main theorem.

2. Preliminaries

In this paper all rings are assumed to be commutative Noetherian and containing an identity element. For a module M over a ring R, we will use $\ell(M)$ to denote the length of M.

Let R be a ring of characteristic $p > 0$. For convenience, we use $q = p^e$ where e may vary. For an ideal I of R, let $I^{[q]}$ be the ideal generated by qth powers of the elements of I. By F_*R we mean R viewed as an R-module via the extension of scalars through the Frobenius endomorphism. If R is reduced, F_*R can be identified with the ring of p-roots $R^{1/p}$. We say that R is F-finite if F_*R is a finitely generated R-module.

Definition 1. Let X be a topological space. A real-valued function f is upper semi-continuous if for any $a \in \mathbb{R}$ the set $\{x \in X \mid f(x) < a\}$ is open in X.

In his papers [Kun69, Kun76] Kunz initiated the study of the Hilbert–Kunz function $f_q(R) = (1/q^d)\ell(R/\mathfrak{m}^{[q]})$. For any q we can define a function on $\operatorname{Spec} R$, the spectrum of R, by setting $f_q(\mathfrak{p}) = f_q(R_{\mathfrak{p}})$. In [Kun76, Proposition 3.3, Corollary 3.4] Kunz obtained the following results.

Theorem 2. *Let R be a locally equidimensional ring. Then for all q:*

(i) $f_q(\mathfrak{p}) \leqslant f_q(\mathfrak{q})$, *if* $\mathfrak{p} \subseteq \mathfrak{q}$;

(ii) $f_q(\mathfrak{p})$ *is upper semi-continuous on* $\operatorname{Spec} R$.

We should note that Kunz claimed this result for an equidimensional ring, but Shepherd-Barron pointed out in [She79] that the theorem is false if R is not locally equidimensional.

Definition 3. Let $(R, \mathfrak{m})$ be a local ring of characteristic $p > 0$. It was shown by Monsky that the limit

$$\mathrm{e}_{HK}(R) = \lim_{q \to \infty} \frac{\ell(R/\mathfrak{m}^{[q]})}{q^{\dim R}}$$

exists and is called the Hilbert–Kunz multiplicity of R.

Defining $\mathrm{e}_{HK}(\mathfrak{p}) := \mathrm{e}_{HK}(R_{\mathfrak{p}})$, we can view the Hilbert–Kunz multiplicity as a function on $\operatorname{Spec} R$. In view of Kunz's result, it was natural to make the following conjecture [ES05].

Conjecture 4. *If R is a locally equidimensional excellent ring, then the Hilbert–Kunz multiplicity is an upper semi-continuous function on* $\operatorname{Spec} R$. *Or, less generally, if R is a locally equidimensional F-finite ring, then the Hilbert–Kunz multiplicity is an upper semi-continuous function on* $\operatorname{Spec} R$.

Note that F-finite rings are excellent by a theorem of Kunz [Kun76, Theorem 2.5]. Also, we note that is natural to restrict to F-finite rings, since F-finiteness arises often as useful finiteness property, especially in geometry. For example, it is still not known whether all excellent rings have a test element.

Remark 5. The reader should be warned that Shepherd-Barron (in [She79, Corollary 2]) claimed a much stronger statement. However, in his proof he used that a descending sequence of open sets stabilizes without a proper justification. In fact Shepherd-Barron's claim implies that e_{HK} attains only finitely many values on $\operatorname{Spec} R$. We provide a counter-example to this claim; see Example 7.

It is also worth pointing out that a stronger property holds for Hilbert–Kunz functions. The proofs of semi-continuity by Kunz [Kun76] and Shepherd-Barron [She79] show the following statement.

Proposition 6. *Let R be a locally equidimensional ring and $\mathfrak{p}$ be a prime ideal. Then for any fixed q and any $a \in \mathbb{R}$ the set*

$$\{\mathfrak{p} \mid f_q(\mathfrak{p}) \leqslant a\}$$

is open in $\operatorname{Spec} R$.

Example 7. Let $R = F[x, y, z, t]/(z^4 + xyz^2 + (x^3 + y^3)z + tx^2y^2)$, where F is the algebraic closure of $\mathbb{Z}/2\mathbb{Z}$. In [BM10], Brenner and Monsky showed that tight closure does not commute with localization in this ring.

Let $\mathfrak{p} = (x, y, z)$; this is a prime ideal of dimension one in R. Building on the work of Monsky [Mon98], the author was able to show that the Hilbert–Kunz multiplicity attains infinitely many values on the set of prime ideals containing $\mathfrak{p}$. Moreover, for any prime ideal $\mathfrak{m}$ containing $\mathfrak{p}$, $\mathrm{e}_{HK}(\mathfrak{m}) > \mathrm{e}_{HK}(\mathfrak{p})$, so the set $\{\mathfrak{q} \mid \mathrm{e}_{HK}(\mathfrak{q}) \leqslant \mathrm{e}_{HK}(\mathfrak{p})\}$ is not open. The details will appear in a future paper.

This rather surprising result shows that, compared to the Hilbert–Samuel multiplicity and a fixed Hilbert–Kunz function, the Hilbert–Kunz multiplicity has a distinctively different global behavior.

We use the following standard terminology. A closed set $V(I)$ consists of all prime ideals containing $I \subseteq R$, and a principal open set D_s consists of all prime ideals not containing $s \in R$. Since D_s can be naturally identified with $\operatorname{Spec} R_s$, we will sometimes abuse notation, and, when speaking of inverting an element s, we will mean considering D_s.

We recall Nagata's criterion of openness in $\operatorname{Spec} R$ [Mat80, 22.B].

Proposition 8. *Let R be a ring. A subset U of* $\operatorname{Spec} R$ *is open if and only if:*

(i) *U is stable under generalization, i.e. if $\mathfrak{q} \in U$ and $\mathfrak{p} \subseteq \mathfrak{q}$, then $\mathfrak{p} \in U$;*
(ii) *U contains a nonempty open subset of $V(\mathfrak{p})$ for any $\mathfrak{p} \in U$.*

Since any open set is a union of principal open sets, the second condition is equivalent to the containment $U \supseteq V(\mathfrak{p}) \cap D_s$ for some $s \notin \mathfrak{p}$.

It follows from Theorem 2 that if $\mathfrak{p} \subseteq \mathfrak{q}$ then $\mathrm{e}_{HK}(\mathfrak{p}) \leqslant \mathrm{e}_{HK}(\mathfrak{q})$, so for any a the set $\{\mathfrak{p} \in \operatorname{Spec} R \mid \mathrm{e}_{HK}(\mathfrak{p}) < a\}$ is stable under generalization. Hence, it is enough to verify only the second condition of the criterion, and we restate Conjecture 4 in the following form.

