Third Meeting, January 12th, 1894.

Dr C. G. KNOTT, President, in the Chair.

On certain Maxima and Minima.

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1. If x, y, z are the distances of the point P from the sides BC, CA, AB of a triangle, to find P so that the product xyz may be a maximum.

If the product xyz has a value very near the critical value, suppose yz to remain constant then x also remains constant, that is, at the required point the curves $yz = \text{constant}$ and $x = \text{constant}$ touch, or the tangent to the hyperbola $yz = \text{constant}$ is parallel to BC. But the intercept on this tangent by the sides AB, AC is bisected at the point of contact, therefore AP produced bisects BC. So also BP bisects AC, and CP bisects AB and therefore P is the centroid of the triangle ABC.

2. The usual method of solution is:—

If xyz is a maximum

$ax \,.\, by \,.\, cz$ is a maximum,

which is true when $ax = by = cz$, that is when

$$\triangle PAB = \triangle PBC = \triangle PCA.$$

3. This latter method may be applied to a figure of any number of sides, for if x, y, z, w, etc., are the perpendiculars then the product $xyzw$, etc., is a maximum when $ax \,.\, by \,.\, cz \,.\, dw$, etc., is a maximum. But ax, by, cz are connected by the symmetrical relation $ax + by + cz + \text{etc.} = 2\triangle$ where $\triangle$ is the area of the polygon and therefore for the critical position $ax = by = cz = \text{etc.}$, that is, all

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the triangles with vertex P and sides of the polygon as bases must be equal.

In general no such solution is possible. For example, in the case of a quadrilateral the point P exists only when one diagonal bisects the quadrilateral and the point P is then the middle point of that diagonal.

4. Treating the case of the quadrilateral by the method of § 1 an interesting result follows.

At the required point P the hyperbolas

$$xy = \text{constant} \text{ and } zw = \text{constant}$$

must touch. Also at the same point the pairs of curves

$$xz = \text{constant} \text{ and } yw = \text{constant}$$

and
$$yz = \text{constant} \text{ and } xw = \text{constant}$$

must touch.

5. Problem:—To find the locus of the point of contact of hyperbolas whose asymptotes are the pairs of sides of a quadrilateral drawn respectively from opposite angular points.

Let $x = 0$, $y = 0$ be two sides drawn from one angular point, and $\frac{x}{a_1} + \frac{y}{b_1} = 1$ and $\frac{x}{a_2} + \frac{y}{b_2} = 1$ the two drawn from the opposite angular point.

The two hyperbolas will be

$$xy = c_1^2 \quad \text{and} \quad \left(\frac{x}{a_1} + \frac{y}{b_1} - 1\right)\left(\frac{x}{a_2} + \frac{y}{b_2} - 1\right) = c_2^2 .$$

The tangents to these curves at the point x_1, y_1 are

$$\frac{x}{x_1} + \frac{y}{y_1} = 2$$

and

$$(x - x_1)\left\{\frac{1}{a_1}\left(\frac{x_1}{a_2} + \frac{y_1}{b_2} - 1\right) + \frac{1}{a_2}\left(\frac{x_1}{a_1} + \frac{y_1}{b_1} - 1\right)\right\}$$

$$+ (y - y_1)\left\{\frac{1}{b_1}\left(\frac{x_1}{a_2} + \frac{y_1}{b_2} - 1\right) + \frac{1}{b_2}\left(\frac{x_1}{a_1} + \frac{y_1}{b_1} - 1\right)\right\} = 0 .$$

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These tangents must be coincident and therefore

$$x_1\left\{\frac{1}{a_1}\left(\frac{x_1}{a_2}+\frac{y_1}{b_2}-1\right)+\frac{1}{a_2}\left(\frac{x_1}{a_1}+\frac{y_1}{b_1}-1\right)\right\}$$

$$=y_1\left\{\frac{1}{b_1}\left(\frac{x_1}{a_2}+\frac{y_1}{b_2}-1\right)+\frac{1}{b_2}\left(\frac{x_1}{a_1}+\frac{y_1}{b_1}-1\right)\right\}$$

$$=\{\text{sum of absolute terms}\}.$$

These two equations are equivalent to one only and therefore the required locus is

$$\left(\frac{x}{a_1}-\frac{y}{b_1}\right)\left(\frac{x}{a_2}+\frac{y}{b_2}-1\right)+\left(\frac{x}{a_2}-\frac{y}{b_2}\right)\left(\frac{x}{a_1}+\frac{y}{b_1}-1\right)=0$$

which is the equation of the centre locus of all conics passing through the four points of intersection of the four sides of the quadrilateral, other than the two through which the asymptotes are drawn.

6. Thus interpreting § 4 by means of this result the point P in a quadrilateral for which the product $xyzw$ is a maximum is *the intersection of the three centre-loci of all conics passing through the three sets of four of the angular points of a complete quadrilateral.*

This point will be real when the quadrilateral is bisected by one of its diagonals, and is the middle-point of that diagonal.

7. The method of § 1 may be extended by finding the curve, the tangent at any point of which is divided by two given lines and by the point of contact in a given ratio.

