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REDUCTION OF DIMENSION OF APPROXIMATE INTERTIAL MANIFOLDS BY SYMMETRY

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In this paper, we study approximate inertial manifolds for nonlinear evolution partial differential equations which possess symmetry. The relationship between symmetry and dimensions of approximate inertial manifolds is established. We demonstrate that symmetry can reduce the dimensions of an approximate inertial manifold. Applications for concrete evolution equations are given.

1. INTRODUCTION

In this paper, we investigate the effect of symmetry on the asymptotic behaviour of solutions to nonlinear evolution partial differential equations. It is now well known that global attractors and inertial manifolds are important concepts in studying the long time behaviour of evolution equations since they contain all the asymptotyc dynamics of a system, see Hale [**10**]; Stuart [**3**] and Temam [**8**].

On the other hand, it is often the case that evolution equations possess symmetry under action of a group, for example, the Kuramoto-Sivashinsky (KS) and Cahn-Hilliard (CH) equations. In that case, it is interesting to discuss whether the permanent regime is also symmetric and how the symmetry affects the asymptotic behaviour of solutions. In this respect, recently it was shown in [**5**] that the global attractor has the same symmetry as the evolution equations. There the author also proved that inertial manifolds constructed by some methods preserve the symmetry, and established the precise relation between the symmetry and the dimensions of inertial manifolds. Based on this fact, in [**2**] the author demonstrated that the Kuramoto-Velarde equation has symmetric inertial manifolds, and the symmetry can indeed reduce the dimension of inertial manifolds.

On the other hand, approximate inertial manifolds (AIM) are used to approximate the dynamics on the global attractors and inertial manifolds, see Jolly, Kevrekidis and Titi [**6**]; Foias, Jolly, Kevrekidis, Sell and Titi [**14**]; Sell [**9**], and the references therein. Since AIMs approximately describe the asymptotic dynamics of a given system, they are used both for dynamical studies and computational purposes. We note that AIMs can

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be proved to exist with much fewer restrictions than those required for the existence of inertial manifolds.

Therefore, in this paper, our goal is to give conditions which imply that AIMs inherit the symmetries of the equations and, on the other hand, to show that the dynamics of symmetric solutions can be described by lower dimensional AIMs.

Although most of the arguments can be carried in an abstract setting and for most known methods of construction of AIMs, for the sake of simplicity, we present here some results on concrete equations and construction methods.

This paper is organised as follows. In the next Section, we first recall some facts about groups and we present the functional form of the equation which will be considered in the paper. Section 3 is devoted to our main results. We show that under some circumstances AIM's possess the same symmetry as the equation. We also establish the precise relationship between the symmetry and the dimensions of AIM's. In Section 4, we apply our abstract results to concrete evolution equations such as the KS equation and CH equation. There we shall be concerned with AIM's constructed by various well known methods: the nonlinear Galerkin method, Gamma method, and Euler-Galerkin method, see [**9**] for more details.

2. Preliminaries

For later purpose, we first recall some notation of groups. In the sequel, we assume that H is a topological space and G is a subgroup of the group of homeomorphisms of H, $G \subset \mathrm{Hom}\,(H)$.

Definition 2.1:

(i) if $A \subset H$, then the isotropy subgroup Σ_A of A is defined by

$$\Sigma_A = \Big\{g \in G : \; g(A) = A\Big\}.$$

(ii) If $\Sigma \subset G$, then the fixed points set $\mathrm{Fix}\,(\Sigma)$ of Σ is defined by

$$\mathrm{Fix}\,(\Sigma) = \Big\{x \in H : \; g(x) = x, \; \forall g \in \Sigma\Big\}.$$

Definition 2.2: Assume that X and Y are subsets of H, $T : X \to Y$ is a mapping and $g \in G$. If $g(X) \subset X$ and $g \circ T = T \circ g$ on X, then we say g and T commute. If g and T commute for all $g \in G$, we say G and T commute.

Hereafter, we denote by H a Banach space and $A : D(A) \subset H \to H$ is a closed linear operator with dense domain, while P denotes a spectral projection of A. Also we assume that G is a group of linear isomorphisms of H, that is, a subgroup of $\mathrm{Iso}\,(H)$. With these notations, we recall the following result from [**5**].