Proposition 9. *Let R be a locally equidimensional ring. Then the Hilbert–Kunz multiplicity is upper semi-continuous on* $\operatorname{Spec} R$ *if and only if for any prime ideal $\mathfrak{p}$ and any $\varepsilon > 0$ there exists $s \notin \mathfrak{p}$ such that for all prime ideals $\mathfrak{q} \in D_s \cap V(\mathfrak{p})$,*

$$\mathrm{e}_{HK}(\mathfrak{q}) < \mathrm{e}_{HK}(\mathfrak{p}) + \varepsilon.$$

It is also easy to show that we can restrict ourselves to domains.

Proposition 10. *Let R be a locally equidimensional ring. If the Hilbert–Kunz multiplicity is upper semi-continuous in R/P for all minimal primes P of R, then the Hilbert–Kunz multiplicity is upper semi-continuous in R.*

Proof. Given ε, we want to find an element $s \notin \mathfrak{p}$ such that $\mathrm{e}_{HK}(\mathfrak{q}) < \mathrm{e}_{HK}(\mathfrak{p}) + \varepsilon$ for any ideal $\mathfrak{q}$ containing $\mathfrak{p}R_s$ of R_s.

Let $P_1, \ldots, P_n$ be the minimal primes of R. Inverting an element if necessary, we may assume that all P_i are contained in $\mathfrak{p}$. By the assumption, there exist elements $s_i \notin \mathfrak{p}$ such that in the corresponding subsets of $\operatorname{Spec} R/P_i$,

$$\mathrm{e}_{HK}(\mathfrak{q}R/P_i) < \mathrm{e}_{HK}(\mathfrak{p}R/P_i) + \varepsilon/(n\ell_{R_{P_i}}(R_{P_i})).$$

Now, if we invert the product s of s_i, we obtain that for any ideal $\mathfrak{q}$ of R_s that contains $\mathfrak{p}$, by the associativity formula for Hilbert–Kunz multiplicity [WY00, (2.3)],

$$\begin{aligned} \mathrm{e}_{HK}(\mathfrak{q}) &= \sum_{i=1}^{n} \mathrm{e}_{HK}(\mathfrak{q}R/P_i)\ell_{R_{P_i}}(R_{P_i}) \\ &< \sum_{i=1}^{n} \left(\mathrm{e}_{HK}(\mathfrak{p}R/\mathfrak{p}_i) + \frac{\varepsilon}{n\ell_{R_{P_i}}(R_{P_i})} \right) \ell_{R_{P_i}}(R_{P_i}) = \mathrm{e}_{HK}(\mathfrak{p}) + \varepsilon. \end{aligned}$$ ◻

Corollary 11. *Conjecture 4 holds if and only if for any excellent domain R, prime ideal $\mathfrak{p}$ of R, and $\varepsilon > 0$, there exists $s \notin \mathfrak{p}$ such that for all prime ideals $\mathfrak{q} \in V(\mathfrak{p}) \cap D_s$,*

$$\mathrm{e}_{HK}(\mathfrak{q}) < \mathrm{e}_{HK}(\mathfrak{p}) + \varepsilon.$$

Proof. We just note that a quotient of an excellent ring is excellent. ◻

A descent of semi-continuity over a faithfully flat extension would be extremely useful for the proof; in fact, it would eliminate the need for § 4. Unfortunately, we do not know how the Hilbert–Kunz multiplicity changes after an arbitrary faithfully flat extension well enough to control it for all localizations. So, we are able to obtain a descent statement only for extensions with regular fibers. The following lemma will be needed in the proof of our main result for algebras of essentially finite type over an excellent local ring.

LEMMA 12. *Let R be a ring and $f\colon R \to S$ be a faithfully flat R-algebra. Moreover, suppose that f has regular fibers. Then the Hilbert–Kunz multiplicity is upper semi-continuous in S if and only if it is upper semi-continuous in R.*

Proof. Let Q be any prime in S and let $\mathfrak{p} = Q \cap R$. Note that $R_\mathfrak{p} \to S_Q$ is faithfully flat with regular fibers, so, by a result of Kunz [Kun76, Proposition 3.9], $\mathrm{e}_{HK}(R_\mathfrak{p}) = \mathrm{e}_{HK}(S_Q)$. Thus, under our assumption the Hilbert–Kunz multiplicity is constant in fibers.

Suppose that upper semi-continuity holds in S. Let a be any real number and consider the closed set $V(I) = \{Q \mid Q \in \operatorname{Spec} S, \mathrm{e}_{HK}(Q) \geqslant a\}$. The argument above tells us that for any $Q \in V(I)$ any minimal prime of $(Q \cap R)S$ is also in $V(I)$. Hence $V(I) = V(JS)$ for $J = I \cap R$.

We claim that $V(J) = \{\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec} R,\ \mathrm{e}_{HK}(\mathfrak{p}) \geqslant a\}$. Note that $J \subseteq \mathfrak{p}$ if and only if $JS \subseteq Q$ for any prime Q in S that contracts to $\mathfrak{p}$, so $\mathrm{e}_{HK}(\mathfrak{p}) = \mathrm{e}_{HK}(Q) \geqslant a$.

For the other direction, note that $f^*\colon \operatorname{Spec} S \to \operatorname{Spec} R$ is surjective, so, since e_{HK} is constant in fibers, we obtain that

$$\{Q \mid Q \in \operatorname{Spec} S, \mathrm{e}_{HK}(Q) < a\} = (f^*)^{-1}\{\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec} R, \mathrm{e}_{HK}(\mathfrak{p}) < a\}.$$

Hence, it is open. □

3. Globally uniform Hilbert–Kunz estimates for F-finite rings

In this section we essentially rebuild Tucker's uniform Hilbert–Kunz estimates from [Tuc12] in order to control the rate of convergence of the Hilbert–Kunz function on an open subset.

We will use some properties of the Hilbert–Samuel multiplicity (see [HS06, Proposition 11.1.10, Theorem 11.2.4, Proposition 11.2.9]) for proofs.

PROPOSITION 13. *Let $(R, \mathfrak{m})$ be a local ring of dimension $d > 0$, $\underline{x}$ be a system of parameters, and I be an arbitrary $\mathfrak{m}$-primary ideal.*

(i) $\ell(R/\underline{x}) \geqslant \mathrm{e}(\underline{x}, R)$. *If $\underline{x}$ is a regular sequence, then equality holds.*
(ii) *(Associativity formula)* $\mathrm{e}(I, R) = \sum_\mathfrak{p} \mathrm{e}(I, R/\mathfrak{p})\ell_{R_\mathfrak{p}}(R_\mathfrak{p})$, *where the sum is taken over all primes $\mathfrak{p}$ such that* $\dim R/\mathfrak{p} = \dim R$.
(iii) *For any numbers $n_1, \dots, n_d$,* $\mathrm{e}((x_1^{n_1}, \dots, x_d^{n_d}), R) = n_1 \cdots n_d\, \mathrm{e}(\underline{x}, R)$.

We also record the following lemma.

