ON THE REPRESENTATION OF INTEGERS AS SUMS OF DISTINCT TERMS FROM A FIXED SEQUENCE

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1. Introduction. Let $A = (a_1, a_2, a_3, \ldots)$ be a sequence of positive integers. We let

$$P(A) = \left\{ \sum_{n=1}^{\infty} \epsilon_n a_n \,\middle|\, \epsilon_n = 0 \text{ or } 1, \text{ almost all } \epsilon_n = 0 \right\}$$

denote the set of integers that are sums of distinct terms of A. If $P(A)$ contains all sufficiently large integers, we say that A is *complete*. We shall show that certain classes of sequences that are characterized by their rate of growth are complete.

THEOREM 1.1. *Let $A = (a_1 \leqslant a_2 \leqslant a_3 \leqslant \ldots)$ be an increasing sequence of positive integers. Suppose that A satisfies*

$$a_n \leqslant Mn^{\alpha} \qquad \textit{for all } n \textit{ where } 0 \leqslant \alpha < 1, \tag{1.1}$$

and

(1.2) *for every integer m, $P(A)$ contains an element from each residue class modulo m.*

Then A is complete.

If we assume that the sequence A is strictly increasing, then condition (1.1) may be weakened considerably.

THEOREM 1.2. *Let $A = (a_1 < a_2 < a_3 < \ldots)$ be a strictly increasing sequence of positive integers that satisfies* (1.2) *and*

$$a_n \leqslant Mn^{1+\alpha} \qquad \textit{for all } n \textit{ where } 0 \leqslant \alpha < 1. \tag{1.3}$$

Then A is complete.

Erdös **(2)** proved Theorem 1.2 in the case where

$$\alpha \leqslant (\sqrt{5} - 1)/2 = 0.6180\ldots,$$

and conjectured that the result was true for $\alpha < 1$.

We shall say that a sequence A is subcomplete if $P(A)$ contains an infinite arithmetic progression. Theorems 1.1 and 1.2 follow easily from condition (1.2),

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once we have established that the restrictions on the rate of growth of A ensure that A is subcomplete.

THEOREM 1.3. *Let A be an increasing sequence of positive integers. If A satisfies* (1.1) *or if A is strictly increasing and satisfies* (1.3), *then A is subcomplete.*

2. Preliminary lemmas. The letters A, B, C, ... will denote sequences of positive integers $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, ... We shall sometimes write $a(n)$ for a_n.

LEMMA 2.1. *Let A be an increasing sequence of positive integers with disjoint subsequences B, C, and D. Suppose that*

$$\text{(2.1)} \qquad \text{for each } m > 0, \qquad \lim_{n\to\infty} \frac{1}{b_{n+m}} \sum_{i=1}^{n} b_i = \infty,$$

and that

$$\text{(2.2)} \qquad \begin{array}{l} c_n > d_n \text{ for each integer } n, \text{ and the sequence} \\ E \text{ defined by } e_n = c_n - d_n \text{ is subcomplete.} \end{array}$$

Then A is subcomplete.

To establish this lemma, we first need another result.

LEMMA 2.2. *Let B be an increasing sequence satisfying* (2.1). *For each integer $r > 0$, there is an integer $m(r)$ such that for any $k \geqslant 0$, at least one of the numbers*

$$(k+1)r, (k+2)r, \ldots, (k+m(r))r$$

is in $P(B)$.

Proof. Let $n > 0$ be an integer. We claim that for some i and j with $0 \leqslant i < j \leqslant r$, the sum

$$s_{ij} = b((n-1)r+i+1) + b((n-1)r+i+2) + \ldots + b((n-1)r+j)$$

is divisible by r. Consider the r sums

$$\begin{aligned} s_{01} &= b((n-1)r+1), \\ s_{02} &= b((n-1)r+1) + b((n-1)r+2), \\ &\;\;\vdots \\ s_{0r} &= b((n-1)r+1) + \ldots + b((n-1)r+r). \end{aligned}$$

If they are distinct (mod r), then one of them, s_{0j}, is divisible by r. On the other hand, if

$$s_{0i} \equiv s_{0j} \pmod r \qquad \text{for } i < j,$$

then $s_{ij} = s_{0j} - s_{0i}$ is divisible by r.