Proposition 2.1. *Assume that g and A commute where $g \in \mathrm{Iso}\,(H)$. Then g commutes with every spectral projection P of A, and $g \in \mathrm{Iso}\Big(R(P)\Big)$.*

In what follows, we consider the semilinear equation in H:

$$u_t + Au + f(u) = 0, \tag{2.1}$$

$$u(0) = u_0 \in H, \tag{2.2}$$

where f is a nonlinear function in H.

We assume that problem (2.1)–(2.2) is well-posed, that is, for all $u_0 \in H$, (2.1)–(2.2) has a unique solution $u(t)$ defined on $\mathbb{R}^+$ such that

$$u \in C\Big([0,\infty), H\Big) \cap C^1\Big((0,\infty), H\Big).$$

Thus, system (2.1)-(2.2) establishes a dynamical system $S(t) : H \to H$ such that $S(t)u_0 = u(t)$, the solution of (2.1)–(2.2). The following consequence is straightforward, see [**5**].

Proposition 2.2. *Assume that G commutes with A and f. Then we have*

$$S(t) \circ g = g \circ S(t), \quad \text{for all } t \geqslant 0 \text{ and } g \in G. \tag{2.3}$$

In particular, if $\Sigma \subset G$, then $V = \mathrm{Fix}\,(\Sigma)$ is a positively invariant linear subspace in the sense that if $u \in V$, then $S(t)u \in V$ for all $t \geqslant 0$.

Definition 2.3: If (2.3) is satisfied, then g is called a symmetry of problem (2.1)–(2.2) and G a group of symmetries.

3. Main results

In this section, we deal with AIM's which possess symmetry. We shall show that AIM's constructed by standard methods preserve the symmetry of the original evolution equations. Also we shall establish the relationship between the symmetry and the dimensions of AIM's. As a result, we shall see the symmetry can reduce the dimensions of AIM's for evolution equations.

We first recall from [**14, 6, 9**] that an AIM, $\mathcal{M}$, is a smooth finite dimensional manifold such that every solution enters and remains in a thin neighbourhood in a finite time, more precisely, there exist an integer m and a positive number ε such that $\mathcal{M}$ is an m-dimensional manifold and

$$\operatorname*{dist}_H\Big(S(t)u_0, \mathcal{M}\Big) \leqslant \varepsilon, \quad \text{for all } t \geqslant \tilde{t},$$

where $\tilde{t}$ depends on R when $\|u_0\| \leqslant R$ and $\mathrm{dist}_H\,(A,B) = \sup_{a\in A}\inf_{b\in B} \|a-b\|_H$. In that case, we call $\mathcal{M}$ an ε-AIM. Then we start with the following.

Theorem 3.1. *Let $\mathcal{M}$ be an m-dimensional ε-AIM for problem (2.1)-(2.2), then for every $g \in G$, $g(\mathcal{M})$ is an m-dimensional $\varepsilon\|g\|$-AIM, where $\|g\|$ is the norm of g in $\mathcal{L}(H)$.*

PROOF: For any $g \in G$ and bounded set B in H, since g is a bounded linear isomorphism of H, we have

$$\operatorname*{dist}_H\Big(S(t)B,\ g(\mathcal{M})\Big) = \operatorname{dist}\Big(S(t)g\big(g^{-1}(B)\big), g(\mathcal{M})\Big)$$
$$\leqslant \|g\| \operatorname{dist}\Big(S(t)\big(g^{-1}(B)\big), \mathcal{M}\Big) \leqslant \varepsilon\|g\|,$$

for all $t \geqslant t(g^{-1}(B))$, where we have used (2.3). Hence $g(\mathcal{M})$ is an $\varepsilon\|g\|$-AIM isomorphic to $\mathcal{M}$. In particular, $g(\mathcal{M})$ is m-dimensional. The proof is complete. ∎

Observe that, as in [**5**], if G is a compact Lie group, by using the Haar measure on G one can construct an equivalent norm in H such that $\|g\| = 1$ for every $g \in G$.