LEMMA 14. *Let R be an excellent ring and $\mathfrak{p}$ be a prime ideal in R. Then there exists an element $f \notin \mathfrak{p}$ such that $R_f/\mathfrak{p}R_f$ is a regular ring and $R_f/\mathfrak{p}^n R_f$ are Cohen–Macaulay R_f-modules for all n.*

Proof. First, invert an element $s \notin \mathfrak{p}$ to make $(R/\mathfrak{p})_s$ regular; this is possible because $R/\mathfrak{p}$ is an excellent domain. Now consider the associated graded ring $\operatorname{gr}_\mathfrak{p}(R_s) = \bigoplus_n \mathfrak{p}^n R_s/\mathfrak{p}^{n+1} R_s$. This is a finitely generated $R_s/\mathfrak{p}R_s$-algebra, so by generic freeness [Mat80, 22.A] we can invert an element $t \notin \mathfrak{p}R_s$ and make it free over the regular ring $(R_s/\mathfrak{p}R_s)_t$.

Let $f = st$. Then it follows that $\mathfrak{p}^n R_f/\mathfrak{p}^{n+1} R_f$ are projective $R_f/\mathfrak{p}R_f$-modules for all n. Hence, by induction, using the sequences

$$0 \to \mathfrak{p}^n R_f/\mathfrak{p}^{n+1} R_f \to R_f/\mathfrak{p}^{n+1} R_f \to R_f/\mathfrak{p}^n R_f \to 0,$$

we get that all residue rings $R_f/\mathfrak{p}^n R_f$ are Cohen–Macaulay in this localization. □

https://doi.org/10.1112/S0010437X15007800 Published online by Cambridge University Press

Lemma 15. *Let R be an excellent ring of characteristic $p>0$ and $\mathfrak{p}$ a prime ideal of R. Let M be a finite R-module. There exist a constant C (independent of $\mathfrak{q}$) and an element $s \notin \mathfrak{p}$ such that for any prime ideal $\mathfrak{q} \in D_s \cap V(\mathfrak{p})$ and for all q,*

$$\ell_{R_\mathfrak{q}}(M_\mathfrak{q}/\mathfrak{q}^{[q]}M_\mathfrak{q}) \leqslant Cq^{\dim M_\mathfrak{q}}.$$

Proof. Assume that $M=R/P$ is a cyclic module for a prime ideal P. If $\mathfrak{p}$ does not contain P, we can invert any $s \in P\backslash\mathfrak{p}$, so $M_s=0$ and the assertion is trivially true. Hence, assume $P \subseteq \mathfrak{p}$.

Via Lemma 14 we can find an element $f \notin \mathfrak{p}$ such that, localizing at f, $R/\mathfrak{p} \cong S/\mathfrak{p}$ becomes regular and $S/\mathfrak{p}^nS$ becomes Cohen–Macaulay for all n. Let $\mathfrak{q}$ be an arbitrary prime ideal in $V(\mathfrak{p}) \cap D_f$. Since $R_\mathfrak{q}/\mathfrak{p}R_\mathfrak{q}$ is a regular local ring, there exists a system of parameters $\underline{x}$ that generates $\mathfrak{q}R_\mathfrak{q}$ modulo $\mathfrak{p}R_\mathfrak{q}$. Suppose that $\mathfrak{p}$ can be generated by t elements in R. Then $(\mathfrak{p}^{tq}, \underline{x}^{[q]})R_\mathfrak{q} \subseteq \mathfrak{q}^{[q]}R_\mathfrak{q}$, so

$$\ell_{R_\mathfrak{q}}(S_\mathfrak{q}/\mathfrak{q}^{[q]}S_\mathfrak{q}) \leqslant \ell_{R_\mathfrak{q}}(S_\mathfrak{q}/(\mathfrak{p}^{tq}, (\underline{x})^{[q]})S_\mathfrak{q}).$$

Since $(S/\mathfrak{p}^{tq}S)_\mathfrak{q}$ is Cohen–Macaulay for any q, we may use Proposition 13 to show that

$$\ell_{R_\mathfrak{q}}(S_\mathfrak{q}/(\mathfrak{p}^{tq}, (\underline{x})^{[q]})S_\mathfrak{q}) = \mathrm{e}((\underline{x})^{[q]}, S_\mathfrak{q}/\mathfrak{p}^{tq}S_\mathfrak{q}) = q^{\operatorname{ht}\mathfrak{q}/\mathfrak{p}}\,\mathrm{e}(\underline{x}, S_\mathfrak{q}/\mathfrak{p}^{tq}S_\mathfrak{q}).$$

Thus, from the associativity formula we obtain

$$\ell_{R_\mathfrak{q}}(S_\mathfrak{q}/(\mathfrak{p}^{tq}, (\underline{x})^{[q]})S_\mathfrak{q}) = q^{\operatorname{ht}\mathfrak{q}/\mathfrak{p}}\,\mathrm{e}(\underline{x}, S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q})\ell_{R_\mathfrak{p}}(S_\mathfrak{p}/\mathfrak{p}^{tq}S_\mathfrak{p}) = q^{\operatorname{ht}\mathfrak{q}/\mathfrak{p}}\ell_{R_\mathfrak{p}}(S_\mathfrak{p}/\mathfrak{p}^{tq}S_\mathfrak{p}).$$

Note that $\mathrm{e}(\underline{x}, S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}) = 1$ because $\underline{x}$ generates the maximal ideal of a regular local ring $S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}$.

To finish the argument, we observe that $\ell_{R_\mathfrak{p}}(S_\mathfrak{p}/\mathfrak{p}^nS_\mathfrak{p}) = \ell_{R_\mathfrak{p}}(R_\mathfrak{p}/(P+\mathfrak{p}^n)R_\mathfrak{p})$ is a polynomial in n of degree $\operatorname{ht}\mathfrak{p}/P$ for all sufficiently large n, so clearly there exists a constant D such that for all q,

$$\ell_{R_\mathfrak{q}}(S_\mathfrak{p}/\mathfrak{p}^{tq}S_\mathfrak{p}) = \ell_{R_\mathfrak{p}}(R_\mathfrak{p}/(P+\mathfrak{p}^{tq})R_\mathfrak{p}) \leqslant D(tq)^{\operatorname{ht}\mathfrak{p}/P}.$$

Thus, we have just obtained a bound

$$\ell_{R_\mathfrak{q}}(S_\mathfrak{q}/\mathfrak{q}^{[q]}S_\mathfrak{q}) \leqslant \ell_{R_\mathfrak{q}}(S_\mathfrak{q}/(\mathfrak{p}^{tq}, (\underline{x})^{[q]})S_\mathfrak{q}) \leqslant q^{\operatorname{ht}\mathfrak{q}/\mathfrak{p}}D(tq)^{\operatorname{ht}\mathfrak{p}/P} = (Dt^{\operatorname{ht}\mathfrak{p}/P})q^{\operatorname{ht}\mathfrak{q}/P} = Cq^{\dim S_\mathfrak{q}}.$$

Hence, if $M=R/P$, the statement has been proved for $C = Dt^{\operatorname{ht}\mathfrak{p}/P}$, a constant independent of $\mathfrak{q}$.

