



MEASURING THE SUBOPTIMALITY OF DIVIDEND CONTROLS IN A BROWNIAN RISK MODEL

JULIA EISENBERG,* *Technische Universität Wien*
PAUL KRÜHNER,** *Wirtschaftsuniversität Wien*

Abstract

We consider an insurance company modelling its surplus process by a Brownian motion with drift. Our target is to maximise the expected exponential utility of discounted dividend payments, given that the dividend rates are bounded by some constant. The utility function destroys the linearity and the time-homogeneity of the problem considered. The value function depends not only on the surplus, but also on time. Numerical considerations suggest that the optimal strategy, if it exists, is of a barrier type with a nonlinear barrier. In the related article of Grandits *et al.* (*Scand. Actuarial J.* **2**, 2007), it has been observed that standard numerical methods break down in certain parameter cases, and no closed-form solution has been found. For these reasons, we offer a new method allowing one to estimate the distance from an arbitrary smooth-enough function to the value function. Applying this method, we investigate the goodness of the most obvious suboptimal strategies—payout on the maximal rate, and constant barrier strategies—by measuring the distance from their performance functions to the value function.

Keywords: Suboptimal control; Hamilton–Jacobi–Bellman equation; dividend payouts; exponential utility function

2020 Mathematics Subject Classification: Primary 93E20
Secondary 91B30; 60H30

1. Introduction

A company's dividend payments are among the most important factors for analytic investors to consider when deciding whether to invest in the firm. Furthermore, dividends serve as a sort of litmus paper, indicating the financial health of the company. Indeed, the reputation, and consequently the commercial success, of a company with a long record of dividend payments will be negatively affected if the company drops the payments. On the other hand, new companies can strengthen their position by declaring dividends. For the sake of fairness, it should be mentioned that there are also some serious arguments against dividend payouts; for example, for tax reasons it might be advantageous to withhold dividend payments. A discussion of the pros and cons of dividend distributions is beyond the scope of the present manuscript. We refer to surveys on the topic by Avanzi [6] or Albrecher and Thonhauser [4].

Because of its importance, the value of expected discounted dividends has long been, and still remains, one of the most popular risk measures in the actuarial literature. Modelling the

Received 19 March 2021; revision received 3 February 2023.

* Postal address: Wiedner Hauptstraße 8, 1040 Wien, Austria. Email address: jeisenbe@fam.tuwien.ac.at

** Postal address: Welthandelsplatz 1, 1020 Wien, Austria. Email address: peisenbe@wu.ac.at

© The Author(s), 2023. Published by Cambridge University Press on behalf of Applied Probability Trust.

entire surplus of an insurance company by a Brownian motion, a compound Poisson process, or a general Lévy process with an infinite or finite time horizon—many papers have been written on maximising expected discounted dividends. The papers of Gerber [13], Bühlmann [10], Azcue and Muler [7], and Albrecher and Thonhauser [3] contain just a few of the results obtained since de Finetti's groundbreaking paper [11]. Shreve, Lehoczky and Gaver [20] considered the problem for a general diffusion process, where the drift and the volatility fulfil some special conditions. Asmussen and Taksar [5] modelled the surplus process via a Brownian motion with drift and found the optimal strategy to be a constant barrier.

All the papers mentioned above deal with linear dividend payments, in the sense that the lump-sum payments or dividend rates are not skewed by a utility function. Hubalek and Schachermayer [16] apply various utility functions to the dividend rates before accumulation. Their result differs significantly from the classical result described in Asmussen and Taksar [5].

An interesting question is to consider the expected 'present utility' of the discounted dividend payments. This means the utility function will be applied to the value of the accumulated discounted dividend payments up to ruin. In this way, one considers as a risk measure the utility of the present value of dividends. The dividend payments are not attributed to a specific owner (the shareholders); they serve as the only cash-flow stream used to evaluate the company's financial health. Therefore, the present utility of the accumulated payments accounts for the company's risk aversion by exercising a dividend payment strategy. The fact that the considerations are stopped at ruin indicates that the negative surplus is considered as a high risk. A higher utility of the present value of future dividend payments makes the company more attractive for potential investors. Early ruin will of course lead to a smaller utility of the present value of dividends. Thus, the event of ruin is a technical feature and does not mean that the company actually goes out of business.

For strategic and reputational reasons, some big companies (such as Munich Re; see [14]) do not reduce their dividends even during periods of crisis. Recently, researchers have started to investigate the problem of non-decreasing dividend payments; some examples are Albrecher *et al.* [1, 2]. In this case, even with a linear utility function, the problem becomes two-dimensional. Adding a nonlinear utility function to this setting further complicates the solution to the problem.

Modelling the surplus by a Brownian motion with drift, Grandits *et al.* [15] applied an exponential utility function to the value of unrestricted discounted dividends. In other words, they considered the expected utility of the present value of dividends and not the expected discounted utility of the dividend rates. In [15], the existence of the optimal strategy was not shown. We will investigate the related problem where the dividend payments are restricted to a finite rate. Note that using a nonlinear utility function increases the dimension of the problem. Therefore, tackling the problem via the Hamilton–Jacobi–Bellman (HJB) approach in order to find an explicit solution seems to be an unsolvable task. Of course, one can prove the value function to be the unique viscosity solution to the corresponding HJB equation and then try to solve the problem numerically. However, on this path one faces two problems that are not easy to tackle. First, the proof that the value function is a (unique) viscosity solution to the corresponding HJB equation can be very complex, time-consuming, and space-consuming. In particular, if one chooses a nonlinear and non-exponential utility function, the value function will depend on three variables: the time t , the surplus x , and the accumulated dividend payments prior to the starting time t . Using an exponential utility allows one to get rid of the third variable. This is also one of the reasons why an exponential utility is considered in the present

paper. Having just two variables to consider allows us to represent the proposed method in a clearer way, avoiding unnecessary details.

Second, if the maximal allowed dividend rate is quite big, then the standard numerical methods such as finite differences and finite elements break down. We discuss some numerical problems in Section 5.

In this paper, we offer a new approach. Instead of proving the value function to be the unique viscosity solution to the corresponding HJB equation, we investigate the ‘goodness’ of suboptimal strategies. In this way, one avoids both problems described above. There is no need to prove that the value function is a classical or a viscosity solution to the HJB equation, and no need to solve the HJB equation numerically. One simply chooses an arbitrary control with an easy-to-calculate return function and compares its performance, or rather an approximation of its performance, against the unknown value function.

The method is based on sharp occupation bounds which we find by a method developed for sharp density bounds in Baños and Krühner [8]. This enables us to make an educated guess and to check whether our pick is indeed almost as good as the optimal strategy. This approach differs drastically from the procedures normally used to calculate the value function, in two ways. First, unlike in most numerical schemes, there is no convergence to the value function; i.e. one gets only a bound for the performance of one given strategy, but no straightforward procedure to get better strategies. Second, our criterion has almost no dependence on the dimension of the problem and is consequently directly applicable in higher dimensions.

The paper is organised as follows. In the next section, we motivate the problem and derive some basic properties of the value function. In Section 3, we consider the case of the maximal constant dividend rate strategy, the properties of the corresponding return function, and the goodness of this strategy (a bound for the distance from the return function to the unknown value function). Section 4 investigates the goodness of a constant barrier strategy. In Section 5, we consider examples illustrating the classical and the new approach. Finally, in the appendix we gather technical proofs and establish occupation bounds.

2. Motivation

We consider an insurance company whose surplus is modelled as a Brownian motion with drift

$$X_t = X_0 + \mu t + \sigma W_t, \quad t \geq 0,$$

where $\mu, \sigma, X_0 \in \mathbb{R}$. We will use the Markov property of X . To be exact, we mean that (Ω, \mathfrak{A}) is a measurable space, $\mathbb{P}_{(t,x)}, x \in \mathbb{R}, t \geq 0$ is a family of measures, $X, W : \Omega \times \mathbb{R}_+ \rightarrow \mathbb{R}$ are continuous sample path processes, and under $\mathbb{P}_{(t,x)}$ we have that $(W_{s+t})_{s \geq 0}$ is a standard Brownian motion $W_u = 0, u \in [0, t], X_s = x + \mu \max\{t - s, 0\} + W_s, s \geq 0$, and $(\mathcal{F}_t)_{t \geq 0}$ is the right-continuous filtration generated by X . In particular, we have $\mathbb{P}_{(t,x)}(X_t = x) = 1$. Note that the process X is defined for all time points $s \geq 0$, but we have $\mathbb{P}_{(t,x)}(X_s = x) = 1$ for $0 \leq s \leq t$, which basically means that X is constant, equal to its starting value x , before its starting time t . We denote by $\mathbb{E}_{(t,x)}$ the expectation corresponding to $\mathbb{P}_{(t,x)}$; also we use the notation $\mathbb{E}_x := \mathbb{E}_{(0,x)}$.

Further, we assume that the company has to pay out dividends, characterised by a dividend rate. Denoting the dividend rate process by C , we can write the ex-dividend surplus process as

$$X_t^C = x + \mu t + \sigma W_t - \int_0^t C_s ds.$$

In the present manuscript we only allow dividend rate processes C which are progressively measurable and satisfy $0 \leq C_s \leq \xi$ for some maximal rate $\xi > 0$ at any time $s \geq 0$. We call these strategies *admissible*. Let $U(x) = \frac{1}{\gamma} - \frac{1}{\gamma}e^{-\gamma x}$, $\gamma > 0$, be the underlying utility function and $\tau^C := \inf\{s \geq t : X_s^C < 0\}$ the ruin time corresponding to the strategy C under the measure $\mathbb{P}_{(t,x)}$. Our objective is to maximise the expected exponential utility of the discounted dividend payments until ruin. Since we can start our observation at every time point $t \in \mathbb{R}_+$, the target functional is given by

$$V^C(t, x) := \mathbb{E}_{(t,x)} \left[U \left(\int_t^{\tau^C} e^{-\delta s} C_s ds \right) \right].$$

Here, $\delta > 0$ denotes the preference rate of the insurer, helping to transfer the dividend payments to the starting time t . Further, we assume that the dividend payout up to t equals 0; for a rigorous simplification see [15], or simply note that with already paid dividends \bar{C} up to time t we have

$$\mathbb{E}_{(t,x)} \left[U \left(\bar{C} + \int_t^{\tau^C} e^{-\delta s} C_s ds \right) \right] = U(\bar{C}) + e^{-\gamma \bar{C}} V^C(t, x).$$

The corresponding value function V is defined by

$$V(t, x) := \sup_C \mathbb{E}_{(t,x)} \left[U \left(\int_t^{\tau^C} e^{-\delta s} C_s ds \right) \right],$$

where the supremum is taken over all admissible strategies C . Note that $V(t, 0) = 0$, because ruin will happen immediately owing to the oscillation of Brownian motion, i.e. $\tau^C = \min\{s \geq t : X_s^C = 0\}$ for any strategy C under $\mathbb{P}_{(t,x)}$. The HJB equation corresponding to the problem can be found similarly as in [15]; for general explanations see for instance [19]:

$$V_t + \mu V_x + \frac{\sigma^2}{2} V_{xx} + \sup_{0 \leq y \leq \xi} \left[y \left(-V_x + e^{-\delta t} (1 - \gamma V) \right) \right] = 0. \tag{1}$$

We would like to stress at this point that we show neither that the value function solves the HJB in some sense, nor that a good-enough solution is the value function. In fact, our approach of evaluating the goodness of a given strategy compared to the unknown optimal strategy does not assume any knowledge about the optimal strategy or its existence.

Assuming that the HJB equation has a classical solution (i.e. that it is smooth enough), one would expect that an optimal strategy C^* is the maximiser in the HJB equation at any given point of time, which would depend on the state of the optimal strategy, i.e.

$$C^*(s, X_s^*) = \begin{cases} 0 & \text{if } -V_x(s, X_s^*) + e^{-\delta s} (1 - \gamma V(s, X_s^*)) < 0, \\ [0, \xi] & \text{if } -V_x(s, X_s^*) + e^{-\delta s} (1 - \gamma V(s, X_s^*)) = 0, \\ \xi & \text{if } -V_x(s, X_s^*) + e^{-\delta s} (1 - \gamma V(s, X_s^*)) > 0, \end{cases}$$

$\mathbb{P}_{(t,x)}$ -almost surely for any $s \geq t$.

Remark 2.1. For every dividend strategy C it holds that

$$V^C(t, x) = \mathbb{E}_{(t,x)} \left[U \left(\int_t^{\tau^C} C_s e^{-\delta s} ds \right) \right] \leq U \left(\xi \int_t^\infty e^{-\delta s} ds \right) = U \left(\frac{\xi}{\delta} e^{-\delta t} \right).$$

We conclude that

$$\lim_{x \rightarrow \infty} V(t, x) \leq U\left(\frac{\xi}{\delta} e^{-\delta t}\right),$$

and V is a bounded function. Consider now a constant strategy $C_t \equiv \xi$; i.e. we always pay on the rate ξ . The ex-dividend process becomes a Brownian motion with drift $\mu - \xi$ and volatility σ . Define further, for $n \geq 1$,

$$\eta_n := \frac{\xi - \mu - \sqrt{(\xi - \mu)^2 + 2\delta\sigma^2 n}}{\sigma^2} < 0, \tag{2}$$

and let $\tau^\xi := \inf\{s \geq t : x + (\mu - \xi)s + \sigma W_s \leq 0\}$; i.e. τ^ξ is the ruin time under the strategy ξ . Here and in the following we define

$$\Delta := \xi\gamma/\delta. \tag{3}$$

With help of a change-of-measure technique (see for example [19, p. 216]), we can calculate the return function V^ξ of the constant strategy $C_t \equiv \xi$ by using the power series representation of the exponential function:

$$\begin{aligned} V^\xi(t, x) &= \mathbb{E}_x \left[U \left(\xi \int_t^{\tau^\xi} e^{-\delta s} ds \right) \right] = \frac{1}{\gamma} - \frac{1}{\gamma} \mathbb{E}_x \left[e^{-\Delta \left(e^{-\delta t} - e^{-\delta(t+\tau^\xi)} \right)} \right] \\ &= \frac{1}{\gamma} - \frac{1}{\gamma} e^{-\Delta e^{-\delta t}} \mathbb{E}_x \left[e^{\Delta e^{-\delta(t+\tau^\xi)}} \right] = \frac{1}{\gamma} - \frac{e^{-\Delta e^{-\delta t}}}{\gamma} \sum_{n=0}^{\infty} \frac{e^{-\delta n \Delta}}{n!} \mathbb{E}_x \left[e^{-\delta \tau^\xi n} \right] \\ &= \frac{1}{\gamma} - \frac{e^{-\Delta e^{-\delta t}}}{\gamma} - \frac{e^{-\Delta e^{-\delta t}}}{\gamma} \sum_{n=1}^{\infty} \frac{e^{-\delta n \Delta}}{n!} e^{\eta_n x}. \end{aligned} \tag{4}$$

It is obvious that in the above power series $\lim_{x \rightarrow \infty}$ and summation can be interchanged, yielding

$$\lim_{x \rightarrow \infty} V^\xi(t, x) = U\left(\frac{\xi}{\delta} e^{-\delta t}\right). \text{ In particular, we can now conclude}$$

$$\lim_{x \rightarrow \infty} V(t, x) = \frac{1}{\gamma} - \frac{1}{\gamma} \exp(-\Delta e^{-\delta t}) = U\left(\frac{\xi}{\delta} e^{-\delta t}\right)$$

uniformly in $t \in [0, \infty)$.