In most methods known so far, see [**14, 6, 9**], an AIM is constructed as a graph $\mathcal{M} = \operatorname{Graph}(\Phi)$ where $\Phi : X_1 \to X_2$ is a Lipschitz mapping and X_1 and X_2 form a spectral decomposition of H. Moreover, $\mathcal{M}$ satisfies that there exists $\varepsilon > 0$ such that

$$\operatorname*{dist}_H\Big(S(t)u_0,\ P\big(S(t)u_0\big) + \Phi\Big(P(S(t)u_0)\Big)\Big) \leqslant \varepsilon, \quad \text{for all} \quad t \geqslant t(u_0), \tag{3.1}$$

where P is the spectral projection from H onto X_1.

We shall see below that if Φ commutes with all elements of G, then $\mathcal{M}$ is G-symmetric. More precisely, we have:

THEOREM 3.2. *With the notation above, assume that $\Phi : X_1 \to X_2$ commutes with all elements of G. Then the AIM $\mathcal{M} = \operatorname{Graph}(\Phi)$ is symmetric in the sense that for every $g \in G$ one has*

$$g(\mathcal{M}) = \mathcal{M}.$$

PROOF: Since X_1 and X_2 are ranges of spectral projections, by Proposition 2.1 we get $g(X_i) = X_i$, $i = 1, 2$, for all $g \in G$. Consequently $g(\mathcal{M}) = \operatorname{Graph}(\Psi)$, where $\Psi = g\Phi g^{-1}$. Therefore, if $\Phi g = g\Phi$ then $g(\mathcal{M}) = \mathcal{M}$, which proves Theorem 3.2. ∎

In what follows, we investigate the impact of symmetry on the dimension of AIMs. Given a subgroup $\Sigma \subset G$, let $V = \operatorname{Fix}(\Sigma)$. As shown in Proposition 2.2, V is invariant for $S(t)$. So symmetry of the initial data is preserved for positive times. Our main results are stated in:

THEOREM 3.3.

(i) *Let $\Sigma \subset G$ be a subgroup and $V = \operatorname{Fix}(\Sigma)$. Assume $\Phi : X_1 \to X_2$ commutes with all elements of G and let $\mathcal{M} = \operatorname{Graph}(\Phi) = (I + \Phi)(X_1)$ be an AIM for (2.1). Then we have*

$$\mathcal{M} \cap V = (I + \Phi)(V \cap X_1). \tag{3.2}$$

(ii) *Assume moreover that Φ satisfies (3.1). Then $\mathcal{M} \cap V$ is an ε-AIM for the flow on V.*

PROOF:

(i) Let $P : H \to X_1$ and $Q : H \to X_2$ be the spectral projections. Since, by Proposition 2.2 we know that for any $g \in G$, $Pg = gP$, $Qg = gQ$, then for every $u \in V$, Pu and Qu also belong to V. On the other hand, if $u \in \mathcal{M} = \text{Graph}(\Phi)$, then u can be written as $u = p + \Phi(p)$ with $p \in X_1$ and since Φ commutes with all elements of G, we have that $u \in V$ if and only if $g(p) = p$ for all $g \in \Sigma$, which completes the proof of (i).

(ii) Since $S(t)u_0 \in V$ for any $u_0 \in V$ and $t \geqslant 0$, we get that $P\big(S(t)u_0\big) \in V \cap X_1$. Hence, by (i) we find that $P\big(S(t)u_0\big) + \Phi\big(P(S(t)u_0)\big) \in \mathcal{M} \cap V$. It follows now from (3.1) that for all $u_0 \in V$

$$\operatorname*{dist}_H\big(S(t)u_0,\ \mathcal{M} \cap V\big) \leqslant \operatorname*{dist}_H\Big(S(t)u_0,\ P\big(S(t)u_0\big) + \Phi\Big(P\big(S(t)u_0\big)\Big)\Big) \leqslant \varepsilon,$$

for all $t \geqslant t(u_0)$ and the proof is finished.