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Set $c_n = s_{ij}$, where s_{ij} is divisible by r. Then

$$b_{(n-1)r+1} \leqslant c_n \leqslant rb_{nr}.$$

Hence,

$$\begin{aligned}\frac{1}{c_{n+1}} \sum_{i=1}^{n} c_i &\geqslant \frac{1}{rb_{(n+1)r}} \sum_{i=1}^{n} b_{(i-1)r+1} \\ &= \frac{1}{r^2} \frac{1}{b_{(n+1)r}} \sum_{i=1}^{n} rb_{(i-1)r+1} \\ &\geqslant \frac{1}{r^2} \frac{1}{b_{(n+1)r}} \sum_{i=1}^{(n-1)r+1} b_i,\end{aligned}$$

which tends to infinity with n by (2.1). Therefore, there is an n_0 such that

$$c_{n+1} \leqslant \sum_{i=1}^{n} c_i \qquad \text{for } n \geqslant n_0.$$

Let

$$M = \sum_{i=1}^{n_0} c_i.$$

If $n \geqslant n_0$ and x is an integer with

$$0 \leqslant x \leqslant \sum_{i=1}^{n} c_i,$$

then there is a $y \in P(\{c_1, \ldots, c_n\})$ such that $x \leqslant y \leqslant x + M$. For $n = n_0$, we take $y = M$. Suppose that the assertion is true for some $n \geqslant n_0$ and we shall prove it for $n + 1$.

Suppose

$$0 \leqslant x \leqslant \sum_{i=1}^{n+1} c_i.$$

If

$$x \leqslant \sum_{i=1}^{n} c_i,$$

the required y exists by assumption. If

$$x > \sum_{i=1}^{n} c_i,$$

then

$$0 \leqslant x - \sum_{i=1}^{n} c_i \leqslant x - c_{n+1} \leqslant \sum_{i=1}^{n+1} c_i - c_{n+1} = \sum_{i=1}^{n} c_i.$$

Hence, there is a $y \in P(\{c_1, \ldots, c_n\})$ with

$$x - c_{n+1} \leqslant y \leqslant x - c_{n+1} + M.$$

Now

$$y + c_{n+1} \in P(\{c_1, \ldots, c_{n+1}\}) \quad \text{and} \quad x \leqslant y + c_{n+1} \leqslant x + M.$$

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We have now shown that if $x \geqslant 0$, there is a $y \in P(C)$ with $x \leqslant y \leqslant x + M$. But $P(C) \subset P(B)$ and every element of $P(C)$ is divisible by r. Hence, we may take $m(r) = M/r + 1$ and the lemma is proved. We can now use this result to prove Lemma 2.1.

Proof of Lemma 2.1. Let r and r_0 be integers such that $r_0 + kr \in P(E)$ for every $k \geqslant 0$. Let $m(r)$ be as in Lemma 2.2. For some n, the integers

$$r_0, r_0 + r, \ldots, r_0 + m(r)r$$

are in $P(\{e_1, \ldots, e_n\})$. Let

$$r_1 = r_0 + m(r)r + \sum_{i=1}^{n} d_i.$$

Let $k \geqslant 0$. By Lemma 2.2, $(k + i)r \in P(B)$ for some i with $1 \leqslant i \leqslant m(r)$. Now

$$r_0 + (m(r) - i)r \in P(\{e_1, \ldots, e_n\}).$$

Let

$$r_0 + (m(r) - i)r = \sum_{j=1}^{n} \epsilon_j(c_j - d_j), \qquad \epsilon_j = 0 \text{ or } 1.$$

Then

$$\begin{aligned} r_0 + (m(r) - i)r + \sum_{j=1}^{n} d_j &= \sum_{j=1}^{n} (d_j + \epsilon_j c_j - \epsilon_j d_j) \\ &= \sum_{j=1}^{n} \epsilon_j c_j + \sum_{j=1}^{n} (1 - \epsilon_j)d_j. \end{aligned}$$