Next we show that for some special values of the maximal rate ξ , with positive probability the ex-dividend surplus process remains positive up to infinity.

Remark 2.2. Let C be an admissible strategy, where X^C is the process under the strategy C . Further, let X^ξ be the process under the constant strategy ξ ; i.e. X^ξ is a Brownian motion with drift $(\mu - \xi)$ and volatility σ . Then it is clear that

$$X_s^\xi \leq X_s^C.$$

If $\mu > \xi$, then it holds (see for example [9, p. 223]) that $\mathbb{P}_{(t,x)}[\tau^C = \infty] \geq \mathbb{P}_{(t,x)}[\tau^\xi = \infty] > 0$.

Finally, we give one structural property of the value function, which, however, is not used later.

Theorem 2.1. *The value function is Lipschitz-continuous, strictly increasing in x , and decreasing in t .*

Proof. • Let $h > 0, \varepsilon > 0$ be arbitrary but fixed. Further, let C be an ε -optimal strategy for $(t, x) \in \mathbb{R}_+^2$, i.e. $V(t, x) \leq V^C(t, x) + \varepsilon$. Define the strategy \tilde{C} for $(t, x + h)$ in the following way:

$$\tilde{C}_s = \begin{cases} C_s & \text{if } t \leq s < \tau^C, \\ \xi & \text{otherwise.} \end{cases}$$

Then \tilde{C} is an admissible strategy and is actually the same as the strategy C until the process $X^{\tilde{C}}$ reaches the level h . Afterwards it pays at maximal rate until ruin, which is strictly later $\tau^C < \tau^{\tilde{C}}$. Note that $U(x + y) = U(x) + e^{-\gamma x}U(y)$, and hence we have

$$\begin{aligned} V(t, x + h) - V(t, x) &\geq V^{\tilde{C}}(t, x + h) - V^C(t, x) - \varepsilon \\ &= \mathbb{E}_{(t, x+h)} \left[U \left(\int_t^{\tau^{\tilde{C}}} \tilde{C}_s e^{-\delta s} ds \right) \right] - \mathbb{E}_{(t, x)} \left[U \left(\int_t^{\tau^C} C_s e^{-\delta s} ds \right) \right] - \varepsilon \\ &= \mathbb{E}_{(t, x+h)} \left[e^{-\gamma \int_t^{\tau^C} \tilde{C}_s e^{-\delta s} ds} U \left(\int_{\tau^C}^{\tau^{\tilde{C}}} \xi e^{-\delta s} ds \right) \right] - \varepsilon \\ &\geq \mathbb{E}_{(t, x+h)} \left[e^{-\gamma \int_t^\infty \xi e^{-\delta s} ds} U \left(\int_{\tau^C}^{\tau^{\tilde{C}}} \xi e^{-\delta s} ds \right) \right] - \varepsilon \\ &\geq K_h - \varepsilon, \end{aligned}$$

where $K_h > 0$ and can be chosen independent of the strategy C . Thus we find that $V(t, x + h) - V(t, x) \geq K_h$.

• Consider further $(t, 0)$ with $t \in \mathbb{R}_+$. Let $h, \varepsilon > 0$ and let C be an arbitrary admissible strategy. Let τ^0 be the ruin time for the strategy which is constant and zero. Define

$$\varrho_n := \frac{\sqrt{\mu^2 + 2\sigma^2\delta n}}{\sigma^2}, \quad \theta_n := \frac{-\mu}{\sigma^2} + \varrho_n, \quad \text{and} \quad \zeta_n := \frac{-\mu}{\sigma^2} - \varrho_n \tag{5}$$

for any $n \in \mathbb{N}$. Using $\mathbb{E}_h[e^{-\delta\tau^0}] = e^{\zeta_1 h}$ (see for instance [9, p. 295]), $X_s^0 \geq X_s^C$, and the convexity of the exponential function, $U(x) = \frac{1 - e^{-\gamma x}}{\gamma} \leq x$, it follows that

$$\begin{aligned} V^C(t, h) &= \mathbb{E}_{(t, h)} \left[U \left(\int_t^{\tau^C} e^{-\delta s} C_s ds \right) \right] \leq \mathbb{E}_h \left[U \left(\xi \int_t^{t+\tau^0} e^{-\delta s} ds \right) \right] \\ &= \mathbb{E}_h \left[U \left(\frac{\xi}{\delta} e^{-\delta t} (1 - e^{-\delta\tau^0}) \right) \right] \leq \frac{\xi}{\delta} e^{-\delta t} (1 - e^{\zeta_1 h}) \leq -\frac{\xi}{\delta} \xi_1 h. \end{aligned} \tag{6}$$

Let $h \geq 0$ and let τ^0 be the ruin time for the strategy which is constant and zero. Let $(t, x) \in \mathbb{R}_+^2$ be arbitrary, and let C be an admissible strategy which is ε -optimal for the starting point $(t, x + h)$, i.e. $V(t, x + h) - V^C(t, x + h) \leq \varepsilon$. Define further $\tilde{\tau} := \inf\{s \geq t : X_s^C = h\}$. Then $X_s^C \geq 0$ for $s \in [t, \tilde{\tau}]$ under $\mathbb{P}_{(t, x)}$ because $X_s^C \geq h$ for $s \in [t, \tilde{\tau}]$ under $\mathbb{P}_{(t, x+h)}$. Then the strategy C , up to $\tilde{\tau}$, is an admissible strategy for (t, x) satisfying

$$V^C(t, x) = \mathbb{E}_{(t, x)} \left[U \left(\int_t^{\tilde{\tau}} e^{-\delta s} C_s ds \right) \right] = \mathbb{E}_{(t, x+h)} \left[U \left(\int_t^{\tilde{\tau}} e^{-\delta s} C_s ds \right) \right].$$

Note that $X_{\tilde{\tau}}^C = h$, and hence we have

$$\begin{aligned} \tau^C - \tilde{\tau} &= \inf \left\{ u \geq 0 : X_{u+\tilde{\tau}}^C = 0 \right\} = \inf \left\{ u \geq 0 : h + (X_{u+\tilde{\tau}} - X_{\tilde{\tau}}) - \int_{\tilde{\tau}}^{u+\tilde{\tau}} C_r \, dr = 0 \right\} \\ &\leq \inf \left\{ u \geq 0 : h + (X_{u+\tilde{\tau}} - X_{\tilde{\tau}}) = 0 \right\} =: \beta^0, \end{aligned}$$

where $\mathbb{P}_{(t,x+h)}^{\beta^0} = \mathbb{P}_{t,h}^{\tau^0}$. Here, $\mathbb{P}_{t,h}^{\tau^0}$ denotes the law of τ^0 under $\mathbb{P}_{t,h}$; analogously, $\mathbb{P}_{(t,x+h)}^{\beta^0}$ is the law of β^0 under $\mathbb{P}_{t,x+h}$. Since U satisfies $U(a + b) \leq U(a) + U(b)$ for any $a, b \geq 0$, we have

$$\begin{aligned} V(t, x + h) &\leq V^C(t, x + h) + \varepsilon = \mathbb{E}_{(t,x+h)} \left[U \left(\int_t^{\tau^C} e^{-\delta s} C_s \, ds \right) \right] + \varepsilon \\ &= \mathbb{E}_{(t,x+h)} \left[U \left(\int_t^{\tilde{\tau}} e^{-\delta s} C_s \, ds + \int_{\tilde{\tau}}^{\tau^C} e^{-\delta s} C_s \, ds \right) \right] + \varepsilon \\ &\leq \mathbb{E}_{(t,x+h)} \left[U \left(\int_t^{\tilde{\tau}} e^{-\delta s} C_s \, ds \right) \right] + \mathbb{E}_{(t,x+h)} \left[U \left(\int_{\tilde{\tau}}^{\tau^C} e^{-\delta s} C_s \, ds \right) \right] + \varepsilon \\ &\leq V^C(t, x) + \mathbb{E}_{(t,x+h)} \left[U \left(\frac{\xi}{\delta} (e^{-\delta \tilde{\tau}} - e^{-\delta \tau^C}) \right) \right] + \varepsilon \\ &\leq V(t, x) + \mathbb{E}_{(t,x+h)} \left[U \left(\frac{\xi}{\delta} (1 - e^{-\delta(\tau^C - \tilde{\tau})}) \right) \right] + \varepsilon \\ &\leq V(t, x) + \mathbb{E}_h \left[U \left(\frac{\xi}{\delta} (1 - e^{-\delta \tau^0}) \right) \right] + \varepsilon. \end{aligned}$$

Because ε was arbitrary, and by (6), we find

$$0 \leq V(t, x + h) - V(t, x) \leq -\frac{\xi}{\delta} \zeta_1 h.$$

Consequently, V is Lipschitz-continuous in the space variable x with Lipschitz constant at most $-\frac{\xi}{\delta} \zeta_1$.

• Next we consider the properties of the value function related to the time variable.

Because $\delta > 0$, it is clear that V is strictly decreasing in t . First we show that the value function is strictly decreasing in time. To this end let $(t, x) \in \mathbb{R}_+^2$, let $h > 0$, and let C be an admissible strategy which is constant and zero before time $t + h$, with τ its ruin time. Since C is measurable with respect to the σ -algebra $\sigma(X_s : s \geq t)$, we can find a measurable function $c : \mathbb{R}_+ \times C(\mathbb{R}_+, \mathbb{R}) \rightarrow \mathbb{R}$ such that $C_s(\omega) = c(s - t, (X_{t+u})_{u \geq 0})$. Defining $\tilde{C}_s := c(s - (t + h), (X_{t+h+u})_{u \geq 0})$, we see that the law of $(X_s, C_s)_{s \geq t}$ under $\mathbb{P}_{(t,x)}$ equals the law of $(X_{s+h}, \tilde{C}_{s+h})_{s \geq t}$ under $\mathbb{P}_{(t+h,x)}$.

The stopping time $\tilde{\tau} := \inf\{s \geq t + h : X_s^{\tilde{C}} = 0\}$ is the corresponding ruin time:

$$\begin{aligned} V^C(t, x) &= \mathbb{E}_{(t,x)} \left[U \left(\int_t^{\tau} C_s e^{-\delta s} \, ds \right) \right] \\ &= \mathbb{E}_{(t+h,x)} \left[U \left(\int_{t+h}^{\tilde{\tau}} \tilde{C}_s e^{-\delta(s-h)} \, ds \right) \right]. \end{aligned}$$

Taking the supremum over all strategies yields

$$V(t, x) = \sup_{\tilde{C}} \mathbb{E}_{(t+h,x)} \left[U \left(e^{\delta h} \int_{t+h}^{\tilde{\tau}^{\tilde{C}}} \tilde{C}_s e^{-\delta s} \, ds \right) \right] > V(t + h, x).$$

Further, let $(t, x) \in \mathbb{R}_+^2$, let $h > 0$, and let C be an admissible strategy. Then the strategy \tilde{C} with $\tilde{C}_s := C_{s-h} \mathbb{1}_{\{s \geq h\}}$ is admissible. Since U is concave, we have

$$\begin{aligned} V(t+h, x) &\geq V^{\tilde{C}}(t+h, x) = \mathbb{E}_{(t+h, x)} \left[U \left(\int_{t+h}^{\tau^{\tilde{C}}+h} e^{-\delta s} C_{s-h} ds \right) \right] \\ &= \mathbb{E}_{(t, x)} \left[U \left(e^{-\delta h} \int_t^{\tau^C} e^{-\delta s} C_s ds \right) \right] \geq e^{-\delta h} V^C(t, x). \end{aligned}$$

Building the supremum over all admissible strategies on the right side of the above inequality and using Remark 2.1 yields

$$0 \geq V(t+h, x) - V(t, x) \geq V(t, x)(e^{-\delta h} - 1) \geq -U\left(\frac{\xi}{\delta}\right)\delta h;$$

consequently, V is Lipschitz-continuous as a function of t , with constant $\delta U(\xi/\delta)$. □

2.1. Heuristics

Heuristically, our approach to compare a given feedback strategy C with the unknown optimal strategy C^* works as follows:

1. We start with the performance function V^C corresponding to some feedback strategy $C_t = c(t, X_t^C)$. If smooth enough, V^C satisfies

$$V_t^C + \mu V_x^C + \frac{\sigma^2}{2} V_{xx}^C + c\{-V_x^C + e^{-\delta t}(1 - \gamma V^C)\} = 0, \quad V^C(t, 0) = 0.$$

2. However, sometimes V^C is not smooth enough or is not known explicitly. In this case one uses a replacement H (simply any $C^{1,2}$ function from $\mathbb{R}_+ \times \mathbb{R}$ to \mathbb{R} with $H(t, 0) = 0$) and defines the mismatch

$$\Psi := H_t + \mu H_x + \frac{\sigma^2}{2} H_{xx} + c\{-H_x + e^{-\delta t}(1 - \gamma H)\},$$

where Ψ as close to zero as possible is desirable.