By choosing a prime filtration of M, we can deduce the general statement from the case considered above. Namely, if P_i are prime ideals appearing in the prime filtration, then

$$\ell_{R_\mathfrak{q}}(M_\mathfrak{q}/\mathfrak{q}^{[q]}M_\mathfrak{q}) \leqslant \sum_i \ell_{R_\mathfrak{q}}((R/P_i)_\mathfrak{q}/\mathfrak{q}^{[q]}(R/P_i)_\mathfrak{q}).$$

Since there are finitely many primes P_i, we can invert finitely many elements in order to force the claim for all R/P_i. Also, note that $\dim M_\mathfrak{q}$ is the maximum of $\dim R_\mathfrak{q}/P_iR_\mathfrak{q}$ over the primes in a prime filtration. So

$$\ell_{R_\mathfrak{q}}(M_\mathfrak{q}/\mathfrak{q}^{[q]}M_\mathfrak{q}) \leqslant \sum_i \ell_{R_\mathfrak{q}}(R_\mathfrak{q}/(P_i+\mathfrak{q}^{[q]})R_\mathfrak{q}) \leqslant \sum_i C_iq^{\operatorname{ht}\mathfrak{q}/P_i} \leqslant \left(\sum_i C_i\right)q^{\dim M_\mathfrak{q}}. \qquad \square$$

Using a standard argument ([Tuc12, Lemma 3.3] or [Mon83, Lemma 1.3]), we derive from Lemma 15 the following result.

Corollary 16. *Let R be an excellent ring of characteristic $p>0$ and $\mathfrak{p}$ be a prime ideal of R. Suppose that M and N are finite R-modules such that their localizations at every minimal prime of R are isomorphic. Then there exist a constant C and an element $s \notin \mathfrak{p}$ such that for*

https://doi.org/10.1112/S0010437X15007800 Published online by Cambridge University Press

any prime ideal $\mathfrak{q} \in D_s \cap V(\mathfrak{p})$ *and for all* q,

$$|\ell_{R_\mathfrak{q}}(M_\mathfrak{q}/\mathfrak{q}^{[q]}M_\mathfrak{q}) - \ell_{R_\mathfrak{q}}(N_\mathfrak{q}/\mathfrak{q}^{[q]}N_\mathfrak{q})| \leqslant Cq^{\operatorname{ht}\mathfrak{q}-1}.$$

Proof. If $\operatorname{ht}\mathfrak{q} = 0$, the claim is trivial, so we may assume that $\operatorname{ht}\mathfrak{q} > 0$. By the assumptions, we have an exact sequence

$$N \to M \to K \to 0,$$

where $K_P = 0$ for every minimal prime P of R. By Lemma 15, there is an element s_1 such that for some constant C_1 and all $\mathfrak{q} \in D_{s_1} \cap V(\mathfrak{p})$,

$$\ell_{R_\mathfrak{q}}(M_\mathfrak{q}/\mathfrak{q}^{[q]}M_\mathfrak{q}) - \ell_{R_\mathfrak{q}}(N_\mathfrak{q}/\mathfrak{q}^{[q]}N_\mathfrak{q}) \leqslant \ell_{R_\mathfrak{q}}(K_\mathfrak{q}/\mathfrak{q}^{[q]}K_\mathfrak{q}) \leqslant C_1 q^{\dim K_\mathfrak{q}}.$$

Since $K_P = 0$ for any minimal prime P, $\dim K_\mathfrak{q} \leqslant \operatorname{ht}\mathfrak{q} - 1$.

To finish the proof, we switch M and N in the first part of the argument, i.e. apply it to the sequence

$$M \to N \to L \to 0.$$

Hence, by inverting an element s_2, we will get

$$\ell_{R_\mathfrak{q}}(N_\mathfrak{q}/\mathfrak{q}^{[q]}M_\mathfrak{q}) - \ell_{R_\mathfrak{q}}(M_\mathfrak{q}/\mathfrak{q}^{[q]}M_\mathfrak{q}) \leqslant \ell_{R_\mathfrak{q}}(L_\mathfrak{q}/\mathfrak{q}^{[q]}L_\mathfrak{q}) \leqslant C_2 q^{\dim L_\mathfrak{q}} \leqslant C_2 q^{\operatorname{ht}\mathfrak{q}-1},$$

and the claim follows for $C = \max(C_1, C_2)$ and $s = s_1 s_2$. □

Definition 17. Let R be a ring of characteristic $p > 0$. For a prime ideal $\mathfrak{p}$ of R, we denote $\alpha(\mathfrak{p}) = \log_p[k(\mathfrak{p}) : k(\mathfrak{p})^p]$, where $k(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ is the residue field of $\mathfrak{p}$.

This number measures the change of length via Frobenius: if M is a finite-length module over a local ring $(R, \mathfrak{m})$, then $\ell(F_* M) = p^{\alpha(\mathfrak{m})}\ell(M)$. In particular,

$$\ell(F_* R \otimes_R R/\mathfrak{m}^{[q]}) = p^{\alpha(\mathfrak{m})}\ell(R/\mathfrak{m}^{[qp]}).$$

We will need the following result of Kunz [Kun76, 2.3].

Proposition 18. *Let R be F-finite and let $\mathfrak{p} \subseteq \mathfrak{q}$ be prime ideals. Then $\alpha(\mathfrak{p}) = \alpha(\mathfrak{q}) + \operatorname{ht}\mathfrak{q}/\mathfrak{p}$.*

Theorem 19. *Let R be an F-finite domain and let $\mathfrak{p}$ be an arbitrary prime ideal. Then there exists an element $s \notin \mathfrak{p}$ such that for any $\varepsilon > 0$ there is q_0 such that for all $q > q_0$,*

$$|\ell_{R_\mathfrak{q}}(R_\mathfrak{q}/\mathfrak{q}^{[q]}R_\mathfrak{q})/q^{\operatorname{ht}\mathfrak{q}} - \mathrm{e}_{HK}(\mathfrak{q})| < \varepsilon$$

for all prime ideals $\mathfrak{q} \in D_s \cap V(\mathfrak{p})$.

Proof. Since R is F-finite, $R^{\oplus p^{\alpha(0)}}$ and $R^{1/p}$ are isomorphic when localized at the minimal prime 0. So, by Corollary 16, we can invert an element and obtain a global bound

$$|\ell_{R_\mathfrak{q}}(R_\mathfrak{q}^{\oplus p^{\alpha(0)}}/\mathfrak{q}^{[q]}R_\mathfrak{q}^{\oplus p^{\alpha(0)}}) - \ell_{R_\mathfrak{q}}(R_\mathfrak{q}^{1/p}/\mathfrak{q}^{[q]}R_\mathfrak{q}^{1/p})| < Cq^{\operatorname{ht}\mathfrak{q}-1}$$

for an arbitrary prime ideal $\mathfrak{q}$ containing $\mathfrak{p}$.