Hence,

$$\begin{aligned} r_1 + kr &= r_0 + m(r)r + \sum_{j=1}^{n} d_j + kr \\ &= (k + i)r + (m(r) - i)r + r_0 + \sum_{j=1}^{n} d_j \\ &= (k + i)r + \sum_{j=1}^{n} \epsilon_j c_j + \sum_{j=1}^{n} (1 - \epsilon_j)d_j. \end{aligned}$$

The first term is in $P(B)$, the second is in $P(C)$, and the third is in $P(D)$. Therefore, the sum is in $P(A)$. This is true for any $k \geqslant 0$, so A is subcomplete.

LEMMA 2.3. *Let A be a sequence and let t be a non-decreasing function from the positive integers to the positive integers. Suppose that for each $r > 0$, either $P(\{a_1, \ldots, a_{t(r)}\})$ contains an element from each residue class* (mod r) *or the sequence $a_1, \ldots, a_{t(r)}$ contains at least r terms not divisible by r. Then for each $r > 0$, $P(\{a_1, \ldots, a_{t(r)}\})$ contains an element from each residue class* (mod r).

Proof. Suppose the contrary. Let r be the smallest integer for which the lemma fails. Then $r > 1$ and the sequence $a_1, \ldots, a_{t(r)}$ contains r terms not divisible by r. Let $X = \{x_1, \ldots, x_s\}$ be representatives for the distinct residue

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classes (mod r) which appear in $P(\{a_1, \ldots, a_{t(r)}\})$. Then $s < r$. By a lemma of Erdös (**2**, Lemma 2), there is a subsequence $b_1, \ldots, b_k$ of $a_1, \ldots, a_{t(r)}$ with $k \leqslant s$ such that every element of X is congruent (mod r) to a sum of distinct terms from the sequence $b_1, \ldots, b_k$.

Since $k \leqslant s < r$, there is a term a_i in the sequence $a_1, \ldots, a_{t(r)}$ that is not in the subsequence and is not divisible by r. Hence, if the residue class of x is in X, so is the residue class of $x + a_i$. By induction, the residue class of $x + pa_i$ is in X for all $p \geqslant 0$.

Let $d = (r, a_i)$. Then $1 \leqslant d < r$ and $d = pa_i + qr$ where p may be chosen to be positive. By the choice of r, the lemma holds for d. Hence, since $d|r$ and $t(d) \leqslant t(r)$, X contains a representative from every residue class (mod d). Let y be any integer. Then

$$y \equiv x_j \pmod{d} \qquad \text{for some } x_j \in X.$$

Therefore,

$$y \equiv x_j + ld \equiv x_j + lpa_i + lqr \equiv x_j + lpa_i \pmod{r}$$

for some l. But the residue class of $x_j + lpa_i$ is in X. This is a contradiction since y is arbitrary.

LEMMA 2.4. *Let A be an increasing sequence satisfying* (1.1). *Then there is an integer $d \geqslant 1$ such that all but a finite number of terms of A are divisible by d, and for each $r > 1$, at least r terms of A are divisible by d but not by rd.*

Proof. Let S be the set of all integers $d \geqslant 1$ such that the number of terms of A not divisible by d is less than d. Now S is non-empty because $1 \in S$. Since $\alpha < 1$, there is an n_0 such that for $n \geqslant n_0$,

$$a_n \leqslant Mn^\alpha < n.$$

Hence if $d \geqslant n_0$, then the first d terms of A are not divisible by d. Therefore, S is finite.

Let d be the largest element of S. Clearly, all but a finite number of terms of A are divisible by d. Let $r > 1$. Then $rd > d$ so $rd \notin S$. Hence, at least rd terms of A are not divisible by rd. At most $d - 1$ of these terms are not divisible by d, so there are at least

$$rd - (d - 1) = (r - 1)d + 1 \geqslant r$$

terms of A which are divisible by d but not by rd.

If A is a sequence and r is an integer, we let $l(r, A)$ denote the number of terms in A that are equal to r. We may have $l(r, A) = \infty$.