3. We consider another strategy C^* and the corresponding controlled process $X^* = X^{C^*}$, as well as their performance $V^* := V^{C^*}$. Its ruin time is denoted by τ , and we obtain from Itô's formula, using $X_\tau^* = 0$ and $H(t, 0) \equiv 0$,

$$\begin{aligned} 0 &= e^{-\gamma \int_t^\tau e^{-\delta u} C_u^* du} \cdot H(\tau, X_\tau^*) \\ &= H(t, X_t^*) + \int_t^\tau e^{-\gamma \int_t^s e^{-\delta u} C_u^* du} \cdot H_x ds \\ &\quad + \int_t^\tau e^{-\gamma \int_t^s e^{-\delta u} C_u^* du} \cdot \left(H_t + \mu H_x + \frac{\sigma^2}{2} H_{xx} + C_s^* \{-H_x - \gamma e^{-\delta s} H\} \right) ds \\ &= H(t, X_t^*) + \int_t^\tau e^{-\gamma \int_t^s e^{-\delta u} C_u^* du} \cdot H_x ds \\ &\quad + \int_t^\tau e^{-\gamma \int_t^s e^{-\delta u} C_u^* du} \cdot \left(H_t + \mu H_x + \frac{\sigma^2}{2} H_{xx} + c(-H_x + e^{-\delta s}(1 - \gamma H)) \right) ds \end{aligned}$$

$$\begin{aligned}
 & + \int_t^\tau e^{-\gamma \int_t^s e^{-\delta u} C_u^* du} \cdot (C_s^* - c) \{-H_x + e^{-\delta s}(1 - \gamma H)\} ds \\
 & - \int_t^\tau e^{-\gamma \int_t^s e^{-\delta u} C_u^* du} \cdot e^{-\delta s} C_s^* ds \\
 = & H(t, X_t^*) + \int_t^\tau e^{-\gamma \int_t^s e^{-\delta u} C_u^* du} \cdot H_x dW_s + \int_t^\tau e^{-\gamma \int_t^s e^{-\delta u} C_u^* du} \cdot \Psi ds \\
 & + \int_t^\tau e^{-\gamma \int_t^s e^{-\delta u} C_u^* du} (C_s^* - c) \{-H_x + e^{-\delta s}(1 - \gamma H)\} ds \\
 & - U\left(\int_t^\tau e^{-\delta u} C_u^* du\right).
 \end{aligned}$$

4. Taking $\mathbb{P}_{(t,x)}$ -expectation (assuming the local martingale from the dW_s -integral is a martingale) and bringing the expectation of the U term on the other side yields

$$\begin{aligned}
 V^*(t, x) = & H(t, x) + \mathbb{E}_{(t,x)} \left[\int_t^\tau e^{-\gamma \int_t^s e^{-\delta u} C_u^* du} \cdot \Psi ds \right] \\
 & + \mathbb{E}_{(t,x)} \left[\int_t^\tau e^{-\gamma \int_t^s e^{-\delta u} C_u^* du} \cdot (C_s^* - c) \{-H_x + e^{-\delta s}(1 - \gamma H)\} ds \right].
 \end{aligned}$$

5. Up to here, this is all standard. The performance function V^* is expressed in terms of a new function H plus two error terms which could have negative sign. Several other stochastic control problems can lead to similar equations. The first error term corresponds to the usage of a function other than the performance function of our initial feedback control C . The second error term corresponds to the suboptimality of the feedback control C compared to the control C^* , measured relatively by the function H .

6. Now we need to control the error terms despite the appearance of the unknown optimal control. The first error term is simply bounded by

$$\left| \mathbb{E}_{(t,x)} \left[\int_t^\tau e^{-\gamma \int_t^s e^{-\delta u} C_u^* du} \Psi ds \right] \right| \leq \mathbb{E}_{(t,x)} \left[\int_t^\tau |\Psi(s, X_s^*)| ds \right],$$

where one has to deal with the unknown process X^* but its control has disappeared. This is the point where occupation bounds as in the appendix yield explicit upper bounds.

7. The appearance of C^* in the second error term can be suppressed by maximising the integrand over all possible values of C^* . Since $C = C^*$ is a possible value, this maximum is positive and we obtain

$$\begin{aligned}
 & \mathbb{E}_{(t,x)} \left[\int_t^\tau \exp\left(-\gamma \int_t^s e^{-\delta u} C_u^* du\right) (C_s^* - c) (-H_x + e^{-\delta s}(1 - \gamma H)) ds \right] \\
 & \leq \mathbb{E}_{(t,x)} \left[\int_t^\tau \sup_{y \in [0, \xi]} ((y - c) (-H_x + e^{-\delta s}(1 - \gamma H))) ds \right].
 \end{aligned}$$

8. Putting these together we obtain

$$\begin{aligned}
 V^*(t, x) \leq & H(t, x) + \mathbb{E}_{(t,x)} \left[\int_t^\tau |\Psi(s, X_s^*)| ds \right] \\
 & + \mathbb{E}_{(t,x)} \left[\int_t^\tau \sup_{y \in [0, \xi]} ((y - c) (-H_x(s, X_s^*) + e^{-\delta s}(1 - \gamma H(s, X_s^*))) ds \right].
 \end{aligned}$$

9. If we have a common upper bound $\Upsilon_{t,x} \geq 0$ for $\mathbb{E}_{(t,x)} \left[\int_t^\tau |\Psi(s, X_s^*)| ds \right]$, then we may take the supremum over all strategies on the left-hand side and obtain

$$V(t, x) \leq H(t, x) + \Upsilon_{t,x} + \mathbb{E}_{(t,x)} \left[\int_t^\tau \sup_{y \in [0, \xi]} ((y - c)(-H_x(s, X_s^*) + e^{-\delta s}(1 - \gamma H(s, X_s^*))) ds \right].$$

To obtain such a common upper bound, we will employ bounds for the expected occupation which are summarised in the appendix. Note that choosing the optimal y , depending on (s, X_s^*) , allows us to employ these common upper bounds for the second summand as well.

Remark 2.3. We note that if $H = V^C$ (i.e. $\Psi = 0$), and if in the maximisation in Item 8 above the maximiser is attained in C , then both error terms vanish and we find

$$V^C \leq V \leq H = V^C,$$

which implies that all of these quantities are the same. This means that if a feedback control C is found such that its performance function V^C satisfies the HJB equation

$$\sup_{y \in [0, \xi]} \left(V_t^C + \mu V_x^C + \frac{\sigma^2}{2} V_{xx}^C + y \left(-V_x^C + e^{-\delta t} (1 - \gamma V^C) \right) \right) = 0, \quad V^C(t, 0) = 0,$$

then we have verified heuristically that $V^C = V$.

3. Payout on the maximal rate

3.1. Could it be optimal to pay on the maximal rate up to ruin?

First we investigate the constant strategy ξ , i.e. the strategy under which the dividends will be paid out at the maximal rate ξ until ruin. In this section we find exact conditions under which this strategy is optimal. We already know from (4) that the corresponding return function is given by

$$V^\xi(t, x) = \frac{1}{\gamma} - \frac{1}{\gamma} e^{-\Delta e^{-\delta t}} - e^{-\Delta e^{-\delta t}} \sum_{n=1}^\infty \frac{\Delta^n}{\gamma n!} e^{-\delta t n} e^{\eta_n x}.$$

It is obvious that V^ξ is increasing and concave in x and decreasing in t . For further considerations we will need the following remark.

Remark 3.1. Consider η_n , defined in (2), as a function of ξ .

1. Since

$$\frac{d}{d\xi} \eta_n = \frac{-\eta_n}{\sqrt{(\xi - \mu)^2 + 2\delta\sigma^2 n}},$$

it is easy to see that $\eta_n(\xi)$ and $\frac{\eta_{n+1}(\xi)n}{\eta_n(\xi)(n+1)}$ are increasing in ξ . Also, we have

$$\lim_{\xi \rightarrow \infty} \frac{\eta_{n+1}(\xi)n}{\eta_n(\xi)(n+1)} = 1.$$

We conclude that $\frac{\eta_{n+1}}{(n+1)} > \frac{\eta_n}{n}$, as $\eta_n, \eta_{n+1} < 0$.

2. Further, we record that

$$\lim_{\xi \rightarrow \infty} \xi \eta_n(\xi) = -\delta n.$$

3. Also, we have

$$\frac{d}{d\xi} (\delta n + \xi \eta_n(\xi)) = \eta_n \left(1 - \frac{\xi}{\sqrt{(\xi - \mu)^2 + 2\delta\sigma^2 n}} \right) \begin{cases} < 0, & \xi < \frac{\mu^2 + 2\delta\sigma^2 n}{2\mu}, \\ \geq 0, & \xi \geq \frac{\mu^2 + 2\delta\sigma^2 n}{2\mu}. \end{cases}$$

Thus, at $\xi = 0$ the function $\xi \mapsto \delta n + \xi \eta_n(\xi)$ attains the value $\delta n > 0$, at its minimum point $\xi^* = \frac{\mu^2 + 2\delta\sigma^2 n}{2\mu}$ we have

$$\delta n + \xi^* \eta_n(\xi^*) = \delta n - \frac{\mu^2 + 2\delta\sigma^2 n}{2\sigma^2} = -\frac{\mu^2}{2\sigma^2} < 0,$$

and, finally, by Item 2 above, for $\xi \rightarrow \infty$ it holds that $\lim_{\xi \rightarrow 0} \delta n + \xi \eta_n(\xi) = 0$. Thus, for

every $n \in \mathbb{N}$ the function $\xi \mapsto 1 + \frac{\eta_n(\xi)\xi}{\delta n}$ has a unique zero at $\frac{\delta n \sigma^2}{2\mu}$.

Furthermore, it is easy to check that in V^ξ summation and differentiation can be interchanged. Differentiation with respect to x yields

$$V_x^\xi(t, x) = -e^{-\Delta e^{-\delta t}} \sum_{n=1}^\infty \frac{\Delta^n}{\gamma n!} e^{-\delta t n} \eta_n e^{\eta_n x}.$$

In order to answer the optimality question, we have to look at the function $-V_x^\xi + e^{-\delta t}(1 - \gamma V^\xi)$ appearing in the crucial condition in the HJB equation (1). If this expression is positive for all $(t, x) \in \mathbb{R}_+^2$, the function V^ξ becomes a candidate for the value function. For simplicity, we multiply the expression $-V_x^\xi + e^{-\delta t}(1 - \gamma V^\xi)$ by $e^{\delta t} e^{\Delta e^{-\delta t}}$ and define

$$\begin{aligned} \psi(t, x) &:= \frac{e^{\Delta t}}{t} \left\{ -V_x^\xi \left(\frac{\ln(t)}{-\delta}, x \right) + t \left(1 - \gamma V^\xi \left(\frac{\ln(t)}{-\delta}, x \right) \right) \right\} \\ &= \sum_{n=0}^\infty t^n \frac{\Delta^n}{n!} \left\{ \frac{\eta_{n+1}\xi}{\delta(n+1)} e^{\eta_{n+1}x} + e^{\eta_n x} \right\}. \end{aligned} \tag{7}$$

If $\psi \geq 0$ on $[0, 1] \times \mathbb{R}_+$, then V^ξ does solve the HJB equation, and as we will see, it is the value function in that case.

Theorem 3.1. V^ξ is the value function if and only if $\xi \leq \frac{\delta\sigma^2}{2\mu}$. In that case V^ξ is a classical solution to the HJB equation (1), and an optimal strategy is constant ξ .

Proof. Since $\frac{\sigma^2}{2} \eta_n^2 = (\xi - \mu)\eta_n + \delta n$ for all $n \geq 1$, it is easy to check, using the power series representation of V^ξ , that V^ξ solves the differential equation

$$V_t^\xi + \mu V_x^\xi + \frac{\sigma^2}{2} V_{xx}^\xi + \xi \left(-V_x^\xi + e^{-\delta t}(1 - \gamma V^\xi) \right) = 0.$$

We first assume that $\xi \leq \frac{\delta\sigma^2}{2\mu}$ and show that V^ξ is the value function. Note that $\xi \leq \frac{\delta\sigma^2}{2\mu}$ is equivalent to $\frac{\eta_1\xi}{\delta} + 1 \geq 0$. We have $\xi \leq n \frac{\delta\sigma^2}{2\mu}$ for any $n \geq 1$, and Item 1 of Remark 3.1 yields, for all $n \geq 2$,

$$\eta_n \frac{\xi}{\delta n} + 1 > \eta_1 \frac{\xi}{\delta} + 1 \geq 0.$$

This gives immediately, for all $(t, x) \in (0, 1] \times \mathbb{R}_+$,

$$\psi(t, x) \geq \sum_{n=0}^{\infty} t^n \frac{\Delta^n}{n!} \left\{ \frac{\eta_{n+1}\xi}{\delta(n+1)} + 1 \right\} e^{\eta_n x} \geq 0,$$

which is equivalent to

$$-V_x^\xi(t, x) + e^{-\delta t}(1 - \gamma V^\xi(t, x)) \geq 0$$

for all $(t, x) \in \mathbb{R}_+^2$. This means that V^ξ solves the HJB equation (1) if $\xi \leq \frac{\delta\sigma^2}{2\mu}$.

Now let C be an arbitrary admissible strategy, τ its ruin time, and $\hat{X}_u := X_u^C$. Applying Itô's formula yields, $\mathbb{P}_{(t,x)}$ -almost surely,

$$\begin{aligned} e^{-\gamma \int_t^{\tau \wedge s} e^{-\delta u} C_u du} V_x^\xi(\tau \wedge s, \hat{X}_{\tau \wedge s}) &= V^\xi(t, x) + \sigma \int_t^{\tau \wedge s} e^{-\gamma \int_t^y e^{-\delta u} C_u du} V_x^\xi dW_y \\ &+ \int_t^{\tau \wedge s} e^{-\gamma \int_t^y e^{-\delta u} C_u du} \left\{ V_t^\xi + (\mu - C_y)V_x^\xi + \frac{\sigma^2}{2} V_{xx}^\xi - \gamma C_y e^{-\delta y} V_y^\xi \right\} dy. \end{aligned}$$

Since V_x^ξ is bounded, the stochastic integral above is a martingale with expectation zero. For the second integral one obtains, using the differential equation for V^ξ ,

$$\begin{aligned} &\int_t^{\tau \wedge s} e^{-\gamma \int_t^y e^{-\delta u} C_u du} \left\{ V_t^\xi + (\mu - C_y)V_x^\xi + \frac{\sigma^2}{2} V_{xx}^\xi - \gamma C_y e^{-\delta y} V_y^\xi \right\} dy \\ &= \int_t^{\tau \wedge s} e^{-\gamma \int_t^y e^{-\delta u} C_u du} \left\{ (C_y - \xi) \left[-V_x^\xi + e^{-\delta y}(1 - \gamma V^\xi) \right] - C_y e^{-\delta y} \right\} dy. \end{aligned}$$

Building the expectations on the both sides and letting $s \rightarrow \infty$, by interchanging the limit and expectation (using the bounded convergence theorem) we obtain

$$\begin{aligned} 0 &= V^\xi(t, x) \\ &+ \mathbb{E}_{(t,x)} \left[\int_t^\tau e^{-\gamma \int_t^y e^{-\delta u} C_u du} (C_y - \xi) \left\{ -V_x^\xi + e^{-\delta y}(1 - \gamma V^\xi) \right\} dy \right] \end{aligned} \tag{8}$$

$$- \mathbb{E}_{(t,x)} \left[\int_t^\tau e^{-\gamma \int_t^y e^{-\delta u} C_u du} \cdot C_y e^{-\delta y} dy \right]. \tag{9}$$

Since $C_u \leq \xi$ and $-V_x^\xi(y, \hat{X}_y) + e^{-\delta y}(1 - \gamma V^\xi(y, \hat{X}_y)) \geq 0$, the expectation in (8) is non-positive.