We follow Tucker's argument from [Tuc12]. Proposition 18 applied to the formula above gives

$$|p^{\operatorname{ht}\mathfrak{q}+\alpha(\mathfrak{q})}\ell_{R_\mathfrak{q}}(R_\mathfrak{q}/\mathfrak{q}^{[q]}R_\mathfrak{q}) - p^{\alpha(\mathfrak{q})}\ell_{R_\mathfrak{q}}(R_\mathfrak{q}/\mathfrak{q}^{[qp]}R_\mathfrak{q})| < Cq^{\operatorname{ht}\mathfrak{q}-1}, \text{so}$$

$$|p^{\operatorname{ht}\mathfrak{q}}\ell_{R_\mathfrak{q}}(R_\mathfrak{q}/\mathfrak{q}^{[q]}R_\mathfrak{q}) - \ell_{R_\mathfrak{q}}(R_\mathfrak{q}/\mathfrak{q}^{[qp]}R_\mathfrak{q})| < p^{-\alpha(\mathfrak{q})}Cq^{\operatorname{ht}\mathfrak{q}-1} \leqslant Cq^{\operatorname{ht}\mathfrak{q}-1}. \tag{1}$$

https://doi.org/10.1112/S0010437X15007800 Published online by Cambridge University Press

We will prove by induction on q' that

$$|(q')^{\operatorname{ht}\mathfrak{q}}\ell_{R_\mathfrak{q}}(R_\mathfrak{q}/\mathfrak{q}^{[q]}R_\mathfrak{q}) - \ell_{R_\mathfrak{q}}(R_\mathfrak{q}/\mathfrak{q}^{[qq']}R_\mathfrak{q})| < C(qq'/p)^{\operatorname{ht}\mathfrak{q}-1}\frac{q'-1}{p-1}. \tag{2}$$

The induction base of $q' = p$ is (1). Now, assume that the claim holds for q' and we want to prove it for $q'p$.

First, (1) applied to qq' gives

$$|p^{\operatorname{ht}\mathfrak{q}}\ell_{R_\mathfrak{q}}(R_\mathfrak{q}/\mathfrak{q}^{[qq']}R_\mathfrak{q}) - \ell_{R_\mathfrak{q}}(R_\mathfrak{q}/\mathfrak{q}^{[qq'p]}R_\mathfrak{q})| < C(qq')^{\operatorname{ht}\mathfrak{q}-1}, \tag{3}$$

and, multiplying the induction hypothesis by $p^{\operatorname{ht}\mathfrak{q}}$, we get

$$|(q'p)^{\operatorname{ht}\mathfrak{q}}\ell_{R_\mathfrak{q}}(R_\mathfrak{q}/\mathfrak{q}^{[q]}R_\mathfrak{q}) - p^{\operatorname{ht}\mathfrak{q}}\ell_{R_\mathfrak{q}}(R_\mathfrak{q}/\mathfrak{q}^{[qq']}R_\mathfrak{q})| < C(qq')^{\operatorname{ht}\mathfrak{q}-1}\frac{q'p-p}{p-1}. \tag{4}$$

Combining (3) and (4) results in

$$|(q'p)^{\operatorname{ht}\mathfrak{q}}\ell_{R_\mathfrak{q}}(R_\mathfrak{q}/\mathfrak{q}^{[q]}R_\mathfrak{q}) - \ell_{R_\mathfrak{q}}(R_\mathfrak{q}/\mathfrak{q}^{[qq'p]}R_\mathfrak{q})| < C(qq')^{\operatorname{ht}\mathfrak{q}-1}\left(\frac{q'p-p}{p-1}+1\right),$$

and the induction step follows.

Now, dividing (2) by $q'^{\operatorname{ht}\mathfrak{q}}$, we obtain

$$|\ell_{R_\mathfrak{q}}(R_\mathfrak{q}/\mathfrak{q}^{[q]}R_\mathfrak{q}) - \frac{1}{q'^{\operatorname{ht}\mathfrak{q}}}\ell_{R_\mathfrak{q}}(R_\mathfrak{q}/\mathfrak{q}^{[qq']}R_\mathfrak{q})| < Cq^{\operatorname{ht}\mathfrak{q}-1}\cdot\frac{q'-1}{p-1}\cdot\frac{1}{q'p^{\operatorname{ht}\mathfrak{q}-1}} \leqslant Cq^{\operatorname{ht}\mathfrak{q}-1}.$$

Thus, letting $q' \to \infty$, we get that

$$|\ell_{R_\mathfrak{q}}(R_\mathfrak{q}/\mathfrak{q}^{[q]}R_\mathfrak{q}) - q^{\operatorname{ht}\mathfrak{q}}\,\mathrm{e}_{HK}(\mathfrak{q})| < Cq^{\operatorname{ht}\mathfrak{q}-1},$$

and the claim follows. $\square$

4. Uniform estimates for a flat extension

In this section we prove convergence estimates of Theorem 19 for algebras of essentially finite type over a complete domain. We will use the existence of a faithfully flat F-finite extension and will relativize the estimates of the previous section for use in the extension.

Lemma 20. *Let R be a locally equidimensional excellent ring and S be an R-algebra. Let I be an ideal in R, let M be a finitely generated S-module such that* $\operatorname{Supp} M \subseteq V(I)$, *and let $\mathfrak{p}$ be a prime ideal in R. Then there exist an element $s \notin \mathfrak{p}$ and a constant C such that for any prime ideal $\mathfrak{q} \in V(\mathfrak{p}) \cap D(s)$ and for any prime ideal Q in S minimal over $\mathfrak{q}S$,*

$$\ell_{S_Q}(M_Q/\mathfrak{q}^{[q]}M_Q) \leqslant Cq^{\operatorname{ht}\mathfrak{q}-\operatorname{ht} I}\ell_{S_Q}(S_Q/\mathfrak{q}S_Q).$$

Proof. If I is not contained in $\mathfrak{p}$ we can invert an element and make M zero, so we may assume that $I \subseteq \mathfrak{p}$.

By Lemma 14 we can invert an element $s \notin \mathfrak{p}$ to make $R/\mathfrak{p}$ regular and $R/(\mathfrak{p}^n + I)$ be Cohen–Macaulay for all n. We claim that the required bound holds for this s.