LEMMA 2.5. *Let A be an increasing sequence satisfying* (1.1). *Suppose that*

$$l(r, A)/r^\alpha, \qquad r \geqslant 1,$$

is unbounded. Then A is subcomplete.

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Proof. If $l(r, A) = \infty$ for some $r \geqslant 1$, the conclusion is immediate. Suppose $l(r, A) < \infty$ for all $r \geqslant 1$. Let d be as in Lemma 2.4. Let N be the number of terms of A not divisible by d. Let $a_{n_1}, a_{n_2}, a_{n_3}, \ldots$ be the subsequence of A consisting of those terms divisible by d. Then $n_k \leqslant k + N$.

We define a sequence B by

$$b_k = a_{n_k}/d.$$

This sequence has the following properties:

$$b_k \leqslant M(N+1)k^\alpha. \tag{2.3}$$

We have

$$b_k \leqslant a_{n_k} \leqslant M(k+N)^\alpha \leqslant M(N+1)\left(\frac{k+N}{N+1}\right)^\alpha \leqslant M(N+1)k^\alpha.$$

If d does not divide r, then $l(r, A) \leqslant N$. Hence,

$$\frac{l(rd, A)}{(rd)^\alpha}, \qquad r \geqslant 1,$$

is unbounded. But

$$\frac{l(rd, A)}{(rd)^\alpha} = \frac{1}{d^\alpha}\frac{l(r, B)}{r^\alpha},$$

so

$$\frac{l(r, B)}{r^\alpha}, \qquad r \geqslant 1, \tag{2.4}$$

is unbounded.

Choose n_0 so that for $n \geqslant n_0$, $M(N+1)n^\alpha < n$. If $r \geqslant n_0$, then $b_1, b_2, \ldots, b_r$ are not divisible by r. By Lemma 2.4, for every $r > 1$, at least r terms of B are not divisible by r. Choose r_0 so that for each r with $1 < r < n_0$, the sequence

$$b_1, b_2, \ldots, b_{r_0}$$

contains at least r terms not divisible by r. If we let $t(r) = \max(r, r_0)$, then the sequence B and the function t satisfy the hypotheses of Lemma 2.3. Hence,

(2.5) if $n = \max(r, r_0)$, then $P(\{b_1, \ldots, b_n\})$ contains an element from each residue class (mod r).

We claim that there is an integer r with the following properties:

$$r \geqslant r_0, \tag{2.6}$$

$$b_n \leqslant \tfrac{1}{4}n \qquad \text{for } n \geqslant r, \tag{2.7}$$

$$\sum_{i=1}^{[n/2]} b_i \geqslant b_{n+1} \qquad \text{for } n \geqslant r, \tag{2.8}$$

$$rl(r, B) \geqslant 2\sum_{i=1}^{r} b_i. \tag{2.9}$$

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By (2.3),

$$b_n \leqslant M(N+1)n^\alpha \leqslant \tfrac{1}{4}n$$

for n sufficiently large. Furthermore,

$$\sum_{i=1}^{[n/2]} b_i \geqslant [n/2] \geqslant M(N+1)(n+1)^\alpha \geqslant b_{n+1}$$

for n sufficiently large. Hence, conditions (2.6)–(2.8) are satisfied by all sufficiently large r.

On the other hand, by (2.4), there are arbitrarily large r that satisfy

$$rl(r, B) \geqslant 2M(N+1)r^{1+\alpha}.$$

But

$$2M(N+1)r^{1+\alpha} \geqslant 2\sum_{i=1}^{r} b_i;$$

so there are arbitrarily large r satisfying (2.9).

Let $l = l(r, B)$ and let m be the integer such that

$$b_{m-1} < b_m = b_{m+1} = \ldots = b_{m+l-1} = r.$$

By (2.7),

$$b_r \leqslant \tfrac{1}{4}r < r = b_m.$$

Therefore, since B is increasing, $r < m$. It now follows from (2.7) that $r = b_m \leqslant \frac{1}{4}m$ or

$$4r \leqslant m. \tag{2.10}$$

The remainder of the proof consists of two assertions, which we prove by induction.