$\square$

We remark that usually X_1 is an eigenspace spanned by the eigenfunctions of A corresponding to a subset σ_1 of $\sigma(A)$. In that case, $X_1 \cap V$ is generated by eigenfunctions of A with eigenvalue in σ_1 and with symmetry Σ. Equation (3.2) shows then in a precise way how symmetry reduces dimensions of an AIM for symmetric solutions.

Note that the main assumption in the preceding results is the fact that Φ commutes with all the symmetries in the group. In the context of inertial manifolds, it was showed in [**5**] that this is actually the case, but the positive invariance of the inertial manifolds is crucially used for the proof. Therefore, in principle, for AIMs specific proofs of this fact must be given depending on the method of construction used. In the next section, rather than working in a general abstract setting, we shall illustrate this question and the consequences of the theorems above on concrete equations and for some well known methods of construction of AIMs.

4. APPLICATIONS

In this section, we apply the abstract results in the previous section to the KS equation and the CH equation. We shall show how the symmetry possessed by these equations can reduce the dimensions of an AIM. We first consider the KS equation with an AIM constructed by the nonlinear Galerkin method. Then we present similar results for the KS equation with an AIM defined by the Gamma method and Euler-Galerkin method. However, in this case, the precise statements of results will be omitted since they are similar. Finally, we discuss the same problem for the CH equation. For the CH model we omit the results similar to the KS equation, and just deal with those corresponding to extra symmetry possessed only by the CH equation but not by the KS equation.

4.1. THE KS EQUATION In this subsection, we consider the following renormalised KS equation:

$$\frac{\partial u}{\partial t} + \nu \frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial x^2} + u \frac{\partial u}{\partial x} = 0, \qquad (x,t) \in \mathbb{R} \times \mathbb{R}^+,$$
$$u(x,0) = u_0(x), \quad x \in \mathbb{R},$$

$\nu > 0$, subject to the periodic condition

$$u(x,t) = u(x+L,t), \quad L > 0.$$

In addition, we impose the zero average condition

$$\int_0^L u(x,t)\,dx = 0, \quad \text{for } t \geqslant 0.$$

We denote by

$$H = \left\{ u \in L^2_{\text{loc}}(\mathbb{R}) : \ u(x+L) = u(x), \ \int_0^L u(x)\,dx = 0 \right\}$$

with the usual norm of $L^2(0,L)$ and $V_k = H^k_{\text{loc}}(\mathbb{R}) \cap H$ endowed with the norm of $H^k(0,L)$. In particular, we let $V_0 = H$ and $V_1 = V$. It is known that for every $u_0 \in H$, there exists a unique solution $u(t) \in V_4$ for all $t > 0$. Also there exists a constant M such that

$$\left\| u(t) \right\|_{H^1} \leqslant M, \quad \text{for } t \geqslant t_0,$$

where t_0 depends only on R when $\|u_0\| \leqslant R$, see [**1**] and [**11**]. So the ball

$$B = \left\{ u \in V : \|u\| \leqslant M \right\} \tag{4.1}$$

is a bounded absorbing set for the KS equation in V. Note that the KS equation is equivalent to the functional differential equation, see [**8**],

$$\frac{du}{dt} + Au - A^{1/2}u + f(u) = 0, \quad u \in H,$$

where $A = \nu \dfrac{\partial^4}{\partial x^4}$ with domain $D(A) = V_4$, and $f(u) = F(u,u)$ where $F(u,v)$ is the bilinear operator defined by

$$F(u,v) = u \frac{\partial v}{\partial x}, \quad \text{for all } u, v \in V.$$

It is easy to see that the eigenvalues of A are $\lambda_n = \nu\big((2\pi/L)n\big)^4$, $n = 1, 2, \cdots$ and the corresponding eigenfunctions are $\sin\big((2\pi/L)nx\big)$ and $\cos\big((2\pi/L)nx\big)$. Let

$$P = P_m : H \to \text{Span}\left\{ \sin\left(\frac{2\pi}{L}x\right), \ \cos\left(\frac{2\pi}{L}x\right), \cdots, \sin\left(\frac{2\pi}{L}mx\right), \ \cos\left(\frac{2\pi}{L}mx\right) \right\}$$

be the spectral projection, and $Q = Q_m = I - P_m$.