As in the proof of Lemma 15, by taking a prime filtration of M we reduce the statement to $M = S/J$, where J is a prime ideal in S that contains IS. So

$$\ell_{S_Q}(S_Q/(\mathfrak{q}^{[q]}S + J)S_Q) \leqslant \ell_{S_Q}(S_Q/(\mathfrak{q}^{[q]} + I)S_Q).$$

After tensoring a prime filtration of $R_\mathfrak{q}/(\mathfrak{q}^{[q]} + I)R_\mathfrak{q}$ with S_Q, we estimate that

$$\ell_{S_Q}(S_Q/(\mathfrak{q}^{[q]} + I)S_Q) \leqslant \ell_{R_\mathfrak{q}}(R_\mathfrak{q}/(\mathfrak{q}^{[q]} + I)R_\mathfrak{q})\ell_{S_Q}(S_Q/\mathfrak{q}S_Q).$$

https://doi.org/10.1112/S0010437X15007800 Published online by Cambridge University Press

Since $R/\mathfrak{p}$ is regular, we can write $\mathfrak{q}R_\mathfrak{q} = (\mathfrak{p} + (\underline{x}))R_\mathfrak{q}$, where $\underline{x}$ are minimal generators of $\mathfrak{q}/\mathfrak{p}$. Suppose that $\mathfrak{p}$ can be generated by t elements in R. Then $\mathfrak{p}^{tq} \subseteq \mathfrak{p}^{[q]}$ and

$$\ell_{R_\mathfrak{q}}(R_\mathfrak{q}/(\mathfrak{q}^{[q]} + I)R_\mathfrak{q}) = \ell_{R_\mathfrak{q}}(R_\mathfrak{q}/(\mathfrak{p}^{[q]} + (\underline{x})^{[q]} + I)R_\mathfrak{q}) \leqslant \ell_{R_\mathfrak{q}}(R_\mathfrak{q}/(\mathfrak{p}^{tq} + (\underline{x})^{[q]} + I)R_\mathfrak{q}).$$

Now, since $R/(\mathfrak{q}^{tq} + I)$ are Cohen–Macaulay, by Proposition 13,

$$\ell_{R_\mathfrak{q}}(R_\mathfrak{q}/(\mathfrak{p}^{tq} + I + (\underline{x})^{[q]})R_\mathfrak{q}) = \mathrm{e}((\underline{x})^{[q]}, R_\mathfrak{q}/(\mathfrak{p}^{tq} + I)R_\mathfrak{q}).$$

Moreover, by the associativity formula,

$$\mathrm{e}((\underline{x})^{[q]}, R_\mathfrak{q}/(\mathfrak{p}^{tq} + I)R_\mathfrak{q}) = \mathrm{e}((\underline{x})^{[q]}, R_\mathfrak{q}/\mathfrak{p}R_\mathfrak{q})\ell_{R_\mathfrak{p}}(R_\mathfrak{p}/(\mathfrak{p}^{tq} + I)R_\mathfrak{p}),$$

and, using again that $R/\mathfrak{p}$ is regular,

$$\mathrm{e}((\underline{x})^{[q]}, R_\mathfrak{q}/\mathfrak{p}R_\mathfrak{q})\ell_{R_\mathfrak{p}}(R_\mathfrak{p}/(\mathfrak{p}^{tq} + I)R_\mathfrak{p}) = q^{\operatorname{ht}\mathfrak{q}/\mathfrak{p}}\ell_{R_\mathfrak{p}}(R_\mathfrak{p}/(\mathfrak{p}^{tq} + I)R_\mathfrak{p}).$$

The Hilbert–Samuel polynomial of $R_\mathfrak{p}/IR_\mathfrak{p}$ has degree $\dim R_\mathfrak{p}/I = \operatorname{ht}\mathfrak{p} - \operatorname{ht} I$. Hence we can find a constant D such that

$$\ell_{R_\mathfrak{p}}(R_\mathfrak{p}/(\mathfrak{p}^{tq} + I)R_\mathfrak{p}) \leqslant D(tq)^{\operatorname{ht}\mathfrak{p} - \operatorname{ht} I} = Cq^{\operatorname{ht}\mathfrak{p} - \operatorname{ht} I}$$

and the claim follows. □

We will need the following lemma obtained via the gamma construction. It is a step in the proof of [HH94, Lemma 6.13], and a more detailed exposition can be found in Hochster's notes [Hoc07, Theorem, p. 139].

Lemma 21. *Let B be a complete local domain and S be a B-algebra of essentially finite type. Suppose that S is a domain. Then there exists a purely inseparable faithfully flat F-finite B-algebra B^Γ such that $S \otimes_B B^\Gamma$ is a domain.*

Theorem 22. *Let B be a complete local domain and R be a domain of essentially finite type over B. If $\mathfrak{p}$ be an arbitrary prime ideal in R, then there exists an element $s \notin \mathfrak{p}$ such that for any $\varepsilon > 0$ there is q_0 such that for all $q > q_0$,*

$$|\ell_{R_\mathfrak{q}}(R_\mathfrak{q}/\mathfrak{q}^{[q]}R_\mathfrak{q})/q^{\operatorname{ht}\mathfrak{q}} - \mathrm{e}_{HK}(\mathfrak{q})| < \varepsilon$$

for all prime ideals $\mathfrak{q} \in D_s \cap V(\mathfrak{p})$.

Proof. We apply Lemma 21 to the quotient field L of R and obtain a B-algebra B^Γ. Note that $S = R \otimes_B B^\Gamma$ is F-finite, so $S^{1/p}$ is a finitely generated S-module.

By the choice of B^Γ, $S \otimes_R L \cong B^\Gamma \otimes_B R \otimes_R L \cong B^\Gamma \otimes_B L$ is a domain. Since B^Γ is purely inseparable over B, $B^\Gamma \otimes_B L$ is integral over a field L, so it is a field. Since taking p-roots commutes with localization, $(S \otimes_R L)^{1/p} \cong (S)^{1/p} \otimes_R L$ is a finite free module over the field $S \otimes_R L \cong B^\Gamma \otimes_B L$. Hence, we can invert an element $f \in R$ to make $S_f^{1/p}$ a free module over S_f. Finally, R is a subring of S and $S \otimes_R L$ is a field, so $S \otimes_R L$ is the quotient field of S and, by definition, the rank of the free module $(S \otimes_R L)^{1/p}$ is $p^{\alpha(0)}$.

Therefore there exist maps

$$0 \to S^{1/p} \to S^{\oplus p^{\alpha(0)}} \to M \to 0$$

and

$$0 \to S^{\oplus p^{\alpha(0)}} \to S^{1/p} \to N \to 0$$

such that $\operatorname{Supp} M, \operatorname{Supp} N \subseteq V(fS)$.

https://doi.org/10.1112/S0010437X15007800 Published online by Cambridge University Press

Thus, after applying Lemma 20 to M and N, we can invert an element $s \notin \mathfrak{p}$ and obtain that for any prime $\mathfrak{q} \supseteq \mathfrak{p}$ and for any minimal prime Q of $\mathfrak{q}S$,

$$\ell_{S_Q}(S_Q^{\oplus p^{\alpha(0)}}/\mathfrak{q}^{[q]}S_Q^{\oplus p^{\alpha(0)}}) - \ell_{S_Q}(S_Q^{1/p}/\mathfrak{q}^{[q]}S_Q^{1/p}) \leqslant \ell_{S_Q}(M_Q/\mathfrak{q}^{[q]}M_Q) \leqslant C_1 q^{\operatorname{ht}\mathfrak{q} - \operatorname{ht}(f)}\ell_{S_Q}(S_Q/\mathfrak{q}S_Q),$$
$$\ell_{S_Q}(S_Q^{1/p}/\mathfrak{q}^{[q]}S_Q^{1/p}) - \ell_{S_Q}(S_Q^{\oplus p^{\alpha(0)}}/\mathfrak{q}^{[q]}S_Q^{\oplus p^{\alpha(0)}}) \leqslant \ell_{S_Q}(N_Q/\mathfrak{q}^{[q]}N_Q) \leqslant C_2 q^{\operatorname{ht}\mathfrak{q} - \operatorname{ht}(f)}\ell_{S_Q}(S_Q/\mathfrak{q}S_Q).$$