For (9) one has

$$\begin{aligned} \mathbb{E}_{(t,x)} \left[\int_t^\tau e^{-\gamma \int_t^y e^{-\delta u} C_u du} \cdot C_y e^{-\delta y} dy \right] &= -\mathbb{E}_{(t,x)} \left[\int_t^\tau d \frac{e^{-\gamma \int_t^y e^{-\delta u} C_u du}}{\gamma} \right] \\ &= \mathbb{E}_{(t,x)} \left[U \left(\int_t^\tau e^{-\delta u} C_u du \right) \right] = V^C(t, x), \end{aligned}$$

giving $V^C(t, x) \leq V^\xi(t, x)$ for all admissible strategies C . Therefore, V^ξ is the value function.

Now let $\xi > \frac{\delta\sigma^2}{2\mu}$, and assume for the sake of contradiction that V^ξ is the value function. Then we have $\psi(0, 0) = 1 + \eta_1 \xi / \delta < 0$. This means in particular that the function ψ is also negative

for some $(t, x) \in (0, 1] \times \mathbb{R}_+$. Consequently, V^ξ does not solve the HJB equation (1). However, V^ξ is smooth enough and has a bounded x -derivative. Thus, classical verification results (see, for instance, [19, Section 2.5.1]) yield that V^ξ solves the HJB equation—a contradiction. \square

In the following, we assume $\xi > \frac{\delta\sigma^2}{2\mu}$.

3.2. The goodness of the strategy ξ

We now provide an estimate on the goodness of the constant payout strategy which relies only on the performance of the chosen strategy ξ and on deterministic constants. Recall from (2) and (5) that

$$\eta_n = \frac{(\xi - \mu) - \sqrt{(\xi - \mu)^2 + 2n\delta\sigma^2}}{\sigma^2},$$

$$\theta_n = \frac{-\mu + \sqrt{\mu^2 + 2n\delta\sigma^2}}{\sigma^2}, \quad \zeta_n = \frac{-\mu - \sqrt{\mu^2 + 2n\delta\sigma^2}}{\sigma^2}.$$

We first present the main inequality of this section, then, in the subsequent remark, discuss the finiteness of the sum.

Theorem 3.2. *Let $t, x \geq 0$. Then we have*

$$V(t, x) \leq V^\xi(t, x) + \xi e^{-\delta t} \sum_{n=0}^\infty e^{-\delta n} \frac{\Delta^n}{n!} \int_0^\infty \left(\frac{-\eta_{n+1}\xi}{\delta(n+1)} e^{\eta_{n+1}y} - e^{\eta_n y} \right)^+ f_{n+1}(x, y) dy,$$

where

$$f_n(x, y) := \frac{2(e^{\theta_n(x \wedge y)} - e^{\zeta_n(x \wedge y)}) e^{\eta_n(x-y)^+}}{\sigma^2((\theta_n - \eta_n)e^{y\theta_n} - (\zeta_n - \eta_n)e^{y\zeta_n})}, \quad y \geq 0.$$

Proof. We know that the return function $V^\xi \in C^{1,2}$. Let C be an arbitrary admissible strategy. Then, using Itô’s formula for $s > t$ under $\mathbb{P}_{(t,x)}$, we have

$$e^{-\gamma \int_t^{s \wedge \tau^C} e^{-\delta u} C_u du} \cdot V^\xi(s \wedge \tau^C, X_{s \wedge \tau^C}^C) = V^\xi(t, x) + \int_t^{s \wedge \tau^C} e^{-\gamma \int_t^r e^{-\delta u} C_u du} \cdot \left\{ V_t^\xi + (\mu - C_r)V_x^\xi + \frac{\sigma^2}{2} V_{xx}^\xi - \gamma e^{-\delta r} C_r V_r^\xi \right\} dr + \sigma \int_t^{s \wedge \tau^C} e^{-\gamma \int_t^r e^{-\delta u} C_u du} \cdot V_x^\xi dW_r.$$

Using the differential equation for V^ξ , one obtains, as in the last proof, using the definition of ψ from (7),

$$e^{-\gamma \int_t^{s \wedge \tau^C} e^{-\delta u} C_u du} \cdot V^\xi(s \wedge \tau^C, X_{s \wedge \tau^C}^C) = V^\xi(t, x) + \int_t^{s \wedge \tau^C} e^{-\gamma \int_t^r e^{-\delta u} C_u du} \cdot (C_r - \xi) \cdot e^{-\delta r} e^{-\Delta e^{-\delta r}} \psi(e^{-\delta r}, X_r^C) dr - \int_t^{s \wedge \tau^C} e^{-\gamma \int_t^r e^{-\delta u} C_u du} \cdot C_r e^{-\delta r} dr + \sigma \int_t^{s \wedge \tau^C} e^{-\gamma \int_t^r e^{-\delta u} C_u du} \cdot V_x^\xi dW_r.$$

Building the $\mathbb{P}_{(t,x)}$ -expectations, letting $s \rightarrow \infty$, and rearranging the terms, one has

$$V^C(t, x) = V^\xi(t, x) + \mathbb{E}_{(t,x)} \left[\int_t^{\tau^C} e^{-\gamma \int_t^r e^{-\delta u} C_u du} \cdot (C_r - \xi) \cdot e^{-\delta r} e^{-\Delta e^{-\delta r}} \psi(e^{-\delta r}, X_r^C) dr \right].$$

Our goal is to find a C -independent estimate for the expectation on the right-hand side above, in order to gain a bound for the difference $V(t, x) - V^\xi(t, x)$. Since $e^{-\gamma \int_t^r e^{-\delta u} C_u du} \leq 1$, $e^{-\Delta e^{-\delta r}} \leq 1$, and $-(C_r - \xi) \leq \xi$, we have

$$\begin{aligned} & \mathbb{E}_{(t,x)} \left[\int_t^{\tau^C} e^{-\gamma \int_t^r e^{-\delta u} C_u du} \cdot (C_r - \xi) \cdot e^{-\delta r} e^{-\Delta e^{-\delta r}} \psi(e^{-\delta r}, X_r^C) dr \right] \\ & \leq -\xi \mathbb{E}_{(t,x)} \left[\int_t^{\tau^C} e^{-\gamma \int_t^r e^{-\delta u} C_u du} \cdot e^{-\delta r} e^{-\Delta e^{-\delta r}} \psi(e^{-\delta r}, X_r^C) \mathbf{1}_{\{\psi(e^{-\delta r}, X_r^C) < 0\}} dr \right] \\ & \leq -\xi \mathbb{E}_{(t,x)} \left[\int_t^{\tau^C} e^{-\delta r} \psi(e^{-\delta r}, X_r^C) \mathbf{1}_{\{\psi(e^{-\delta r}, X_r^C) < 0\}} dr \right]. \end{aligned}$$

Now, inserting the power series representation of ψ from (7), one gets

$$\begin{aligned} & -\xi \mathbb{E}_{(t,x)} \left[\int_t^{\tau^C} e^{-\delta r} \psi(e^{-\delta r}, X_r^C) \mathbf{1}_{\{\psi(e^{-\delta r}, X_r^C) < 0\}} dr \right] \\ & \leq \xi \sum_{n=0}^{\infty} e^{-\delta t(n+1)} \frac{\Delta^n}{n!} \mathbb{E}_{(t,x)} \left[\int_t^{\tau^C} e^{-\delta(r-t)(n+1)} \left(\frac{-\eta_{n+1}\xi}{\delta(n+1)} e^{\eta_{n+1}X_r^C} - e^{\eta_n X_r^C} \right)^+ dr \right] \\ & \leq \xi e^{-\delta t} \sum_{n=0}^{\infty} e^{-\delta n} \frac{\Delta^n}{n!} \int_0^\infty \left(\frac{-\eta_{n+1}\xi}{\delta(n+1)} e^{\eta_{n+1}y} - e^{\eta_n y} \right)^+ f_{n+1}(x, y) dy, \end{aligned}$$

where the last inequality follows from Theorem A.1. □

Remark 3.2. One might wonder whether the infinite sum appearing on the right-hand side of Theorem 3.2 is finite. In order to see its finiteness we try to find an upper bound of the form A^n for the integral. To this end we split the integral into two parts: the part from 0 to x and the remaining part. Since $\theta_n \rightarrow \infty$ while $\eta_n, \zeta_n \rightarrow -\infty$ for $n \rightarrow \infty$, we have for $0 \leq y \leq x$ that

$$\begin{aligned} f_n(x, y) & := \frac{2(1 - e^{(\zeta_n - \theta_n)y}) e^{\eta_n(x-y)}}{\sigma^2 ((\theta_n - \eta_n) - (\zeta_n - \eta_n)e^{(\zeta_n - \theta_n)y})} \\ & \leq K_1 \frac{1}{\theta_n - \eta_n} \\ & \leq K_1 \end{aligned}$$

for some suitable constant $K_1 > 0$ (not depending on n and y). For $0 \leq x \leq y$ we find

$$\begin{aligned} f_n(x, y) & := \frac{2(e^{\theta_n(x-y)} - e^{\zeta_n x - \theta_n y})}{\sigma^2 ((\theta_n - \eta_n) - (\zeta_n - \eta_n)e^{y(\zeta_n - \theta_n)})} \\ & \leq K_2 e^{\theta_n(x-y)} \end{aligned}$$

for some suitable constant $K_1 > 0$ (not depending on n and y). The bracket appearing inside the integral before f_{n+1} is bounded by some constant $K_3 > 0$. We find that

$$\int_0^\infty \left(\frac{-\eta_{n+1}\xi}{\delta(n+1)} e^{\eta_{n+1}y} - e^{\eta_n y} \right)^+ f_{n+1}(x, y) dy \leq xK_1K_3 + \frac{K_2K_3}{\theta_n} \leq K_4$$

for some suitable constant $K_4 > 0$. Hence, the sum is bounded by

$$\exp(\Delta e^{-\delta t})K_4.$$

4. The goodness of constant barrier strategies

Shreve *et al.* [20] and Asmussen and Taksar [5] considered the problem of dividend maximisation for a surplus described by a Brownian motion with drift. The optimal strategy there turned out to be a barrier strategy with a constant barrier.

Let $q \in \mathbb{R}_+$ and let C be given by $C_s = \xi \mathbb{1}_{\{X_s^C > q\}}$; i.e., C is a barrier strategy with a constant barrier q and ruin time $\tau^C = \inf\{s \geq 0 : X_s^C = 0\}$. By the Markov property of X^C , the corresponding return function satisfies

$$V^C(t, x) = \frac{1}{\gamma} - \frac{1}{\gamma} \mathbb{E}_x \left[e^{-\gamma \int_t^{t+\tau^C} e^{-\delta s} C_s ds} \right].$$

Note that for every $a > 0$ we have

$$\mathbb{E}_x \left[e^{a \int_t^{t+\tau^C} e^{-\delta s} C_s ds} \right] \leq e^{a \int_t^\infty e^{-\delta s} \xi ds} = e^{\frac{a\xi}{\delta} e^{-\delta t}} < \infty.$$

This means the moment generating function of $\int_t^{t+\tau^C} e^{-\delta s} C_s ds$ is infinitely often differentiable and all moments of $\int_t^{t+\tau^C} e^{-\delta s} C_s ds$ exist.

Aiming to find the performance function of a barrier strategy with a constant barrier q , we use the classical ansatz of calculating the performance ‘above the barrier’ and ‘below the barrier’ and putting these two solutions together via the smooth fit at the barrier (in our case a $\mathcal{C}^{(1,1)}$ fit). We define

$$M_n(q) := \mathbb{E}_q \left[\left(\Delta - \gamma \int_0^{\tau^C} e^{-\delta s} C_s ds \right)^n \right] > 0.$$

Since a barrier strategy depends on the surplus, but not on the time, we pretend to start at time 0, accounting for a different starting time $t > 0$ by shifting the corresponding stopping times by t . Starting at $x > q$, one pays at the maximal rate ξ up to $\tau^{q,\xi}$ and then follows the barrier strategy with the starting value q . Starting at $x < q$, one does not pay dividends until $\tau^{q,0}$, i.e. until the surplus hits the level q or ruins. If the level q will be hit before ruin, then one follows the barrier strategy starting at q . This means in particular that after the level q is hit, the strategy will be exactly the same regardless of whether one starts at $x > q$ or at $x < q$. We will use this fact to enforce a smooth fit ($\mathcal{C}^{(1,1)}$ fit) at the barrier. A $\mathcal{C}^{(1,2)}$ fit can usually be achieved simply by a barrier strategy, which turns out to be the optimal strategy and whose performance function is the value function. See for instance [5] and [19] for details; further explanations are given in Section 5.1. Figure 1 illustrates the $\mathcal{C}^{(1,1)}$ fit of the return function corresponding to the 5-barrier. The grey and black areas correspond to the ‘above the barrier’

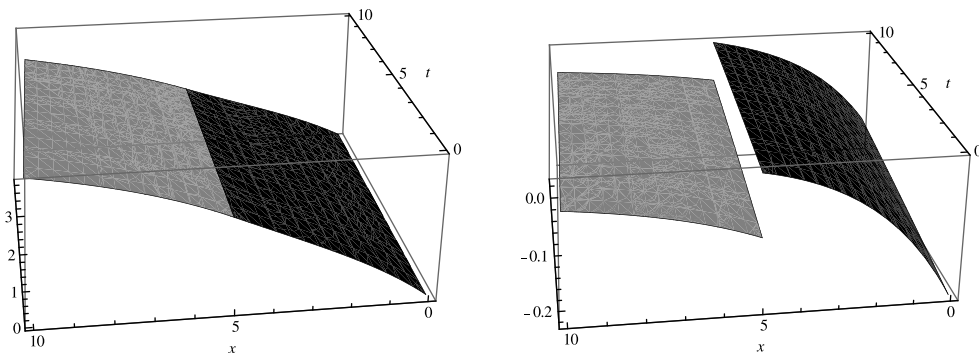


FIGURE 1. The return function corresponding to a 5-barrier strategy and its second derivative with respect to x .

and ‘below the barrier’ solutions. The right panel shows that the second derivative with respect to x of the performance function is not continuous at the barrier.