Assertion A. *Let* $r \leqslant n \leqslant m-1$. *If x is an integer satisfying*

$$\sum_{i=1}^{r} b_i \leqslant x \leqslant lr + \sum_{i=r+1}^{n} b_i,$$

then $x \in P(\{b_1, \ldots, b_n, b_m, b_{m+1}, \ldots, b_{m+l-1}\})$.

First let $n = r$. By (2.5) and (2.6), there is a $y \in P(\{b_1, \ldots, b_r\})$ with $x - y \equiv 0 \pmod r$. Now

$$0 \leqslant x - \sum_{i=1}^{r} b_i \leqslant x - y \leqslant x \leqslant lr.$$

Therefore,

$$x - y \in \{0, r, 2r, \ldots, lr\} = P(\{b_m, b_{m+1}, \ldots, b_{m+l-1}\}),$$

and the conclusion follows.

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Assume that the assertion is true for some n with $r \leqslant n < m - 1$, and we shall prove it for $n + 1$. We may assume that

$$lr + \sum_{i=r+1}^{n} b_i < x,$$

since otherwise our assertion follows from the inductive assumption. Hence,

$$x > lr + \sum_{i=r+1}^{n} b_i \geqslant 2\sum_{i=1}^{r} b_i + \sum_{i=r+1}^{n} b_i = \sum_{i=1}^{r} b_i + \sum_{i=1}^{n} b_i$$

by (2.9). Therefore,

$$x - b_{n+1} > \sum_{i=1}^{r} b_i + \sum_{i=1}^{n} b_i - b_{n+1} \geqslant \sum_{i=1}^{r} b_i$$

by (2.8).

On the other hand,

$$x - b_{n+1} \leqslant lr + \sum_{i=r+1}^{n+1} b_i - b_{n+1} = lr + \sum_{i=r+1}^{n} b_i.$$

Thus, $x - b_{n+1} \in P(\{b_1, \ldots, b_n, b_m, b_{m+1}, \ldots, b_{m+l-1}\})$ by the inductive assumption. The conclusion now follows.

Assertion B. *Let $m + l - 1 \leqslant n$. If x is an integer satisfying*

$$\sum_{i=1}^{r} b_i \leqslant x \leqslant \sum_{i=r+1}^{n} b_i,$$

then $x \in P(\{b_1, \ldots, b_n\})$.

If $n = m + l - 1$, the conclusion follows from Assertion A with $n = m - 1$. Assume that the assertion is valid for some $n \geqslant m + l - 1$; then we shall prove it for $n + 1$.

In view of the inductive assumption, we may as well assume that

$$x > \sum_{i=r+1}^{n} b_i.$$

By (2.10), $n \geqslant m \geqslant 4r$. Therefore,

$$\begin{aligned} x > \sum_{i=r+1}^{n} b_i &= \sum_{i=r+1}^{2r} b_i + \sum_{i=2r+1}^{n} b_i \\ &\geqslant \sum_{i=1}^{r} b_i + \sum_{i=[n/2]+1}^{n} b_i \\ &\geqslant \sum_{i=1}^{r} b_i + \sum_{i=1}^{[n/2]} b_i. \end{aligned}$$

Hence, by (2.8),

$$x - b_{n+1} \geqslant \sum_{i=1}^{r} b_i.$$

But

$$x - b_{n+1} \leqslant \sum_{i=r+1}^{n+1} b_i - b_{n+1} = \sum_{i=r+1}^{n} b_i,$$

so $x - b_{n+1} \in P(\{b_1, \ldots, b_n\})$. The conclusion now follows.

Assertion B implies that B is complete. If $x \in P(B)$, then $dx \in P(A)$, so A is subcomplete. Lemma 2.5 has now been proved.

LEMMA 2.6. *Let A be a sequence of positive integers. There is an increasing sequence B such that $P(B) \subset P(A)$ and*

$$\sum_{i=1}^{n} b_i \leqslant \sum_{i=1}^{n} a_i$$

for any n.