For $\tau \in \mathbb{R}$, we define $g_\tau, g_* : H \to H$ such that for $u \in H$, $g_\tau(u)(x) = u(x+\tau)$. and $g_*(u)(x) = -u(-x)$ with $x \in \mathbb{R}$. Then we see that for all $\tau \in \mathbb{R}$, g_τ and g_* commute with all terms of the KS equation, which means that

$$G = \Big\langle \{g_\tau, \tau \in \mathbb{R}\},\ g_* \Big\rangle$$

is a (noncommutative) group of symmetries of the KS equation. Note that g_* is of order 2, and $T = \{g_\tau,\ \tau \in \mathbb{R}\}$ is a commutative group whose effective action on H is given by the quotient of T over the set of elements which act trivially on H, that is, the quotient over $T_0 = \{g_\tau,\ \tau = kL,\ k \in Z\}$ and

$$\frac{T}{T_0} \approx \frac{\mathbb{R}}{LZ} \approx \Big\{g_\tau,\ \tau \in [0, L)\Big\} \approx \mathbb{S}^1 \approx SO(2),$$

which is a compact and commutative Lie group ($SO(2)$ is the group of the orthogonal transformations on the plane which preserve the orientation). Also, let

$$G_0 = \Big\langle \Big\{g_\tau, \tau \in [0, L)\Big\},\ g_* \Big\rangle.$$

Then $G_0 \approx O(2)$, where $O(2)$ is the orthogonal group of the plane. Note that for $\tau \in \mathbb{R}$, it is easy to see that $\operatorname{Fix}(g_\tau)$ consists of the functions in H which are τ-periodic, and $\operatorname{Fix}(g_*)$ consists of the odd functions in H, and $\operatorname{Fix}\Big(\langle g_*, g_\tau \rangle\Big)$ consists of odd τ-periodic functions. Finally note that all elements in G_0 are isometries in the spaces V_k.

In what follows, we first show AIMs constructed by the nonlinear Galerkin method are G_0-symmetric. For this method, see [**6**], we define $\Phi^s : B_m \to B_m^\perp$ with $B_m = P_m H \cap B$ and $B_m^\perp = QH \cap B$, where B is given in (4.1), such that for $p \in B_m$, $\Phi^s(p)$ satisfies

$$A\Phi^s(p) - A^{1/2}\Phi^s(p) + Qf\Big(p + \Phi^s(p)\Big) = 0. \tag{4.2}$$

Equation (4.2) is an implicit equation and can be solved by the contraction principle. More precisely, for each fixed $p \in B_m$, let

$$T_p(q) = A^{-1/2}q - A^{-1}Qf(p+q), \quad \text{for all } q \in B_m^\perp.$$

One can then show that there exists an integer N such that if $m \geqslant N$ then T_p maps $B_m^\perp$ into itself and is a contraction, see [**6**]. Thus the contraction principle implies that for each $p \in B_m$, T_p has a unique fixed point in $B_m^\perp$ which is the solution of (4.2). As shown in [**6**], $\mathcal{M}^s = \operatorname{Graph}(\Phi^s)$ is a $2m$-dimensional AIM satisfying (3.1).

Now we prove that $\mathcal{M}^s$ has the same symmetry as the KS equation, that is, that the assumption on Theorem 3.2 holds true. First we have

THEOREM 4.1. *The mapping Φ^s defined by (4.2) commutes with all elements of G_0, that is,*

$$g\Phi^s = \Phi^s g, \quad \textit{for every } g \in G_0.$$

PROOF: For every $p \in B_m$ and $g \in G_0$, by (4.2) we have

$$(gA\Phi^s)(p) - \big(gA^{1/2}\Phi^s\big)(p) + (gQ)f\big(p + \Phi^s(p)\big) = 0.$$