For $F(t, x) := V^C(t, x)$, $x > q$, and for $G(t, x) := V^C(t, x)$, $x < q$, it holds that

$$\begin{aligned}
 F(t, x) &= \frac{1}{\gamma} - \frac{1}{\gamma} \mathbb{E}_x \left[e^{-\gamma \xi \int_t^{t+\tau^{q,\xi}} e^{-\delta s} ds} - \gamma \int_t^{t+\tau^C} e^{-\delta s} C_s ds \right] \\
 &= \frac{1}{\gamma} - \frac{1}{\gamma} \mathbb{E}_x \left[\exp \left(e^{-\delta t} \left(-\Delta (1 - e^{-\delta \tau^{q,\xi}}) - \gamma \int_{\tau^{q,\xi}}^{\tau^C} e^{-\delta s} C_s ds \right) \right) \right] \\
 &= \frac{1}{\gamma} - \frac{1}{\gamma} e^{-\Delta e^{-\delta t}} \mathbb{E}_x \left[\exp \left(e^{-\delta t} e^{-\delta \tau^{q,\xi}} \left(\Delta - \gamma \int_0^{\tau^C - \tau^{q,\xi}} e^{-\delta s} C_{s+\tau^{q,\xi}} ds \right) \right) \right] \\
 &= \frac{1}{\gamma} - \frac{1}{\gamma} e^{-\Delta e^{-\delta t}} - \frac{1}{\gamma} e^{-\Delta e^{-\delta t}} \sum_{n=1}^{\infty} \frac{e^{-\delta t n}}{n!} \mathbb{E}_x [e^{-\delta n \tau^{q,\xi}}] \mathbb{E}_q \left[\left(\Delta - \gamma \int_0^{\tau^C} e^{-\delta s} C_s ds \right)^n \right] \\
 &= \frac{1}{\gamma} - \frac{1}{\gamma} e^{-\Delta e^{-\delta t}} - \frac{1}{\gamma} e^{-\Delta e^{-\delta t}} \sum_{n=1}^{\infty} \frac{e^{-\delta t n}}{n!} e^{\eta n(x-q)} M_n(q) \\
 &= -\frac{1}{\gamma} \sum_{n=1}^{\infty} \frac{e^{-\delta t n}}{n!} \sum_{k=0}^n \binom{n}{k} (-\Delta)^{n-k} M_k(q) e^{\eta k(x-q)}, \tag{10}
 \end{aligned}$$

$$\begin{aligned}
 G(t, x) &= \mathbb{E}_x [F(t + \tau^{q,0}, q); X_{\tau^{q,0}}^0 = q] \\
 &= -\frac{1}{\gamma} \sum_{n=1}^{\infty} \frac{e^{-\delta t n}}{n!} \cdot \frac{e^{\theta n x} - e^{\zeta n x}}{e^{\theta n q} - e^{\zeta n q}} \sum_{k=0}^n \binom{n}{k} (-\Delta)^{n-k} M_k(q). \tag{11}
 \end{aligned}$$

Here, for the fourth equality, we expand the first exponential function in the expectation into its power series and use the Markov property to see that the $\mathbb{P}_{0,x}$ -law given $\mathcal{F}_{\tau^{q,\xi}}$ of $\tau^C - \tau^{q,\xi}$ equals the $\mathbb{P}_{0,q}$ -law of τ^C . Also, for the last equality for G , we insert the formula for F and use the identities given in Borodin and Salminen [9, p. 309, Formula 3.0.5(b)]. The notation used in G means $\mathbb{E}_x[Y_t; A] = \mathbb{E}_x[Y_t \mathbb{1}_A]$ for some process Y .

In order to analyse the performance function of a barrier strategy, we will expand the performance function into integer powers of $e^{-\delta t}$ with x -dependent coefficients and truncate at some N . This will result in an approximation for the performance function which is much easier

to handle, but which incurs an additional truncation error. Inspecting Equations (10) and (11) motivates the approximations

$$F^N(t, x) = \sum_{n=1}^N e^{-\delta t n} \sum_{k=0}^n A_{n,k} e^{\eta k(x-q)}, \tag{12}$$

$$G^N(t, x) := \sum_{n=1}^N D_n e^{-\delta t n} \frac{e^{\theta_n x} - e^{\zeta_n x}}{e^{\theta_n q} - e^{\zeta_n q}}, \tag{13}$$

for $x, t \geq 0$, where $\eta_0 := 0$. In order to achieve a $C^{(1,1)}$ fit we choose $D_n := \sum_{k=0}^n A_{n,k}$ and

$$A_{n,n} := \frac{\sum_{k=0}^{n-1} (v_n - \eta_k) A_{n,k}}{\eta_n - v_n}, \quad v_n := \frac{\theta_n e^{\theta_n q} - \zeta_n e^{\zeta_n q}}{e^{\theta_n q} - e^{\zeta_n q}}.$$

This leaves the choice for $A_{n,0}, \dots, A_{n,k-1}$ open, which we now motivate by inspecting the dynamics equations for F, G ; these should be

$$\begin{aligned} G_t(t, x) + \mu G_x(t, x) + \frac{\sigma^2}{2} G_{xx}(t, x) &= 0, \\ F_t(t, x) + \mu F_x(t, x) + \frac{\sigma^2}{2} F_{xx}(t, x) &= \xi \left(F_x(t, x) + e^{-\delta t} (\gamma F(t, x) - 1) \right), \end{aligned}$$

with boundary condition $G(t, 0) = 0$ for $t, x \geq 0$.

It is easy to verify that $G^N(t, 0) = 0$ and $G_t^N(t, x) + \mu G_x^N(t, x) + \frac{\sigma^2}{2} G_{xx}^N(t, x) = 0$. However, since $H_k(x) := e^{\eta_k x}$ solves the equation

$$\delta k H_k(x) = (\mu - \xi) \partial_x H_k(x) + \frac{\sigma^2}{2} \partial_x^2 H_k(x),$$

we find that

$$\begin{aligned} F_t^N(t, x) + (\mu - \xi) F_x^N(t, x) + \frac{\sigma^2}{2} F_{xx}^N(t, x) &= \sum_{n=1}^N e^{-\delta t n} \sum_{k=0}^{n-1} \delta(k-n) A_{n,k} e^{\eta_k(x-q)}, \\ e^{-\delta t} \xi (\gamma F^N(t, x) - 1) &= -e^{-\delta t} \xi + \sum_{n=2}^{N+1} e^{-\delta t n} \sum_{k=0}^{n-1} \gamma \xi A_{n-1,k} e^{\eta_k(x-q)}. \end{aligned}$$

We will treat the term

$$e^{-\delta t(N+1)} \xi \gamma \sum_{k=0}^N A_{N,k} e^{\eta_k(x-q)}$$

as an error term and otherwise equate the two expressions above. This allows us to define the remaining coefficients, which are given by

$$\begin{aligned} A_{n,k} &:= \frac{\gamma \xi A_{n-1,k}}{\delta(k-n)} = \left(-\frac{\gamma \xi}{\delta} \right)^{n-k} \frac{A_{k,k}}{(n-k)!} = (-\Delta)^{n-k} \frac{A_{k,k}}{(n-k)!}, \\ A_{n,0} &:= \left(-\frac{\gamma \xi}{\delta} \right)^{n-1} \frac{\xi}{\delta n!} = \frac{(-\gamma)^{n-1} \xi^n}{\delta^n n!} = \frac{(-\Delta)^n}{-\gamma n!} \end{aligned}$$

for $n \geq k \geq 1$; the last line is also valid for $n = 0$.

The following lemma shows that F^N solves ‘almost’ the same equation as F is thought to solve. We see an error term which, instead of being zero, converges faster than $e^{-\delta t N}$ for time going to infinity.

Lemma 4.1. *We have*

$$G_t^N(t, x) + \mu G_x^N(t, x) + \frac{\sigma^2}{2} G_{xx}^N(t, x) = 0,$$

$$F_t^N(t, x) + \mu F_x^N(t, x) + \frac{\sigma^2}{2} F_{xx}^N(t, x) + \xi \psi^N(e^{-\delta t}, x) = -e^{-\delta t(N+1)} \xi \gamma \sum_{k=0}^N A_{N,k} e^{\eta k(x-q)},$$

for any $t \geq 0, x \geq q$, where

$$\psi^N(e^{-\delta t}, x) := -F_x^N(t, x) + e^{-\delta t}(1 - \gamma F^N(t, x)).$$

Proof. The claim follows from inserting the definitions of G^N and F^N . □

We define

$$V^N(t, x) := \mathbb{1}_{\{x \geq q\}} F^N(t, x) + \mathbb{1}_{\{x < q\}} G^N(t, x), \tag{14}$$

$$\psi^N(e^{-\delta t}, x) := -V_x^N(t, x) + e^{-\delta t}(1 - \gamma V^N(t, x))$$

for any $t, x \geq 0$. We now want to compare the approximate performance function V^N for the barrier strategy with level q to the unknown value function. We proceed by first bounding ψ^N in terms of a double power series in $e^{-\delta t}$ and x -dependent exponentials.

Lemma 4.2. *With the preceding definitions, for $x \geq q$ we have*

$$-\psi^N(e^{-\delta t}, x) \mathbb{1}_{\{\psi^N(e^{-\delta t}, x) < 0\}} \leq \sum_{n=1}^{N+1} e^{-\delta t n} \left(\sum_{k=0}^n e^{\eta k(x-q)} \{ \mathbb{1}_{\{n=1, k=0\}} - \mathbb{1}_{\{n \neq N+1\}} \eta k A_{n,k} - \mathbb{1}_{\{n \neq 1, k \neq n\}} \gamma A_{n-1,k} \} \right)^+,$$

and for $0 \leq x < q$ we have

$$\psi^N(e^{-\delta t}, x) \mathbb{1}_{\{\psi^N(e^{-\delta t}, x) > 0\}} \leq \sum_{n=1}^{N+1} e^{-\delta t n} \left(\mathbb{1}_{\{n=1\}} - D_n h'_n(x) \mathbb{1}_{\{n \neq N+1\}} - \gamma D_{n-1} h_{n-1}(x) \mathbb{1}_{\{n \neq 1\}} \right)^+,$$

where

$$h_n(x) := \frac{e^{\theta_n x} - e^{\zeta_n x}}{e^{\theta_n q} - e^{\zeta_n q}}.$$

Proof. Inserting the definition of ψ^N and the definitions of F^N and G^N found in Equations (12) and (13) respectively, for $x \geq q$ (with $\eta_0 = 0$) we obtain

$$\psi^N(e^{-\delta t}, x) = \sum_{n=1}^{N+1} e^{-\delta t n} \sum_{k=1}^n e^{\eta k(x-q)} \left(\mathbb{1}_{\{n=1, k=0\}} - \mathbb{1}_{\{n \neq N+1\}} \eta k A_{n,k} - \mathbb{1}_{\{n \neq 1, k \neq n\}} \gamma A_{n-1,k} \right),$$

and for $0 \leq x < q$ we obtain

$$\psi^N(e^{-\delta t}, x) = \sum_{n=1}^{N+1} e^{-\delta t n} \left(\mathbb{1}_{\{n=1\}} - D_n h'_n(x) \mathbb{1}_{\{n \neq N+1\}} - \gamma D_{n-1} h_{n-1}(x) \mathbb{1}_{\{n \neq 1\}} \right).$$

Using the inequality $\left(\sum_{n=1}^N e^{-\delta t n} c_n \right)^+ \leq \sum_{n=1}^N e^{-\delta t n} (c_n)^+$ for $c \in \mathbb{R}^N$, for $x \geq q$ we obtain

$$\begin{aligned} & -\psi^N(e^{-\delta t}, x) \mathbb{1}_{\{\psi^N(e^{-\delta t}, x) < 0\}} \\ & \leq \sum_{n=1}^{N+1} e^{-\delta t n} \left(\sum_{k=1}^n e^{\eta_k(x-q)} \left\{ \mathbb{1}_{\{n=1, k=0\}} - \mathbb{1}_{\{n \neq N+1\}} \eta_k A_{n,k} - \mathbb{1}_{\{n \neq 1, k \neq n\}} \gamma A_{n-1,k} \right\} \right)^+ \end{aligned}$$

and for $0 \leq x < q$ we obtain

$$\begin{aligned} \psi^N(e^{-\delta t}, x) \mathbb{1}_{\{\psi^N(e^{-\delta t}, x) > 0\}} & \leq \sum_{n=1}^{N+1} e^{-\delta t n} \left(\mathbb{1}_{\{n=1\}} - D_n h'_n(x) \mathbb{1}_{\{n \neq N+1\}} \right. \\ & \quad \left. - \gamma D_{n-1} h_{n-1}(x) \mathbb{1}_{\{n \neq 1\}} \right)^+, \end{aligned}$$

as claimed. □

We will employ the same method as in Section 3.2 and rely on the occupation bounds from Theorem A.1. We have in mind that $V^N \approx V^C \leq V$. The three error terms appearing on the right-hand side of the following proposition are, in order, the error for behaving suboptimally above the barrier, the error for behaving suboptimally below the barrier, and the approximation error.

Proposition 4.1. *We have*

$$\begin{aligned} V(t, x) & \leq V^N(t, x) + \sum_{n=1}^{N+1} e^{-\delta t n} \xi \left[\right. \\ & \left(\sum_{k=0}^n \left\{ \mathbb{1}_{\{n=1, k=0\}} - \mathbb{1}_{\{n \neq N+1\}} \eta_k A_{n,k} - \mathbb{1}_{\{n \neq 1, k \neq n\}} \gamma A_{n-1,k} \right\} \cdot \int_q^\infty e^{\eta_k(y-q)} f_n(x, y) dy \right)^+ \\ & + \int_0^q \left(-D_n \frac{\theta_n e^{\theta_n y} - \zeta_n e^{\zeta_n y}}{e^{\theta_n q} - e^{\zeta_n q}} + \left(\mathbb{1}_{\{n=1\}} - \gamma \mathbb{1}_{\{n \neq 1\}} D_{n-1} \frac{e^{\theta_{n-1} y} - e^{\zeta_{n-1} y}}{e^{\theta_{n-1} q} - e^{\zeta_{n-1} q}} \right)^+ \cdot f_n(x, y) dy \right) \\ & \left. + e^{-\delta t(N+1)} \xi \gamma \int_0^\infty \sum_{k=0}^N |A_{N,k}| e^{\eta_k(y-q)} f_{N+1}(x, y) dy \right] \end{aligned}$$

for any $t, x \geq 0$, where the f_k are defined in Theorem 3.2.

Proof. Observe that V^N is analytic outside the barrier q and $\mathcal{C}^{(1,\infty)}$ on $\mathbb{R}_+ \times \mathbb{R}_+$, and the second space derivative is a bounded function. Thus, we can apply the change-of-variables formula; see [17].