Proof. Let b_n be equal to the nth smallest term of A, where the smaller of two terms with the same value is taken to be the one with the smaller index. Clearly B is increasing. We have $P(B) \subset P(A)$ because B is a permutation of a subsequence of A.

Since $b_1, b_2, \ldots, b_n$ are equal to the n smallest terms of A, their sum is less than or equal to the sum of any n terms of A. In particular,

$$\sum_{i=1}^{n} b_i \leqslant \sum_{i=1}^{n} a_i.$$

3. Proofs of the theorems.

Proof of Theorem 1.3. First suppose that A is increasing and satisfies (1.1). Let I denote the set of α, $0 \leqslant \alpha < 1$, for which the theorem holds. If $\alpha = 0$, then A is bounded, so it contains infinitely many terms with the same value. In this case A is clearly subcomplete, so $0 \in I$. If $0 \leqslant \beta \leqslant \alpha$ and $\alpha \in I$, then $\beta \in I$ because $n^\beta \leqslant n^\alpha$ for all $n \geqslant 1$. Hence, if $\alpha_0 = \sup I$, it suffices to show that $\alpha_0 = 1$.

Suppose $0 \leqslant \alpha_0 < 1$. Let

$$\alpha = \tfrac{2}{3}\alpha_0 + \tfrac{1}{3}.$$

Then $0 \leqslant \alpha < 1$, but $\alpha \notin I$ because $\alpha > \alpha_0$. Hence, there is an increasing sequence A that is not subcomplete but satisfies

(3.1) $$a_n \leqslant Mn^\alpha \qquad \text{for all } n$$

for some M.

In view of Lemma 2.5, $l(r, A)/r^\alpha$ is bounded. Hence, there is an N such that

(3.2) $$l(r, A) \leqslant Nr^\alpha \qquad \text{for all } r.$$

We define disjoint subsequences B, C, and D of A as follows:

$$b_n = a_{3n+2},$$
$$c_n = a(3[n + NM^\alpha n^{\alpha^2} + 1] + 1),$$
$$d_n = a_{3n}.$$

Here $[x]$ denotes the greatest integer in x. For each m,

$$\frac{1}{b_{n+m}} \sum_{i=1}^{n} b_i \geqslant \frac{n}{M(3n + 3m + 2)^\alpha}.$$

The right-hand side tends to infinity with n, so B satisfies (2.1).

We have $c_n \geqslant d_n$ because A is increasing. Suppose that $c_n = d_n$ for some n. Then by (3.1) and (3.2),

$$\begin{aligned} l(a_{3n}, A) &\geqslant 3[n + NM^\alpha n^{\alpha^2} + 1] + 1 - 3n \\ &= 3[NM^\alpha n^{\alpha^2} + 1] + 1 \\ &\geqslant 3NM^\alpha n^{\alpha^2} + 1 \\ &\geqslant N(M(3n)^\alpha)^\alpha + 1 \\ &\geqslant N(a_{3n})^\alpha + 1 \\ &\geqslant l(a_{3n}, A) + 1. \end{aligned}$$

This is a contradiction, so $c_n > d_n$ for all n.

Let $e_n = c_n - d_n$ and let F be the increasing sequence obtained from E by Lemma 2.6. Then for each $n > 0$,

$$\begin{aligned} n f_n &\leqslant \sum_{i=n+1}^{2n} f_i \leqslant \sum_{i=1}^{2n} f_i \leqslant \sum_{i=1}^{2n} e_i \\ &= \sum_{i=1}^{2n} c_i - \sum_{i=1}^{2n} d_i \\ &\leqslant \sum_{i=1}^{[2n+NM^\alpha(2n)^{\alpha^2}+1]} a_{3i+1} - \sum_{i=1}^{2n} a_{3i} \\ &\leqslant \sum_{i=1}^{2n-1} (a_{3i+1} - a_{3i+3}) + \sum_{i=2n}^{[2n+NM^\alpha(2n)^{\alpha^2}+1]} a_{3i+1} \\ &\leqslant [NM^\alpha(2n)^{\alpha^2} + 2]a(3[2n + NM^\alpha(2n)^{\alpha^2} + 1] + 1) \\ &\leqslant Qn^{\alpha^2}M(Rn)^\alpha = MQR^\alpha n^{\alpha+\alpha^2}, \end{aligned}$$

where $Q = NM^\alpha 2^{\alpha^2} + 2$ and $R = 10 + 3NM^\alpha 2^{\alpha^2}$. Hence, for each n,

$$f_n \leqslant MQR^\alpha n^{\alpha+\alpha^2-1} \leqslant MQR^\alpha n^{2\alpha-1}.$$