Taking $C = \max(C_1, C_2)$ and noting that $\operatorname{ht}(f) = 1$, we estimate that

$$|\ell_{S_Q}(S_Q^{\oplus p^{\alpha(0)}}/\mathfrak{q}^{[q]}S_Q^{\oplus p^{\alpha(0)}}) - \ell_{S_Q}(S_Q^{1/p}/\mathfrak{q}^{[q]}S_Q^{1/p})| < Cq^{\operatorname{ht}\mathfrak{q}-1}\ell_{S_Q}(S_Q/\mathfrak{q}S_Q).$$

So, since $\alpha(0) = \operatorname{ht} Q + \alpha(Q)$ by Proposition 18,

$$|p^{\operatorname{ht} Q + \alpha(Q)}\ell_{S_Q}(S_Q/\mathfrak{q}^{[q]}S_Q) - p^{\alpha(Q)}\ell_{S_Q}(S_Q/\mathfrak{q}^{[qp]}S_Q)| < Cq^{\operatorname{ht}\mathfrak{q}-1}\ell_{S_Q}(S_Q/\mathfrak{q}S_Q).$$

Because S_Q is flat over $R_\mathfrak{q}$ and $\mathfrak{q}S_Q$ is Q-primary, for any artinian $R_\mathfrak{q}$-module M,

$$\ell_{S_Q}(M \otimes_{R_\mathfrak{q}} S_Q) = \ell_{R_\mathfrak{q}}(M)\ell_{S_Q}(S_Q/\mathfrak{q}S_Q).$$

Therefore, the estimate above can be rewritten as

$$|p^{\operatorname{ht} Q + \alpha(Q)}\ell_{R_\mathfrak{q}}(R_\mathfrak{q}/\mathfrak{q}^{[q]}R_\mathfrak{q}) - p^{\alpha(Q)}\ell_{R_\mathfrak{q}}(R_\mathfrak{q}/\mathfrak{q}^{[qp]}R_\mathfrak{q})| < Cq^{\operatorname{ht}\mathfrak{q}-1}.$$

Since S is flat, $\operatorname{ht} Q = \operatorname{ht}\mathfrak{q}$, so we obtain (1) from Theorem 19,

$$|p^{\operatorname{ht}\mathfrak{q}}\ell_{R_\mathfrak{q}}(R_\mathfrak{q}/\mathfrak{q}^{[q]}R_\mathfrak{q}) - \ell_{R_\mathfrak{q}}(R_\mathfrak{q}/\mathfrak{q}^{[qp]}R_\mathfrak{q})| < Cp^{-\alpha(Q)}q^{\operatorname{ht}\mathfrak{q}-1} \leqslant Cq^{\operatorname{ht}\mathfrak{q}-1},$$

and the rest of the proof follows the argument in Theorem 19. □

5. Proof of the main result and concluding remarks

We can now finish the proof of upper semi-continuity of the Hilbert–Kunz multiplicity for F-finite rings and algebras of essentially finite type over an excellent local ring. To do this, we will verify the second equivalent condition of Proposition 9.

Theorem 23. *Let R be a locally equidimensional ring. Suppose that R either is F-finite or is an algebra of essentially finite type over an excellent local ring B. If $\mathfrak{p}$ be a prime ideal of R, then for any $\varepsilon > 0$ there exists $s \notin \mathfrak{p}$ such that for all prime ideals $\mathfrak{q} \in D_s \cap V(\mathfrak{p})$,*

$$\mathrm{e}_{HK}(\mathfrak{q}) < \mathrm{e}_{HK}(\mathfrak{p}) + \varepsilon.$$

Proof. If R is not F-finite, consider the extension $R \to R \otimes_B \widehat{B}$. Since B is excellent, the natural map $B \to \widehat{B}$ is regular. So, by [Mat80, Lemma 4, p. 253], $R \to R \otimes_B \widehat{B}$ satisfies the conditions of Lemma 12. Hence, by Proposition 9 and Lemma 12, we assume that B is complete.

Note that the considered classes of rings are stable under taking quotients. So, by Proposition 10 we can assume that R is a domain.

By Theorems 19 and 22 there exist an element $s \notin \mathfrak{p}$ and a fixed power $q_0 = p^{e_0}$ such that for all $\mathfrak{q} \in D_s \cap V(\mathfrak{p})$,

$$|\ell_{R_\mathfrak{q}}(R_\mathfrak{q}/\mathfrak{q}^{[q_0]}R_\mathfrak{q})/q_0^{\operatorname{ht}\mathfrak{q}} - \mathrm{e}_{HK}(\mathfrak{q})| < \varepsilon/2.$$

In particular,

$$|\ell_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}^{[q_0]}R_\mathfrak{p})/q_0^{\operatorname{ht}\mathfrak{p}} - \mathrm{e}_{HK}(\mathfrak{p})| < \varepsilon/2.$$

We can now use Proposition 6 and obtain a nonempty open subset $\mathfrak{p} \in U \subseteq V(\mathfrak{p})$ such that for any $\mathfrak{q} \in U$,

$$\ell_{R_\mathfrak{q}}(R_\mathfrak{q}/\mathfrak{q}^{[q_0]}R_\mathfrak{q})/q_0^{\operatorname{ht}\mathfrak{q}} = f_{q_0}(\mathfrak{q}) = f_{q_0}(\mathfrak{p}) = \ell_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}^{[q_0]}R_\mathfrak{p})/q_0^{\operatorname{ht}\mathfrak{p}}.$$

For completeness we give a construction of such U below. The argument is inspired by the proof of [HY02, Theorem 3.3].

Since $R/\mathfrak{p}^{[q_0]}$ is excellent, its Cohen–Macaulay locus is open [EGAIV, 7.8.3(iv)]. Thus we can find an open subset $\mathfrak{p} \in U \subseteq V(\mathfrak{p})$ such that $(R/\mathfrak{p}^{[q_0]})_\mathfrak{q}$ is Cohen–Macaulay and $(R/\mathfrak{p})_\mathfrak{q}$ is regular for all $\mathfrak{q} \in U$.