Since g is linear and commutes with A, Q and f, we have that

$$A\Big(g\big(\Phi^s(p)\big)\Big) - A^{1/2}\Big(g\big(\Phi^s(p)\big)\Big) + Qf\Big(g(p) + g\big(\Phi^s(p)\big)\Big) = 0,$$

which implies that $g\big(\Phi^s(p)\big)$ is the solution of

$$Az - A^{1/2}z + Qf\big(g(p) + z\big) = 0. \tag{4.3}$$

Therefore, since $g(p) \in B_m$ and by the definition of Φ^s, (4.2), we know that $\Phi^s\big(g(p)\big)$ is the unique solution of (4.3). So it follows that

$$g\big(\Phi^s(p)\big) = \Phi^s\big(g(p)\big), \quad \text{for every } p \in B_m,$$

which concludes Theorem 4.1. □

Now we can apply Theorems 3.1 and 3.2 to get

THEOREM 4.2. *The AIM $\mathcal{M}^s = \text{Graph}\,(\Phi^s)$ with Φ^s given by (4.2) is G_0-symmetric. that is,*

$$g(\mathcal{M}^s) = \mathcal{M}^s, \quad \textit{for every } g \in G_0.$$

The following results are immediate consequences of Theorem 3.3.

THEOREM 4.3. *Let $\tau_k = L/k$, $k = 1, 2, \cdots$, $G_k = \langle g_{\tau_k}\rangle$ and $Z_k = \text{Fix}\,(G_k)$ (L/k-periodic functions). Then $\mathcal{M}^s \cap Z_k$ is an AIM for the flow on Z_k which is a graph over $X_1 \cap Z_k$, spanned by*

(i) $\Big\{\sin\big(jk(2\pi/L)x\big),\ \cos\big(jk(2\pi/L)x\big),\ jk \leqslant m,\ j \geqslant 1\Big\}$, *if* $k < m$,
(ii) $\Big\{\sin\big(m(2\pi/L)x\big),\ \cos\big(m(2\pi/L)x\big)\Big\}$, *if* $k = m$,
(iii) $\{0\}$, *if* $k > m$.

In particular, if $k = m$, then $\mathcal{M}^s \cap Z_m$ is a two dimensional AIM for the L/m-periodic solutions; if $k > m$, then $\mathcal{M}^s \cap Z_k = \{0\}$ is an AIM for the L/k-periodic solutions.

THEOREM 4.4. *Let $G_* = \langle g_*\rangle$ and $Z_* = \text{Fix}\,(G_*)$ (odd functions). Then $\mathcal{M}^s \cap Z_*$ is an m-dimensional AIM for the flow on Z_* which is a graph over $X_1 \cap Z_*$, spanned by*

$$\Big\{\sin\Big(k\frac{2\pi}{L}x\Big),\quad 1 \leqslant k \leqslant m\Big\}.$$

THEOREM 4.5. *Let $\tau_k = L/k$, $k = 1, 2, \cdots$, $G_{*,k} = \langle g_*,\ g_{\tau_k}\rangle$ and $Z_{*,k} = \text{Fix}\,(G_{*,k})$ (odd L/k-periodic functions). Then $\mathcal{M}^s \cap Z_{*,k}$ is an AIM for the flow on $Z_{*,k}$ which is the intersection of the manifolds in Z_k and Z_* constructed above, and a graph over $X_1 \cap Z_{*,k}$, spanned by*

(i) $\{\sin(jk(2\pi/L)x),\quad jk \leqslant m,\ j \geqslant 1\}$, *if* $k < m$,
(ii) $\{\sin(m(2\pi/L)x),\ \}$, *if* $k = m$,
(iii) $\{0\}$, *if* $k > m$.

In particular, if $k = m$, *then* $\mathcal{M}^s \cap Z_{*,m}$ *is a one dimensional AIM for the odd* L/m-*periodic solutions.*

THEOREM 4.6. *Let* $G_{*\circ 2} = \langle g_* \circ g_{L/2} \rangle$ *and* $Z_{*\circ 2} = \mathrm{Fix}\,(G_{*\circ 2}) = \{u \in H:\ u(x) = -u((L/2) - x)\}$. *Then* $\mathcal{M}^s \cap Z_{*\circ 2}$ *is an m-dimensional AIM for the flow on* $Z_{*\circ 2}$ *which is a graph over* $X_1 \cap Z_{*\circ 2}$, *spanned by*

$$\left\{\sin\left(2k\frac{2\pi}{L}x\right),\ \cos\left((2j+1)\frac{2\pi}{L}x\right),\ \ 1 \leqslant 2k \leqslant m,\ 1 \leqslant 2j+1 \leqslant m\right\}.$$