We have $f_n \geqslant 1$ for all n, so $2\alpha - 1 \geqslant 0$. On the other hand,

$$2\alpha - 1 = 2(\tfrac{2}{3}\alpha_0 + \tfrac{1}{3}) - 1 = \tfrac{4}{3}\alpha_0 - \tfrac{1}{3} = \alpha_0 + \tfrac{1}{3}(\alpha_0 - 1) < \alpha_0.$$

Therefore, $2\alpha - 1 \in I$, so F is subcomplete. Now $P(F) \subset P(E)$, so E is subcomplete. By Lemma 2.1, A is subcomplete, which is a contradiction.

Now suppose A is strictly increasing and satisfies (1.3). Define disjoint subsequences B, C, and D by

$$b_n = a_{3n+2}, \qquad c_n = a_{3n+1}, \qquad d_n = a_{3n}.$$

Since B is strictly increasing, for each m

$$\frac{1}{b_{n+m}} \sum_{i=1}^{n} b_i \geqslant \frac{(1/2)n(n+1)}{M(3n+3m+2)^{1+\alpha}}.$$

Hence, B satisfies (2.1).

Now $c_n = a_{3n+1} > a_{3n} = d_n$. Let $e_n = c_n - d_n$ and let F be the monotonic sequence obtained from E by Lemma 2.6. By Lemma 2.1 and what we have already proved, it now suffices to show that for some N,

$$f_n \leqslant Nn^{\alpha} \qquad \text{for all } n.$$

We have

$$\begin{aligned} nf_n &\leqslant \sum_{i=n+1}^{2n} f_i \leqslant \sum_{i=1}^{2n} e_i \\ &= \sum_{i=1}^{2n} a_{3i+1} - \sum_{i=1}^{2n} a_{3i} \\ &\leqslant a_{6n+1} + \sum_{i=1}^{2n-1} (a_{3i+1} - a_{3i+3}) \\ &\leqslant a_{6n+1} \leqslant M(6n+1)^{1+\alpha} \leqslant 7^{1+\alpha} Mn^{1+\alpha}. \end{aligned}$$

Therefore,

$$f_n \leqslant 7^{1+\alpha} Mn^{\alpha},$$

and the proof of Theorem 1.3 is complete.

Proof of Theorem 1.1 *and* 1.2. Let A be an increasing sequence satisfying (1.2). Suppose that either A satisfies (1.1) or A is strictly increasing and satisfies (1.3). We shall call these two situations Case I and Case II, respectively.

Suppose we can find sequences B and C that are disjoint subsequences of A and have the properties that $P(B)$ contains an element from each residue class (mod r) for each r, and C is subcomplete. Let r_0 and r be integers such that

$$r_0 + rk \in P(C) \qquad \text{for each } k \geqslant 0.$$

Let $\{x_1, x_2, \ldots, x_r\} \subset P(B)$ where $x_i \equiv i \pmod{r}$. If x is an integer and

$$x \geqslant r_0 + \max(x_1, x_2, \ldots, x_r),$$

then $x - r_0 \equiv x_i \pmod{r}$ for some i, so

$$x = x_i + r_0 + r\left(\frac{x - r_0 - x_i}{r}\right) \in P(A).$$

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Hence, to show that A is complete it suffices to construct the sequences B and C.

Choose n_0 so large that

$$4M(4n)^\alpha < n \qquad \text{for } n \geqslant n_0.$$

By (1.2), $P(A)$ contains an element from each residue class (mod r) for each r. Hence, we can choose r_0 so that $P(\{a_1, a_2, \ldots, a_{r_0}\})$ contains an element from each residue class (mod r) for $1 \leqslant r < n_0$.