Let $\mathfrak{q}$ be an arbitrary prime in U. Since $R_\mathfrak{q}/\mathfrak{p}R_\mathfrak{q}$ is regular, $\mathfrak{q}R_\mathfrak{q}$ is generated by a regular sequence $\underline{x}$ modulo $\mathfrak{p}R_\mathfrak{q}$. Then, by the associativity formula,

$$\ell_{R_\mathfrak{q}}(R_\mathfrak{q}/\mathfrak{q}^{[q_0]}R_\mathfrak{q}) = \ell_{R_\mathfrak{q}}(R_\mathfrak{q}/(\mathfrak{p}^{[q_0]}, (\underline{x})^{[q_0]})R_\mathfrak{q}) = \mathrm{e}((\underline{x})^{[q_0]}, R_\mathfrak{q}/\mathfrak{p}^{[q_0]}) = q_0^{\operatorname{ht}\mathfrak{q}/\mathfrak{p}}\ell_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}^{[q_0]}R_\mathfrak{p}).$$

Thus, we obtain that on $U \cap D_s$, $\ell_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}^{[q_0]}R_\mathfrak{p})/q_0^{\operatorname{ht}\mathfrak{p}}$ is within $\varepsilon/2$ of both $\mathrm{e}_{HK}(\mathfrak{p})$ and $\mathrm{e}_{HK}(\mathfrak{q})$, and the statement follows. □

Corollary 24. *Let R be a locally equidimensional ring. Moreover, suppose that R either is F-finite or is an algebra of essentially finite type over an excellent local ring B. Then the Hilbert–Kunz multiplicity is upper semi-continuous on* $\operatorname{Spec} R$.

We note the following corollary of semi-continuity.

Corollary 25. *Let R be a Noetherian ring and suppose the Hilbert–Kunz multiplicity is upper semi-continuous on $\operatorname{Spec} R$. Then the Hilbert–Kunz multiplicity satisfies the ascending chain condition on $\operatorname{Spec} R$, i.e. any increasing sequence $e_1 = \mathrm{e}_{HK}(\mathfrak{p}_1) \leqslant e_2 = \mathrm{e}_{HK}(\mathfrak{p}_2) \leqslant \cdots$ stabilizes. In particular, the Hilbert–Kunz multiplicity attains its maximum on $\operatorname{Spec} R$.*

Proof. Since e_{HK} is upper semi-continuous, $U_i = \{\mathfrak{p} \mid \mathrm{e}_{HK}(\mathfrak{p}) < e_i\}$ form an increasing sequence of open sets, so it stabilizes. □

Remark 26. In [Tuc12], Tucker asked if F-signature is lower semi-continuous in F-finite rings. One might hope that the ideas of this paper are extendable to F-signature, but, at the present moment, we know nothing about the convergence rate of the F-signature of a local ring.

In fact, one could even ask if the splitting numbers can be written as

$$a_e = r_F q^h + O(q^{h-1}),$$

where r_F is the F-splitting ratio, $h = \alpha(R) + \dim(R/P)$, and P is the splitting prime of R; see [Tuc12] for more details.

Remark 27. We conclude by discussing the further difficulties that prevent us from proving Conjecture 4 in full generality, for an arbitrary excellent ring.

The problem stems from the known proof of existence of the Hilbert–Kunz multiplicity; both the original paper [Mon83] and its refinement [Tuc12] prove the existence of the limit for a local ring by reducing to a faithfully flat F-finite extension obtained by extending the residue field. Thus, there is not much connection between these objects for different localizations, so the results and methods of the present paper cannot be applied.

Furthermore, it is not enough to have a global faithfully flat F-finite extension; we needed to use the gamma construction in order to have an extension with suitable properties.

https://doi.org/10.1112/S0010437X15007800 Published online by Cambridge University Press

Acknowledgements
The author would like to thank Craig Huneke for suggesting the problem and continuous support, Kevin Tucker for suggestions and discussion of the results, and Ian Aberbach for comments.

References

AE08 I. M. Aberbach and F. Enescu, *Lower bounds for Hilbert–Kunz multiplicities in local rings of fixed dimension*, Michigan Math. J. **57** (2008), 1–16; Special volume in honor of Melvin Hochster; MR 2492437 (2010h:13028).

BE04 M. Blickle and F. Enescu, *On rings with small Hilbert–Kunz multiplicity*, Proc. Amer. Math. Soc. **132** (2004), 2505–2509 (electronic); MR 2054773 (2005b:13029).

BM10 H. Brenner and P. Monsky, *Tight closure does not commute with localization*, Ann. of Math. (2) **171** (2010), 571–588; MR 2630050 (2011d:13005).

ES05 F. Enescu and K. Shimomoto, *On the upper semi-continuity of the Hilbert–Kunz multiplicity*, J. Algebra **285** (2005), 222–237; MR 2119113 (2005j:13017).

EGAIV A. Grothendieck, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. II*, Publ. Math. Inst. Hautes Études Sci. **24** (1965); MR 0199181 (33 #7330).

Hoc07 M. Hochster, *Foundations of tight closure*, Lecture Notes for Math 711, Fall 2007, University of Michigan (2007); available at http://www.math.lsa.umich.edu/~hochster/711F07/711.html.

HH94 M. Hochster and C. Huneke, *F-regularity, test elements, and smooth base change*, Trans. Amer. Math. Soc. **346** (1994), 1–62; MR 1273534 (95d:13007).

HS06 C. Huneke and I. Swanson, *Integral closure of ideals, rings, and modules*, London Mathematical Society Lecture Note Series, vol. 336 (Cambridge University Press, Cambridge, 2006); MR 2266432 (2008m:13013).

HY02 C. Huneke and Y. Yao, *Unmixed local rings with minimal Hilbert–Kunz multiplicity are regular*, Proc. Amer. Math. Soc. **130** (2002), 661–665.

Kun69 E. Kunz, *Characterizations of regular local rings of characteristic p*, Amer. J. Math. **91** (1969), 772–784; MR 0252389 (40 #5609).

Kun76 E. Kunz, *On Noetherian rings of characteristic p*, Amer. J. Math. **98** (1976), 999–1013; MR 0432625 (55 #5612).

Mat80 H. Matsumura, *Commutative algebra*, Mathematics Lecture Note Series, vol. 56, second edition (Benjamin/Cummings Publishing, Reading, MA, 1980); MR 575344 (82i:13003).

Mon83 P. Monsky, *The Hilbert–Kunz function*, Math. Ann. **263** (1983), 43–49; MR 697329 (84k:13012).

Mon98 P. Monsky, *Hilbert–Kunz functions in a family: point-S_4 quartics*, J. Algebra **208** (1998), 343–358; MR 1644019 (99k:13005).

She79 N. I. Shepherd-Barron, *On a problem of Ernst Kunz concerning certain characteristic functions of local rings*, Arch. Math. (Basel) **31** (1978/79), 562–564; MR 531569 (81e:13012).

Tuc12 K. Tucker, *F-signature exists*, Invent. Math. **190** (2012), 743–765; MR 2995185.

WY00 K. Watanabe and K. Yoshida, *Hilbert–Kunz multiplicity and an inequality between multiplicity and colength*, J. Algebra **230** (2000), 295–317; MR 1774769 (2001h:13032).

Ilya Smirnov ismirnov@umich.edu
Department of Mathematics, University of Virginia, Charlottesville, VA 22904-4137, USA

Current address: Department of Mathematics, University of Michigan, Ann Arbor, MI 48109-1043, USA