In the sequel, we consider the Gamma method. First note that the KS equation is equivalent to the system

$$\frac{dp}{dt} + Ap - A^{1/2}p + P_m f(p+q) = 0,$$

$$\frac{dq}{dt} + Aq - A^{1/2}q + Q_m f(p+q) = 0,$$

where $p = P_m u$, $q = Q_m u$.

The idea of the Gamma method is to construct an AIM as an inertial manifold of the perturbed system

$$\frac{dp}{dt} + Ap - A^{1/2}p + P_m f(p+q) = 0, \tag{4.4}$$

$$\frac{dq}{dt} + \gamma Aq - A^{1/2}q + Q_m f(p+q) = 0, \tag{4.5}$$

where $\gamma > 1$, see [**14, 9**]. By using a variation of the argument in [**4**] or [**7**] one concludes that there are constants K_0 and K_1 such that if the eigenvalues λ_n and λ_{n+1} satisfy $\lambda_n \geqslant K_0$, together with the gap condition

$$\gamma\lambda_{n+1} - \lambda_n \geqslant K_1\left(\gamma^{1/2}\lambda_{n+1}^{1/2} + \lambda_n^{1/2}\right), \tag{4.6}$$

then (4.4)–(4.5) has an inertial manifold of the form

$$\mathcal{M}^\gamma = \mathrm{Graph}\,(\Phi^\gamma),$$

where Φ^γ is a Lipschitz function from $P_m H$ to $Q_m H \cap D(A)$. It turns out then that $\mathcal{M}^\gamma$ is an AIM for the original system.

For the KS equation, since $\lambda_n = \nu(2\pi n/L)^4$, it is easy to verify that (4.6) is satisfied for some m. Then (4.4)–(4.5) has a 2m-dimensional inertial manifold $\mathcal{M}^\gamma = \mathrm{Graph}\,(\Phi^\gamma) = (I + \Phi^\gamma)(X_1)$, where X_1 is spanned by

$$\left\{\sin\left(\frac{2\pi}{L}nx\right),\ \cos\left(\frac{2\pi}{L}nx\right),\ \ 1 \leqslant n \leqslant m\right\}.$$

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From [5], it follows that Φ^γ is G-symmetric. So Theorems 3.1 and 3.3 apply to $\mathcal{M}^\gamma$ and similar results to the ones above can also be obtained. Details are omitted here.

We now consider the Euler-Galerkin method. The main step of that method is to define $\Phi^\tau : B_m \to B_m^\perp$ such that for $p \in B_m$ $\Phi^\tau(p)$ is the solution of the equation

$$\Phi^\tau(p) + \tau A\Phi^\tau(p) - \tau A^{1/2}\Phi^\tau(p) + \tau Qf\big(p + \Phi^\tau(p)\big) = 0 \tag{4.7}$$

where $\tau > 0$ is a constant. Similarly to the nonlinear Galerkin method, using the contraction principle one can show that there exists N such that when $m \geqslant N$, equation (4.7) has a unique solution $\Phi^\tau(p) \in B_m^\perp$ for any $p \in B_m$. Setting $\mathcal{M}^\tau = \text{Graph}\,(\Phi^\tau)$, then it follows from [14] and [13] that $\mathcal{M}^\tau$ is an AIM for the KS equation. As for Theorem 4.1 we can deduce that Φ^τ is G-symmetric. And thus all results in Section 3 are valid for $\mathcal{M}^\tau$. We omit the details here again.