Define sequences B and C by

$$b_n = \begin{cases} a_n & \text{if } n \leqslant 2r_0, \\ a_{2(n-r_0)-1} & \text{if } n > 2r_0, \end{cases}$$

and

$$c_n = a_{2(n+r_0)}.$$

Then B and C are disjoint subsequences of A. We have

$$c_n = a_{2(n+r_0)} \leqslant M(2(n + r_0))^\gamma \leqslant M(2 + 2r_0)^\gamma n^\gamma$$

where $\gamma = \alpha$ in Case I, and in Case II, C is strictly increasing and $\gamma = 1 + \alpha$. By Theorem 1.3, C is subcomplete.

Let $t(r) = \max(r_0, 4r)$. We claim that the sequence B and the function t satisfy the hypotheses of Lemma 2.3. If $r < n_0$, then

$$P(\{b_1, \ldots, b_{t(r)}\}) \supset P(\{a_1, \ldots, a_{r_0}\}),$$

which contains an element from each residue class (mod r). Suppose $r \geqslant n_0$. Note that

$$a_{2i-1} = \begin{cases} b_{2i-1} & \text{if } \;\; 2i - 1 \leqslant 2r_0, \\ b_{i+r_0} & \text{if } \;\; 2i - 1 > 2r_0. \end{cases}$$

Furthermore, if $2i - 1 > 2r_0$, then $i + r_0 \leqslant 2i - 1$. Hence, the sequence $\bar{A} = (a_1, a_3, a_5, \ldots, a_{4r-1})$ is a subsequence of $(b_1, b_2, \ldots, b_{4r})$ which is a subsequence of $(b_1, b_2, \ldots, b_{t(r)})$.

In Case I each term of $\bar{A}$ is less than or equal to a_{4r}, and

$$a_{4r} \leqslant M(4r)^\alpha \leqslant 4M(4r)^\alpha < r.$$

Hence, each of the $2r$ terms in $\bar{A}$ is not divisible by r.

Now suppose we are in Case II. If fewer than r terms of $\bar{A}$ are not divisible by r, then more than r terms of $\bar{A}$ are divisible by r. The terms of $\bar{A}$ are distinct because A is strictly increasing, so for some $a_i \in \bar{A}$, $a_i > r^2$. Therefore,

$$r^2 < a_i < a_{4r} \leqslant M(4r)^{1+\alpha} = 4M(4r)^\alpha r < r^2.$$

This is a contradiction.

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By Lemma 2.3, $P(B)$ contains an element from each residue class (mod r) for each r. This completes the proof.

4. Remarks. Let $\alpha > 1$. It is easy to construct an increasing sequence A that satisfies

$$a_n \leqslant n^\alpha$$

but is such that

$$\sup_n a_{n+1} - \sum_{i=1}^{n} a_i = \infty. \tag{4.1}$$

Such a sequence clearly is not subcomplete. A similar construction yields a strictly increasing sequence A that satisfies (4.1) and

$$a_n \leqslant n^{1+\alpha}.$$

These examples show that our theorems are false for $\alpha > 1$. Cassels **(1)** constructs counter-examples to Theorem 1.2 and Theorem 1.3 in the strictly increasing case for $\alpha > 1$. His sequences satisfy the additional regularity condition that

$$a_{n+1} = a_n + o(a_n^{\frac{1}{2}+\epsilon}),$$

where ϵ is an arbitrary preassigned positive number. Hence, these results are false for $\alpha > 1$, even in the presence of rather strong "smoothness" conditions.

The following questions remain open:

If A is an increasing sequence satisfying

$$a_n \leqslant Mn \qquad \text{for all } n,$$

then is A subcomplete?

If A is a strictly increasing sequence satisfying

$$a_n \leqslant Mn^2 \qquad \text{for } n \geqslant n_0$$

where $M \leqslant 1/2$, then is A subcomplete? (We must require $M \leqslant 1/2$ in this case to ensure that A does not satisfy (4.1).)

References

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