Choose an arbitrary strategy \bar{C} and denote its ruin time by τ . Following the heuristics from Section 2.1 up to Step 4 with $H = V^N$ for the strategy \bar{C} yields

$$\begin{aligned} V^{\bar{C}}(t, x) &= V^N(t, x) + \mathbb{E}_{(t,x)} \left[\int_t^\tau e^{-\gamma \int_t^r e^{-\delta u} \bar{C}_u du} \left(\bar{C}_r - \xi \mathbb{I}_{\{X_r^{\bar{C}} > q\}} \right) \psi^N(e^{-\delta r}, X_r^{\bar{C}}) dr \right] \\ &\quad - \xi \gamma \mathbb{E}_{(t,x)} \left[\int_t^\tau e^{-\gamma \int_t^r e^{-\delta u} \bar{C}_u du} e^{-\delta r(N+1)} \mathbb{I}_{\{X_r^{\bar{C}} > q\}} \sum_{k=0}^N A_{N,k} e^{\eta k (X_r^{\bar{C}} - q)} dr \right] \\ &\leq V^N(t, x) + \xi \gamma \mathbb{E}_{(t,x)} \left[\int_t^\tau e^{-\delta r(N+1)} \mathbb{I}_{\{X_r^{\bar{C}} > q\}} \sum_{k=0}^N |A_{N,k}| e^{\eta k (X_r^{\bar{C}} - q)} dr \right] \\ &\quad + \mathbb{E}_{(t,x)} \left[\int_t^\tau \left(-\xi \mathbb{I}_{\{X_r^{\bar{C}} > q, \psi^N(e^{-\delta r}, X_r^{\bar{C}}) < 0\}} + \xi \mathbb{I}_{\{X_r^{\bar{C}} < q, \psi^N(e^{-\delta r}, X_r^{\bar{C}}) > 0\}} \right) \psi^N(e^{-\delta r}, X_r^{\bar{C}}) dr \right], \end{aligned}$$

where we used that $0 \leq \bar{C}_r \leq \xi$. Applying Lemma 4.2 to the last summand, pulling out the sum, and applying Theorem A.1 yields

$$\begin{aligned} V^{\bar{C}}(t, x) &\leq V^N(t, x) \\ &\quad + \sum_{n=1}^{N+1} e^{-\delta t n} \xi \left[\left(\sum_{k=0}^n \left\{ \mathbb{I}_{\{n=1, k=0\}} - \mathbb{I}_{\{n \neq N+1\}} \eta_k A_{n,k} - \mathbb{I}_{\{n \neq 1, k \neq n\}} \gamma A_{n-1,k} \right\} \right. \right. \\ &\qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \left. \left. \times \int_q^\infty e^{\eta k (y-q)} f_n(x, y) dy \right)^+ \right. \\ &\quad \left. + \int_0^q \left(\mathbb{I}_{\{n=1\}} - D_n h'_n(y) \mathbb{I}_{\{n \neq N+1\}} - \gamma D_{n-1} h_{n-1}(y) \mathbb{I}_{\{n \neq 1\}} \right)^+ f_n(x, y) dy \right] \\ &\quad + e^{-\delta t(N+1)} \xi \gamma \int_q^\infty \sum_{k=0}^N |A_{N,k}| e^{\eta k (y-q)} f_{N+1}(x, y) dy, \end{aligned}$$

where

$$h_n(y) := \frac{e^{\theta_n y} - e^{\xi_n y}}{e^{\theta_n q} - e^{\xi_n q}}.$$

Since \bar{C} was an arbitrary strategy and the right-hand side does not depend on \bar{C} , the claim follows. □

Now we quantify the notion $V^N \approx V^C$. Here, we see a single error term which corresponds to the approximation error (third summand) in Proposition 4.1.

Lemma 4.3. *Let $t, x \geq 0$. Then we have*

$$|V^N(t, x) - V^C(t, x)| \leq e^{-\delta t(N+1)} \xi \gamma \int_q^\infty \sum_{k=0}^N |A_{N,k}| e^{\eta k (y-q)} f_{N+1}(x, y) dy.$$

Proof. By following the lines of the proof of Proposition 4.1 with the specific strategy $\bar{C}_t = C_t = \xi \mathbb{I}_{\{X_t^C > q\}}$ until estimates are used, we obtain

$$\begin{aligned} V^C(t, x) &= V^N(t, x) + \mathbb{E}_{(t,x)} \left[\int_t^\tau e^{-\gamma \int_t^r e^{-\delta u} C_u du} (C_r - \xi \mathbb{I}_{\{X_r^C > q\}}) \psi^N(r, X_r^C) dr \right] \\ &\quad - \xi \gamma \mathbb{E}_{(t,x)} \left[\int_t^\tau e^{-\gamma \int_t^r e^{-\delta u} C_u du} e^{-\delta r(N+1)} \mathbb{I}_{\{X_r^C > q\}} \sum_{k=0}^N A_{N,k} e^{\eta_k (X_r^C - q)} dr \right] \\ &= V^N(t, x) \\ &\quad - \xi \gamma \mathbb{E}_{(t,x)} \left[\int_t^\tau e^{-\gamma \int_t^r e^{-\delta u} C_u du} e^{-\delta r(N+1)} \mathbb{I}_{\{X_r^C > q\}} \sum_{k=0}^N A_{N,k} e^{\eta_k (X_r^C - q)} dr \right]. \end{aligned}$$

Hence, we find

$$\begin{aligned} |V^C(t, x) - V^N(t, x)| &\leq \xi \gamma \mathbb{E}_{(t,x)} \left[\int_t^\tau e^{-\delta r(N+1)} \mathbb{I}_{\{X_r^C > q\}} \sum_{k=0}^N |A_{N,k}| e^{\eta_k (X_r^C - q)} dr \right] \\ &= \xi \gamma e^{-\delta t(N+1)} \int_{\mathbb{R}} \mathbb{I}_{\{X_r^C > q\}} \sum_{k=0}^N |A_{N,k}| e^{\eta_k (y_r - q)} f_{N+1}(x, y) dy \end{aligned}$$

by Theorem A.1. □

5. Examples

Here, we consider two examples. The first one illustrates how the value function and the optimal strategy can be calculated using a straightforward approach under various unproven assumptions. In fact, we will assume (without proof) that the value function is smooth enough, that the optimal strategy is of barrier type, and that the barrier, the value function above the barrier, and the value function below the barrier have suitable power series representations. In [15] it has been observed that similar power series—if they exist—have very large coefficients for certain parameter choices. This could mean that the power series do not converge or that insufficient computing power is available.

In the second subsection, we will illustrate the new approach and calculate the distance from the performance function of a constant barrier strategy to the value function. The key advantages of this approach are that we do not rely on properties of the value function, nor do we need to know what it looks like. From a practical perspective, if the value function cannot be found, one should simply choose any strategy with an easy-to-calculate return function. Then it is good to know how large the error to the optimal strategy is.

5.1. The straightforward approach

In this example we let $\mu = 0.15$, $\delta = 0.05$, $\gamma = 0.2$, and $\sigma = 1$. We attempt to find the value function numerically. However, we do not know whether the assumptions which we will make do actually hold true for any possible parameters—or even for the parameters we chose.

We conjecture and assume that the optimal strategy is of a barrier type where the barrier is given by a time-dependent curve, say α ; the value function $V(t, x)$ is assumed to be a $C^{1,2}(\mathbb{R}_+^2)$ function, and we define

$$\begin{aligned} h(t, x) &:= V(t, x), & t \geq 0, \quad x \in [\alpha(t), \infty), \\ g(t, x) &:= V(t, x), & t \geq 0, \quad x \in [0, \alpha(t)]. \end{aligned}$$

This means we assume that h solves the HJB equation (1) on $\mathbb{R}_+ \times [\alpha(t), \infty)$ and that g solves (1) on $\mathbb{R}_+ \times [0, \alpha(t)]$. In particular, the functions h and g satisfy

$$\begin{aligned} h_t + (\mu - \xi)h_x + \frac{\sigma^2}{2}h_{xx} + \xi e^{-\delta t}(1 - \gamma h) &= 0, & \lim_{x \rightarrow \infty} h(t, x) &= U\left(\frac{\xi e^{-\delta t}}{\delta}\right), \\ g_t + \mu g_x + \frac{\sigma^2}{2}g_{xx} &= 0, & g(t, 0) &\equiv 0. \end{aligned}$$

Similarly to the derivations of the functions F and G in Section 4, we assume that

$$\begin{aligned} h(t, x) &:= \frac{1}{\gamma} - \frac{1}{\gamma}e^{-\Delta e^{-\delta t}} + e^{-\Delta e^{-\delta t}} \sum_{n=1}^{\infty} J_n e^{-\delta m} e^{\eta_n x}, \\ g(t, x) &:= \sum_{n=1}^{\infty} L_n e^{-\delta m} (e^{\theta_n x} - e^{\zeta_n x}), \\ \alpha(t) &:= \sum_{n=0}^{\infty} \frac{a_n}{n!} e^{-\delta m} \end{aligned}$$

for some coefficients. Note that we do not investigate the question of whether the functions h , g , and α have power series representations. We define further auxiliary coefficients $b_{k,n}$, $p_{k,n}$, and $q_{k,n}$:

$$e^{\eta_n \alpha(t)} =: \sum_{k=0}^{\infty} \frac{b_{k,n}}{k!} e^{-\delta tk}, \quad e^{\theta_n \alpha(t)} =: \sum_{k=0}^{\infty} \frac{p_{k,n}}{k!} e^{-\delta tk}, \quad e^{\zeta_n \alpha(t)} =: \sum_{k=0}^{\infty} \frac{q_{k,n}}{k!} e^{-\delta tk}.$$

Since we assume that the value function is twice continuously differentiable with respect to x , we have, using smooth fit,

$$h(t, \alpha(t)) = g(t, \alpha(t)), \quad g_x(t, \alpha(t)) = h_x(t, \alpha(t)), \quad g_{xx}(t, \alpha(t)) = h_{xx}(t, \alpha(t)). \tag{15}$$

Note that (15) yields $h_t(t, \alpha(t)) = g_t(t, \alpha(t))$. Therefore, we can conclude that $h_x(t, \alpha(t)) = e^{-\delta t}(1 - \gamma h(t, \alpha(t)))$. Alternatively to (15), one can consider at $(t, \alpha(t))$ the equations

$$-h_x + e^{-\delta t}(1 - \gamma h) = 0, \quad -g_x + e^{-\delta t}(1 - \gamma g) = 0, \quad h = g. \tag{16}$$

Thus, we can find the coefficients a_n , J_n , and L_n from the three equations (16).

First, we calculate the coefficients of the power series resulting from the functions $e^{\eta_n \alpha(t)}$, $e^{\theta_n \alpha(t)}$, $e^{\zeta_n \alpha(t)}$. This is done using the general Leibniz rule:

$$\begin{aligned}
 b_{k+1,n} &= \eta_n \sum_{j=0}^k \binom{k}{j} a_{k-j+1} b_{j,n}, & b_{0,n} &= e^{\eta_n \alpha(0)}, \\
 p_{k+1,n} &= \theta_n \sum_{j=0}^k \binom{k}{j} a_{k-j+1} p_{j,n}, & p_{0,n} &= e^{\theta_n \alpha(0)}, \\
 q_{k+1,n} &= \zeta_n \sum_{j=0}^k \binom{k}{j} a_{k-j+1} q_{j,n}, & q_{0,n} &= e^{\zeta_n \alpha(0)}.
 \end{aligned}$$

Now, in order to calculate the coefficients in the power series representation of $h(t, \alpha(t))$ and $g(t, \alpha(t))$ and their derivatives, we define auxiliary coefficients for $m \in \{1, 2\}$:

$$\begin{aligned}
 X_{m,j} &:= \sum_{n=1}^j J_n \eta_n^{m-1} \frac{b_{j-n,n}}{(j-n)!}, & Z_{m,k} &:= \sum_{j=1}^k \frac{\Delta^{k-j}}{(k-j)!} X_{m,j}, \\
 W_{m,k,j} &:= L_j (\theta_j^{m-1} p_{k,j} - \zeta_j^{m-1} q_{k,j}), & Y_{m,k} &:= \sum_{n=1}^k \frac{W_{m,k-n,n}}{(k-n)!}.
 \end{aligned}$$

Then we can write the functions g and h along with their derivatives as power series:

$$\begin{aligned}
 g(t, \alpha(t)) &= \sum_{k=1}^{\infty} e^{-\delta tk} Y_{1,k}, & h(t, \alpha(t)) &= \sum_{k=1}^{\infty} e^{-\delta tk} Z_{1,k} - \frac{1}{\gamma} \sum_{k=1}^{\infty} (-\Delta)^k \frac{e^{-\delta tk}}{k!}, \\
 g_x(t, \alpha(t)) &= \sum_{k=1}^{\infty} e^{-\delta tk} Y_{2,k}, & h_x(t, \alpha(t)) &= \sum_{k=1}^{\infty} e^{-\delta tk} Z_{2,k}.
 \end{aligned}$$

Equating coefficients yields

$$a_0 = \frac{\log\left(\frac{\eta_1 - \zeta_1}{\eta_1 - \theta_1} \cdot \frac{\zeta_1}{\theta_1}\right)}{\theta_1 - \zeta_1}, \quad L_1 = \frac{1}{\theta_1 e^{\theta_1 a_0} - \zeta_1 e^{\zeta_1 a_0}}, \quad J_1 = \frac{e^{-\eta_1 a_0}}{\eta_1},$$

and for $k \geq 2$,

$$X_{2,k} = -\gamma X_{1,k-1}, \quad Y_{2,k} = -\gamma Y_{1,k-1}, \quad Y_{1,k} = Z_{1,k} - \frac{(-\Delta)^k}{\gamma k!}. \tag{17}$$

Note that the equations in (17) specify L_k , J_k , and a_{k-1} at the k th step. The coefficients given above have a recursive structure. Because of this fact, the method presented turns out to be very time- and memory-consuming. Numerical calculations show that the above procedure yields well-defined power series for relative small values of ξ (see Figure 2). However, for big ξ the coefficients explode, which makes the calculations unstable and imprecise, especially for t close to zero.

In Figure 3 we see the functions h (black) and g (grey) meeting at the barrier $\alpha(t)$ in the left panel. The right panel illustrates the crucial functions $-h_x + e^{-\delta t}(1 - \gamma h)$ (black) and

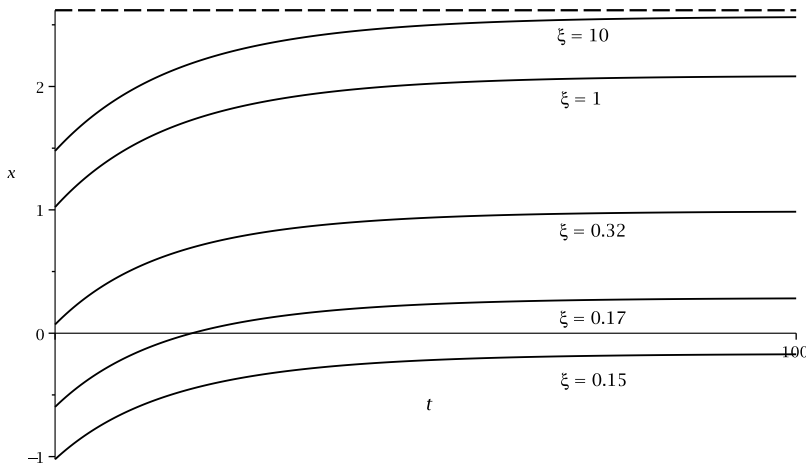


FIGURE 2. The optimal strategies for different values of ξ . The dashed line corresponds to the Asmussen–Taksar strategy [5] (unrestricted dividend case).