4.2. The CH equation The nonlinear CH equation is a continuous model for the description of the dynamics of pattern formation in a phase transition, see [12]. This equation involves an unknown function $u : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ such that

$$\frac{\partial u}{\partial t} - \Delta K(u) = 0, \qquad (x,t) \in \mathbb{R} \times \mathbb{R}^+,$$

where $K(u) = -\nu\Delta u + f(u)$, $\nu > 0$, and $f(x) = \sum_{j=1}^{2p-1} a_j x^j$, $a_{2p-1} > 0$. This equation is supplemented with the boundary and initial conditions

$$\begin{aligned} u(x,t) &= u(x+L,t), \quad x \in \mathbb{R}, \\ u(x,0) &= u_0(x), \quad x \in \mathbb{R}. \end{aligned}$$

Here we also impose the zero average condition

$$\int_0^L u(x,t)\,dx = 0, \quad \text{for } t \geqslant 0.$$

Obviously, with the same notations as above, the CH equation is equivalent to

$$\frac{du}{dt} + \nu A u - \Delta f(u) = 0, \quad \text{in } H.$$

It is known, see [8], that the CH model is dissipative, more precisely, there exists a constant $\widetilde{M}$ such that the ball

$$\widetilde{B} = \left\{u \in V_2;\ \|u\|_{H^2} \leqslant \widetilde{M}\right\}$$

is a bounded absorbing set.

Proceeding as before one can show that $\mathcal{M}^s$, $\mathcal{M}^\gamma$ and $\mathcal{M}^\tau$ constructed as above are AIM's for the CH equation with B replaced by $\widetilde{B}$.

Now let $g^* = -g_*$, that is, $g^*(u)(x) = u(-x)$, then we find that g^* is also a symmetry for the CH equation. So now the group of symmetry is

$$G = \left\langle \left\{ g_\tau,\ \tau \in [0, L) \right\},\ g_*,\ g^* \right\rangle.$$

Hence besides the cases discussed above we see that $Z^* = \operatorname{Fix}(g^*)$, that is, the set of even functions in H, is also invariant for the CH equation. Similar results to the KS equation are not repeated here again, so we restrict ourselves to the new symmetry g^*. For g^* we have

Theorem 4.7. *Let $\mathcal{M}^s$ is the AIM defined above and $G^* = \langle g^* \rangle$, $Z^* = \operatorname{Fix}(G^*)$ (even functions). Then $\mathcal{M}^s \cap Z^*$ is an m-dimensional AIM for the flow on Z^* which is a graph over $X_1 \cap Z^*$, spanned by*

$$\left\{ \cos\left(k\frac{2\pi}{L}x\right),\quad 1 \leqslant k \leqslant m \right\}.$$

Theorem 4.8. *Let $\tau_k = L/k$, $k = 1, 2, \cdots$, $G_k^* = \langle g^*,\ g_{\tau_k} \rangle$ and $Z_k^* = \operatorname{Fix}(G_k^*)$ (even L/k-periodic functions). Then $\mathcal{M}^s \cap Z_k^*$ is an AIM for the flow on Z_k^*, a graph over $X_1 \cap Z_k^*$, spanned by*

(i) $\left\{ \cos\big(jk(2\pi/L)x\big),\quad jk \leqslant m,\ j \geqslant 1 \right\}$, *if $k < m$,*
(ii) $\left\{ \cos\big(m(2\pi/L)x\big),\ \right\}$, *if $k = m$,*
(iii) $\{0\}$, *if $k > m$.*

In particular, if $k = m$, then $\mathcal{M}^s \cap Z_m^*$ is a one dimensional AIM for the even L/m-periodic solutions.

Theorem 4.9. *Let $G^{*\circ 2} = \langle g^* \circ g_{L/2} \rangle$, and $Z^{*\circ 2} = \operatorname{Fix}(G^{*\circ 2}) = \big\{ u \in H :\ u(x) = u(L/2 - x) \big\}$. Then $\mathcal{M}^s \cap Z^{*\circ 2}$ is an m-dimensional AIM for the flow on $Z^{*\circ 2}$, a graph over $X_1 \cap Z^{*\circ 2}$, spanned by*

$$\left\{ \sin\left((2k+1)\frac{2\pi}{L}x\right),\ \cos\left(2j\frac{2\pi}{L}x\right),\quad 1 \leqslant 2k+1 \leqslant m,\ 2j \leqslant m \right\}.$$

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