Let Ox, Oy (Fig. 10) be the given lines, and XPP′Y a line passing through two consecutive points on the curve.

$$\frac{\delta y}{\delta x}=\frac{\text{P}'\text{N}}{\text{PN}}=\pm\frac{y}{\text{OX}-x}=\pm\frac{y}{x}\cdot\frac{x}{\text{OX}-x}=\pm\frac{y}{x}\cdot\frac{\text{YP}}{\text{PX}}$$

$=\pm\dfrac{y}{x}\cdot\dfrac{l}{m}$ where $l:m$ is the given ratio.

Taking the negative sign $l\dfrac{\delta x}{x}+m\dfrac{\delta y}{y}=0$ or $x^l y^m=\kappa$.

Whence, by § 1, the maximum value of $x^l y^m z^n$ in a triangle is at the point P where AP divides BC in the ratio of $n:m$, BP divides CA in the ratio $l:n$ and CP divides AB in ratio $m:l$. (Fig. 11.)

By taking the positive sign in the above equation we find $x^{-l}y^m z^n$ a maximum for a point similarly found, but BP, CP divide AC, AB externally.

Examples :—For incentre $x^a y^b z^c$ is a maximum.

For the Gergonne point $x^{\frac{1}{s-a}}y^{\frac{1}{s-b}}z^{\frac{1}{s-c}}$ is a maximum.

8. It is not difficult to show that any point in the plane of a triangle is the maximum or minimum position for an infinite number of functions of the distances of the point from the sides of a triangle.

For example, to find the critical position for

$$\phi(\alpha, \beta, \gamma) \equiv u\alpha^2 + v\beta^2 + w\gamma^2 + 2u'\beta\gamma + 2v'\gamma\alpha + 2w'\alpha\beta.$$

Applying the ordinary criterion we find

$$\frac{\partial\phi}{\partial\alpha_0}\Big/ a = \frac{\partial\phi}{\partial\beta_0}\Big/ b = \frac{\partial\phi}{\partial\gamma_0}\Big/ c$$

which are also the equations for finding the centre $\alpha_0, \beta_0, \gamma_0$ of the conic $\phi(\alpha, \beta, \gamma) = 0$.

Applying still further the criterion for a true maximum or minimum we find the same condition as that $\phi(\alpha, \beta, \gamma) = 0$ should be an ellipse. So that to find various functions for which a given point is the critical position it is necessary only to find the equations to the various ellipses with centre at the given point. For this we have two equations among five unknown quantities and therefore there is a triply infinite number of functions.

To take a few particular cases, let $u' = v' = w' = 0$ then u, v, w are found from the equation

$$\frac{u\alpha_0}{a} = \frac{v\beta_0}{b} = \frac{w\gamma_0}{c}$$

$\alpha_0, \beta_0, \gamma_0$ being a given point.

For the centre of gravity we have $a^2\alpha^2 + b^2\beta^2 + c^2\gamma^2$ a minimum.

For the positive Brocard point $b^2a^2\alpha^2 + c^2b^2\beta^2 + a^2c^2\gamma^2$ is a minimum, and for the orthocentre and the Gergonne point we have

$a\alpha^2\cos A + b\beta^2\cos B + c\gamma^2\cos C$ and $(s-a)a^2\alpha^2 + (s-b)b^2\beta^2 + (s-c)c^2\gamma^2$

respectively the minimum values.

In the first example given we have a function of the second degree of $a\alpha$, $b\beta$, $c\gamma$ as a critical value at the centroid. This point would be the critical point equally well for any symmetrical function whatever of $a\alpha$, $b\beta$, $c\gamma$ since we have the symmetrical relation $a\alpha + b\beta + c\gamma = 2\Delta$.

9. The following are two theorems on the critical value of $r_1^m + r_2^m + r_3^m$ where r_1, r_2, r_3 are the distances of a point from the vertices of a triangle.

(1) If near the point $r_2^m + r_3^m$ remains constant r_1^m also remains constant and therefore the direction of r_1 is normal to the curve $r_2^m + r_3^m = \text{constant}$.

In this curve by differentiating we have (Fig. 12)

$$\frac{dr_2}{ds} : -\frac{dr_3}{ds} = r_3^{m-1} : r_2^{m-1}$$

but $$\frac{dr_2}{ds} = \sin AOB \quad \text{and} \quad -\frac{dr_3}{ds} = \sin AOC$$

therefore $\sin BOC : \sin AOC : \sin AOB = r_1^{m-1} : r_2^{m-1} : r_3^{m-1}$.

Example :—If $m = 1$, $\sin AOC = \sin AOB = \sin BOC$
that is, $r_1 + r_2 + r_3$ is a minimum when the angles AOB, AOC, BOC are equal, which is Fermat's theorem.

(2) Again, if AO meet BC in L

$$BL : LC = r_2 \sin AOB : r_3 \sin AOC$$
$$= 1/r_2^{m-2} : 1/r_3^{m-2}.$$

If $m = 2$ then $BL = LC$ and we have the known result that $r_1^2 + r_2^2 + r_3^2$ is a minimum for the centre of gravity.

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