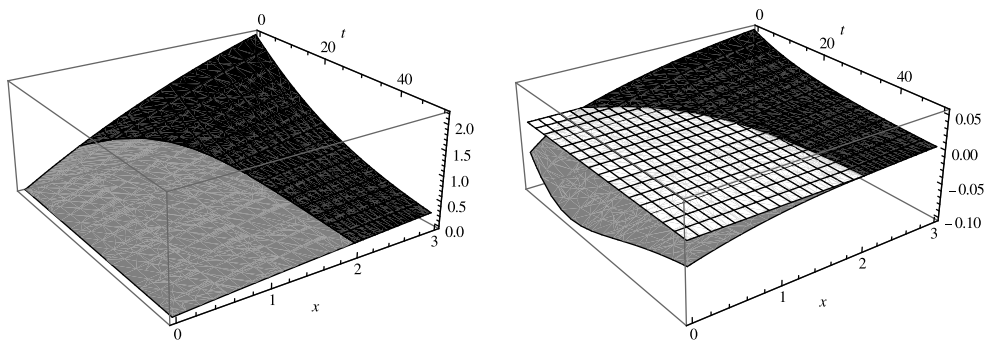


FIGURE 3. Left, the functions $h(t, x)$ (black) and $g(t, x)$ (grey); right, the functions $-h_x + e^{-\delta t}(1 - \gamma h)$ (black), $-g_x + e^{-\delta t}(1 - \gamma g)$ (grey), and 0 (white), for $\xi = 1$.

$-g_x + e^{-\delta t}(1 - \gamma g)$ (grey), along with the zero-plane (white). One sees that the zero-plane cuts $-h_x + e^{-\delta t}(1 - \gamma h)$ and $-g_x + e^{-\delta t}(1 - \gamma g)$ exactly along the curve α .

Note that the numerical procedure used here works well only for small values of ξ . Because of the recursive structure of the coefficients, the bigger ξ values cause the coefficients to explode and enforce an early truncation of the power series representations.

It should be noted here once again that the obtained functions h and g do not represent the value function. And the optimal strategy cannot yet be claimed to be of a barrier type with the barrier given by α ; first, one has to prove a verification theorem.

5.2. The distance to the value function

We use the same parameters as in the previous section, i.e. $\mu = 0.15$, $\delta = 0.05$, $\gamma = 0.2$, and $\sigma = 1$. We illustrate the error bound given by Proposition 4.1 for $N = 20$ summands and

four different values of ξ , namely 0.15, 0.17, 0.32, and 1. We will compare the unknown value function to the performance of the barrier strategy with barrier at

$$q = \left(\frac{\log(-\zeta_1) + \log(\zeta_1 + \eta_1) - \log(\theta_1) - \log(\theta_1 - \eta_1)}{\theta_1 - \zeta_1} \right)^+; \tag{18}$$

i.e. we employ the strategy $C_s = \xi \mathbb{I}_{\{X_s^C \geq q\}}$. Recall the definition of η_1 , θ_1 , and ζ_1 from (2) and (5).

Mathematica code for the calculation of the coefficients J_n, L_n , and a_n .

```
(*The optimal strategy for xi=1, (g=gamma)*)
mu = 0.15; xi = 1; sigma = 1; delta = 0.05; g = 0.2; n = 500;

w = -xi*g/delta;

eta[k_] = xi - mu - Sqrt[(xi - mu)^2 + 2*delta*k];
zeta[k_] = -mu - Sqrt[mu^2 + 2*delta*k];
theta[k_] = -mu + Sqrt[mu^2 + 2*delta*k];

(*The coefficients*)
Array[L, n]; Array[J, n]; Array[a, n];

(*Auxiliary functions*)
p[k_, m_] := p[k, m] = If[m == 0, Exp[theta[k]*a[0]],
  theta[k]*Sum[Binomial[m - 1, j]*a[m - j]*p[k, j], {j, 0, m - 1}]];
q[k_, m_] := q[k, m] = If[m == 0, Exp[zeta[k]*a[0]],
  zeta[k]*Sum[Binomial[m - 1, j]*a[m - j]*q[k, j], {j, 0, m - 1}]];
b[k_, m_] := b[k, m] = If[m == 0, Exp[eta[k]*a[0]],
  eta[k]*Sum[Binomial[m - 1, j]*a[m - j]*b[k, j], {j, 0, m - 1}]];

X1[k_] := X1[k] = Sum[J[j]*b[j, k - j]/((k - j)!), {j, 1, k}];
X2[k_] := X2[k] = Sum[J[j]*b[j, k - j]/((k - j)!), {j, 2, k - 1}];
X3[k_] := X3[k] = Sum[J[j]*eta[j]*b[j, k - j]/((k - j)!), {j, 2, k - 1}];
X4[k_] := X4[k] = Sum[J[j]*eta[j]*b[j, k - j]/((k - j)!), {j, 1, k - 1}];
Z[k_] := Z[k] = Sum[w^(k - j)/(k - j)!*X1[j], {j, 1, k - 1}];

Y1[k_] := Y1[k] = Sum[L[j]*(p[j, k - j] - q[j, k - j])/((k - j)!), {j, 1, k}];
Y2[k_] := Y2[k] = Sum[L[j]*(p[j, k - j] - q[j, k - j])/((k - j)!), {j, 2, k - 1}];
Y3[k_] := Y3[k] = Sum[L[j]*(theta[j]*p[j, k - j] - zeta[j]*q[j, k - j])/((k - j)!), {j, 2, k - 1}];
Y4[k_] := Y4[k] = Sum[L[j]*(theta[j]*p[j, k - j] - zeta[j]*q[j, k - j])/((k - j)!), {j, 1, k - 1}];

r1[k_, j_] := r1[k, j] = (p[k, j] - q[k, j]);
r2[k_, j_] := r2[k, j] = (theta[k]*p[k, j] - zeta[k]*q[k, j]);
r3[k_, j_] := r3[k, j] = (theta[k]^2*p[k, j] - zeta[k]^2*q[k, j]);

(*The first coefficients*)
a[0] = 1/(theta[1] - zeta[1])*Log[(eta[1] - zeta[1])/(eta[1] - theta[1])*zeta[1]/theta[1]];
L[1] = 1/(theta[1]*p[1, 0] - zeta[1]*q[1, 0]);
J[1] = 1/(eta[1]*b[1, 0]);
L[2] = (w - eta[2]/eta[1]*w + w^2/(2*g)*eta[2])/(r2[2,0] - eta[2]*r1[2,0]);
J[2] = (L[2]*r2[2,0] - w)/(eta[2]*b[2, 0]);
a[1] = (-g*J[1]*b[1, 0] - J[2]*eta[2]*b[2, 0])/eta[1];

(*Recursions for the calculation of the remaining coefficients*)
f[k_] := f[k] = (-g*Y1[k - 1] - Y4[k])/r2[k,0];
h[k_] := h[k] = (-g*X1[k - 1] - X4[k])/(eta[k]*b[k, 0]);
v[k_] := v[k] = (k - 1)!/(L[1]*(r2[1, 0] - r1[k, 0]/r2[k, 0]*r3[1, 0])
  - J[1]*eta[1]*b[1, 0]*(1 - eta[1]/eta[k]))
*(Z[k] - w^k/(g*k!) + X2[k] - g/eta[k]*X1[k - 1] - 1/eta[k]*X3[k]
  - Y2[k] + g*r1[k, 0]/r2[k, 0]*Y1[k - 1] + r1[k, 0]/r2[k, 0]*Y3[k]
  - Sum[Binomial[k - 2, j]*a[k - 1 - j]/(k - 1)!*(L[1]*(r2[1, j] - r1[k, 0]/r2[k, 0]*r3[1, j])
  - J[1]*b[1, j]*(eta[1] - eta[1]^2/eta[k])], {j, 1, k - 2}]);

For[i = 3, i <= n, i++, a[i - 1] = v[i]; L[i] = f[i]; J[i] = h[i]];
```

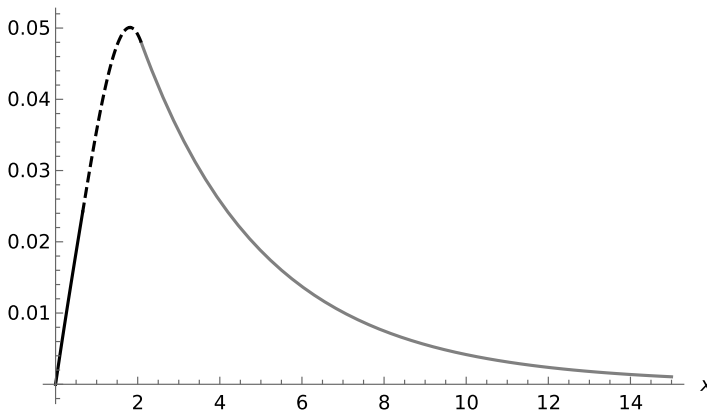


FIGURE 4. The difference of the value function and an approximation of the performance function corresponding to a constant barrier strategy at $t = 0$: $V(0, x) - V^N(0, x)$ for $\xi = 1$ with V^N given in (14) and the barrier q given in (18).

The barrier strategy with the barrier q has been shown to be optimal if no utility function is applied; see [19, p. 97]. In the case of $\xi = 0.15$ one finds $q = 0$, i.e. we pay out at maximal rate all the time, which is optimal by Proposition 3.1. Therefore, this case is left with only an approximation error. For the other values of ξ , it is non-optimal to follow a barrier strategy, and hence we do have a substantial error which cannot disappear in the limit. The corresponding panels in Figure 5 show this error, as for $N = 20$ summands the approximation error is already several magnitudes smaller than the error incurred by following a suboptimal strategy.

Figure 4 illustrates for $\xi = 1$ the difference between the value function $V(x)$ and the approximation V^N , given in (14), of the performance function corresponding to the barrier strategy with the barrier q given in (18) at $t = 0$. Note that the difference $V(0, x) - V^N(0, x)$ consists of three subfunctions:

$$V(0, x) - V^N(0, x) = \begin{cases} F(0, x) - F^N(0, x), & x \geq q, \text{ grey line in Figure 4,} \\ F(0, x) - G^N(0, x), & x \in [\alpha(0), q], \text{ dashed line in Figure 4,} \\ G(0, x) - G^N(0, x), & x \in [0, \alpha[0]], \text{ solid black line in Figure 4.} \end{cases}$$

It is clear that for any fixed x the maximal difference $V(t, x) - V^N(t, x)$ is attained at $t = 0$, as the curve α is increasing and converges to q for $t \rightarrow \infty$. Thus, the difference $q - \alpha(t)$ attains its maximum at $t = 0$, leading to a bigger difference between the performance functions.

Appendix

In this appendix we provide deterministic upper bounds for the expected discounted occupation of a process whose drift is not precisely known. This allows us to derive an upper bound for the expect discounted and cumulated positive functional of the process. These bounds are summarised in Theorem A.1.

Let $a, b \in \mathbb{R}$ with $a \leq b$, $I := [a, b]$, $\sigma > 0$, $\delta \geq 0$, W a standard Brownian motion, and consider the process

$$dX_t = C_t dt + \sigma dW_t$$

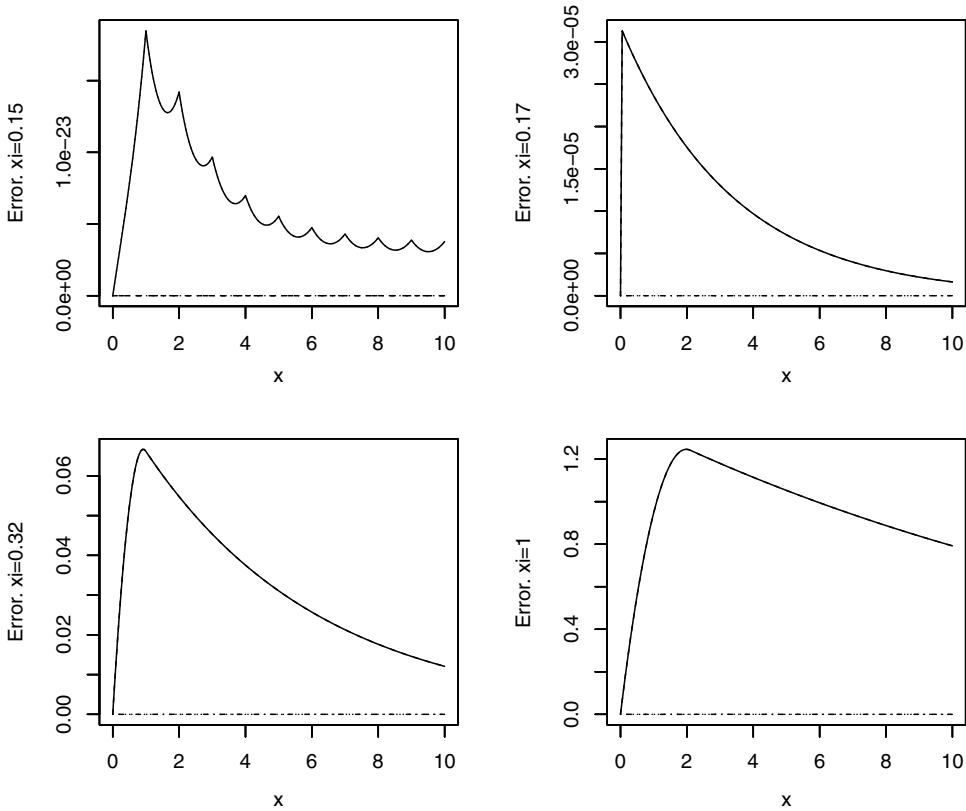


FIGURE 5. The plots show numerical evaluations of the error bounds given in Proposition 4.1 for the barrier strategy with parameters $\xi = 0.15$, $\xi = 0.17$, $\xi = 0.32$, and $\xi = 1$ respectively, as indicated at the side of each panel. The error bound is shown at time $t = 0$, where it is largest, across several values of x .

where C is some I -valued progressively measurable process. We recall that we denote by \mathbb{P}_x a measure with $\mathbb{P}_x[X_0 = x]$. The local time of X at level y and time t is denoted by L_t^y , and $\tau := \inf\{t \geq 0 : X_t = 0\}$. Furthermore, for $x, y \geq 0$ we define

$$\alpha := \frac{a + \sqrt{a^2 + 2\delta\sigma^2}}{\sigma^2}, \quad \beta_+ := \frac{\sqrt{b^2 + 2\delta\sigma^2} - b}{\sigma^2}, \quad \beta_- := \frac{-\sqrt{b^2 + 2\delta\sigma^2} - b}{\sigma^2},$$

$$f(x, y) := \frac{2(e^{\beta_+(x \wedge y)} - e^{\beta_-(x \wedge y)})e^{-\alpha(x-y)^+}}{\sigma^2((\beta_+ + \alpha)e^{y\beta_+} - (\beta_- + \alpha)e^{y\beta_-})}.$$

Theorem A.1. We have $\mathbb{E}_x \left[\int_0^\tau e^{-\delta s} dL_s^y \right] \leq \sigma^2 f(x, y)$. In particular, for any measurable function $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ we have

$$\mathbb{E}_x \left[\int_0^\tau e^{-\delta s} \psi(X_s) ds \right] \leq \int_0^\infty \psi(y) f(x, y) dy.$$

The proof is given at the end of this section.

Lemma A.1. *The function f is absolutely continuous in its first variable, with derivative*

$$f_x(x, y) := \begin{cases} \frac{2(\beta_+ e^{x\beta_+} - \beta_- e^{x\beta_-})}{\sigma^2((\beta_+ + \alpha)e^{y\beta_+} - (\beta_- + \alpha)e^{y\beta_-})}, & x \leq y, \\ \frac{2(-\alpha e^{y\beta_+} + \alpha e^{y\beta_-})e^{-\alpha(x-y)}}{\sigma^2((\beta_+ + \alpha)e^{y\beta_+} - (\beta_- + \alpha)e^{y\beta_-})}, & x > y. \end{cases}$$

For any $y \geq 0$, the function $f_x(\cdot, y)$ is of finite variation, and

$$df_x(x, y) = -\frac{2}{\sigma^2} \delta_y(dx) + \left(\frac{2\delta}{\sigma^2} f(x, y) - \frac{2(b\mathbb{1}_{\{x < y\}} + a\mathbb{1}_{\{x > y\}})}{\sigma^2} f_x(x, y) \right) dx,$$

where δ_y denotes the Dirac measure in y . Moreover, if we denote by $f_{xx}(x, y)$ the second derivative of f with respect to the first variable for $x \neq y$, then we get

$$\sup_{u \in [a, b]} \left(\frac{\sigma^2}{2} f_{xx}(x, y) + uf_x(x, y) - \delta f(x, y) \right) = 0, \quad x \neq y.$$

Proof. Obtaining the derivative and the associated measure is straightforward. If $\delta = 0$, then the statement of the lemma is trivial. This is due to the fact that $\beta_+ = 0$, $\beta_- = -\frac{2b}{\sigma^2}$, $\alpha = \frac{2a}{\sigma^2}$. The function f in this case satisfies $f_x(x, y) > 0$ if $x < y$ and $f_x(x, y) < 0$ if $x > y$.

Now, assume that $\delta > 0$. We have $\alpha, \beta_+ > 0 > \beta_-$, which immediately yields $f_x(x, y) > 0$ for $x < y$ and $f_x(x, y) < 0$ for $x > y$. The last equality follows. \square

Lemma A.2. *Let $y \geq 0$ and assume that $C_t = a\mathbb{1}_{\{X_t > y\}} + b\mathbb{1}_{\{X_t \leq y\}}$. Then*

$$\mathbb{E}_x \left[\int_0^\tau e^{-\delta s} dL_s^y \right] = \sigma^2 f(x, y).$$

Proof. The Itô–Tanaka formula together with the occupation time formula yields

$$\begin{aligned} f(X_{t \wedge \tau}, y) &= f(x, y) + \int_0^t \sigma f_x(X_{s \wedge \tau}, y) dW_s - \frac{1}{\sigma^2} L_{t \wedge \tau}^y \\ &\quad + \int_0^t C_s f_x(X_{s \wedge \tau}, y) + \frac{\sigma^2}{2} f_{xx}(X_{s \wedge \tau}, y) ds \\ &= f(x, y) + \int_0^t \sigma f_x(X_{s \wedge \tau}, y) dW_s - \frac{1}{\sigma^2} L_{t \wedge \tau}^y + \delta \int_0^t f(X_{s \wedge \tau}, y) ds. \end{aligned}$$

Using the product formula yields

$$e^{-\delta t} f(X_{t \wedge \tau}, y) = f(x, y) + \int_0^t \sigma e^{-\delta s} f_x(X_{s \wedge \tau}, y) dW_s - \frac{1}{\sigma^2} \int_0^{t \wedge \tau} e^{-\delta s} dL_s^y.$$

Since $f_x(\cdot, y)$ is bounded we see that the second summand is a martingale. If $\delta > 0$, then we find that

$$\lim_{t \rightarrow \infty} \mathbb{E}_x [e^{-\delta t} f(X_{t \wedge \tau}, y)] = 0.$$

If $\delta = 0$ and $a \leq 0$, then $\tau < \infty$ \mathbb{P} -almost surely, and the boundedness of f yields

$$\lim_{t \rightarrow \infty} \mathbb{E}_x [f(X_{t \wedge \tau}, y)] = 0.$$

If $\delta = 0$ and $a > 0$, then $\lim_{t \rightarrow \infty} X_{t \wedge \tau}$ takes values in $\{0, \infty\}$ and $\lim_{x \rightarrow \infty} f(x, y) = 0$; thus the boundedness of f again yields

$$\lim_{t \rightarrow \infty} \mathbb{E}_x[f(X_{t \wedge \tau}, y)] = 0.$$

Thus, we find by monotone convergence that

$$0 = f(x, y) - \frac{1}{\sigma^2} \lim_{t \rightarrow \infty} \mathbb{E}_x \left[\int_0^{t \wedge \tau} e^{-\delta s} dL_s^y \right] = f(x, y) - \frac{1}{\sigma^2} \mathbb{E}_x \left[\int_0^\tau e^{-\delta s} dL_s^y \right]. \quad \square$$

The next lemma is a simple variation of the occupation times formula.

Lemma A.3. *Let $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be continuous, let τ be a random time, and let $\psi : \mathbb{R} \rightarrow \mathbb{R}_+$ be Borel measurable. Then*

$$\int_0^\tau g(s) \psi(X_s) \sigma^2 ds = \int_{\mathbb{R}} \psi(y) Z^y dy,$$

where $Z^y := \int_0^\tau g(s) dL_s^y$.

Proof. If the claim is proved for bounded stopping times, then an arbitrary stopping time τ can be approximated by bounded stopping times via $\tau = \lim_{N \rightarrow \infty} \min\{N, \tau\}$ and monotone convergence yields the claim. For the remainder of the proof we assume that τ is a bounded stopping time. Additionally, we start with bounded and Lebesgue-integrable ψ . Once the claim is proved for bounded and Lebesgue-integrable ψ , it follows for the remaining ψ by monotone convergence.

Let $\epsilon > 0$. Since g is continuous it is uniformly continuous on $[0, \tau]$. Hence, there is $\delta > 0$ such that $|g(x) - g(y)| < \epsilon$ for any $x, y \in [0, \tau]$ with $|x - y| < \delta$. For an integer $N > \tau/\delta$ we find, with

$$F_N := \sum_{k=1}^N \int_{(k-1)\frac{\tau}{N}}^{k\frac{\tau}{N}} (g(s) - g((k-1)\tau/N)) \psi(X_s) \sigma^2 ds,$$

that

$$\begin{aligned} \int_0^\tau g(s) \psi(X_s) \sigma^2 ds &= \sum_{k=1}^N \int_{(k-1)\frac{\tau}{N}}^{k\frac{\tau}{N}} g((k-1)\tau/N) \psi(X_s) \sigma^2 ds \\ &\quad + \sum_{k=1}^N \int_{(k-1)\frac{\tau}{N}}^{k\frac{\tau}{N}} (g(s) - g((k-1)\tau/N)) \psi(X_s) \sigma^2 ds \\ &= \int_{\mathbb{R}} \psi(y) \sum_{k=1}^N g((k-1)\tau/N) (L_{k\frac{\tau}{N}}^y - L_{(k-1)\frac{\tau}{N}}^y) dy + F_N, \end{aligned}$$

where we use [18, Corollary VI.1.6] for the second equality. We have $|F_N| \leq \epsilon \int_0^\tau \psi(X_s) \sigma^2 ds$ by the choice of δ , and

$$\sum_{k=1}^N g((k-1)\tau/N) (L_{k\frac{\tau}{N}}^y - L_{(k-1)\frac{\tau}{N}}^y) \rightarrow \int_0^\tau g(s) dL_s^y, \quad N \rightarrow \infty.$$

It holds that

$$\left| \sum_{k=1}^N g((k-1)\tau/N) (L_{k\frac{\tau}{N}}^y - L_{(k-1)\frac{\tau}{N}}^y) \right| \leq \sup_{a \in [0, \tau]} g(a) \cdot L_\tau^y,$$

and hence, Lebesgue’s dominated convergence result yields

$$\int_{\mathbb{R}} \psi(y) \sum_{k=1}^N g((k-1)\tau/N)(L_{k\tau/N}^y - L_{(k-1)\tau/N}^y) dy \rightarrow \int_{\mathbb{R}} \psi(y) Z^y dy$$

as required. □

Proof of Theorem A.1. Fix $y \geq 0$. For any progressively measurable process η with values in I we define

$$Y_t^\eta := X_0 + \int_0^t \eta_s ds + \sigma W_t \quad \text{and} \quad V(x) := \sup_{\eta} \mathbb{E}_x \left[\int_0^\tau e^{-\delta s} dL_s^{y,\eta} \right],$$

where $\tau^\eta := \inf\{t \geq 0 : Y_t^\eta = 0\}$ and $L^{\cdot,\eta}$ denotes a continuous version of the local time of Y^η . Clearly, we have

$$\mathbb{E}_x \left[\int_0^\tau e^{-\delta s} dL_s^{y,\eta} \right] \leq V(x).$$

Moreover, the previous two lemmas yield that Y^{η^*} with

$$\eta_t^* = a \mathbb{1}_{\{Y_t^{\eta^*} > y\}} + b \mathbb{1}_{\{Y_t^{\eta^*} \leq y\}}$$

is the optimally controlled process, and we get $V(x) = \sigma^2 f(x, y)$. (The process η^* exists because the corresponding stochastic differential equation admits pathwise uniqueness; see [18, Theorem IX.3.5].) This proves the inequality for the local time. The additional inequality follows now from Lemma A.3. □

Acknowledgement

Both authors would like to thank the anonymous referees for their helpful comments and suggestions.

Funding information

The research of J. Eisenberg was funded by the Austrian Science Fund (FWF), Project No. V 603-N35.

Competing interests

There are no competing interests to declare.

References

- [1] ALBRECHER, H., AZCUE, P. AND MULER, N. (2020). Optimal ratcheting of dividends in insurance. *SIAM J. Control Optimization* **58**, 1822–1845.
- [2] ALBRECHER, H., BÄUERLE, N. AND BLADT, M. (2018). Dividends: from refracting to ratcheting. *Insurance Math. Econom.* **83**, 47–58.
- [3] ALBRECHER, H. AND THONHAUSER, S. (2007). Dividend maximization under consideration of the time value of ruin. *Insurance Math. Econom.* **41**, 163–184.
- [4] ALBRECHER, H. AND THONHAUSER, S. (2009). Optimality results for dividend problems in insurance. *RACSAM* **103**, 295–320.
- [5] ASMUSSEN, S. AND TAKSAR, M. (1997). Controlled diffusion models for optimal dividend pay-out. *Insurance Math. Econom.* **20**, 1–15.

- [6] AVANZI, B. (2009). Strategies for dividend distribution: a review. *N. Amer. Actuarial J.* **13**, 217–251.
- [7] AZCUE, P. AND MULER, N. (2005). Optimal reinsurance and dividend distribution policies in the Cramér–Lundberg model. *Math. Finance* **15**, 261–308.
- [8] BAÑOS, D. AND KRÜHNER, P. (2016). Optimal density bounds for marginals of Itô processes. *Commun. Stoch. Anal.* **10**, 131–150.
- [9] BORODIN, A. N. AND SALMINEN, P. (2002). *Handbook of Brownian Motion—Facts and Formulae*, 2nd edn. Birkhäuser, Basel.
- [10] BÜHLMANN, H. (1970). *Mathematical Methods in Risk Theory*. Springer, New York.
- [11] DE FINETTI, B. (1957). Su un'impostazione alternativa della teoria collettiva del rischio. In *Transactions of the XVth International Congress of Actuaries*, Vol. II, Actuarial Society of America, New York, pp. 433–443.
- [12] FLEMING, W. H. AND SONER, H. M. (1993). *Controlled Markov processes and viscosity solutions*, 1st edn. Springer, New York.
- [13] GERBER, H. U. (1969). Entscheidungskriterien für den zusammengesetzten Poisson-Prozess. *Mitt. Verein. Schweiz. Versicherungsmath.* **69**, 185–228.
- [14] GOULD, J. (2008). Munich Re pledges stable dividend after profit drop. Available at <https://www.reuters.com/article/sppage012-15116107-oisbn-idUSL511610720080806>.
- [15] GRANDITS, P., HUBALEK, F., SCHACHERMAYER, W. AND ZIGO, M. (2007). Optimal expected exponential utility of dividend payments in Brownian risk model. *Scand. Actuarial J.* **2**, 73–107.
- [16] HUBALEK, F. AND SCHACHERMAYER, W. (2004). Optimization expected utility of dividend payments for a Brownian risk process and a peculiar nonlinear ode. *Insurance Math. Econom.* **34**, 193–225.
- [17] PESKIR, G. (2005). A change-of-variable formula with local time on curves. *J. Theoret. Prob.* **18**, 499–535.
- [18] REVUZ, D. AND YOR, M. (2005). *Continuous Martingales and Brownian Motion*, 3rd edn. Springer, Berlin, Heidelberg.
- [19] SCHMIDLI, H. (2008). *Stochastic Control in Insurance*. Springer, London.
- [20] SHREVE, S. E., LEHOCZKY, J. P. AND GAVAR, D. P. (1984). Optimal consumption for general diffusions with absorbing and reflecting barriers. *SIAM J. Control Optimization* **22**, 55–75.