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# COMPACT TOEPLITZ OPERATORS ON WEIGHTED HARMONIC BERGMAN SPACES

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**Abstract**

We consider the Bergman spaces consisting of harmonic functions on the unit ball in $\mathbb{R}^n$ that are square-integrable with respect to radial weights. We will describe compactness for certain classes of Toeplitz operators on these harmonic Bergman spaces.

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## 1. Introduction

Let $n \geq 2$ be a fixed integer. We use the notation $B = \{x \in \mathbb{R}^n : |x| < 1\}$ and $S = \partial B = \{x \in \mathbb{R}^n : |x| = 1\}$. Let $\sigma$ denote the rotation-invariant area measure on $S$, normalized so that $\sigma(S) = 1$. If $u$ is harmonic on $B$, then $u$ has the mean value property

$$u(y) = \int_S u(y + r\zeta)\, d\sigma(\zeta), \tag{1.1}$$

whenever the ball $B(y, r) \subset B$. For $-1 < \alpha < \infty$ let $dV_\alpha(x) = (1 - |x|^2)^\alpha\, dV(x)$, where $V$ denotes Lebesgue volume measure on $\mathbb{R}^n$. The *harmonic Bergman space* $b^{2,\alpha}(B)$ is the space of all harmonic functions $u$ which are in $L^2(B, V_\alpha)$. Integrating (1.1) with respect to $r$ we obtain

$$u(y) = \frac{1}{V_\alpha(B(y, r))} \int_{B(y,r)} u(x)\, dV_\alpha(x),$$

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whenever $B(y, r) \subset B$. Consequently, if $u$ is harmonic on $B$, then

$$(1.2)\qquad |u(y)| \le \frac{1}{V_\alpha(B(y,r))^{1/2}} \left( \int_{B(y,r)} |u(x)|^2 \, dV_\alpha(x) \right)^{1/2}$$

$$(1.3)\qquad \le \frac{1}{V_\alpha(B(y,r))^{1/2}} \|u\|.$$

whenever $B(y, r) \subset B$. Inequality (1.3) implies that $b^{2,\alpha}(B)$ is a closed subspace of $L^2(B, V_\alpha)$, thus a Hilbert space. We denote the orthogonal projection of $L^2(B, V_\alpha)$ onto $b^{2,\alpha}(B)$ by $Q_\alpha$. For $f \in L^\infty(B)$ the *Toeplitz operator* $T_f : b^{2,\alpha}(B) \to b^{2,\alpha}(B)$ is defined by

$$T_f u = Q_\alpha[fu], \qquad u \in b^{2,\alpha}(B).$$

The operator $T_f$ is clearly bounded on $b^{2,\alpha}(B)$: $\|T_f\| \le \|f\|_\infty$. For certain classes of symbols we will describe when these operators are compact.

## 2. Preliminaries

Inequality (1.3) shows that for fixed $y \in B$ the linear functional $u \mapsto u(y)$ is bounded on $b^{2,\alpha}(B)$. By the Riesz-Fischer Theorem, there exists a unique function $R_\alpha(\cdot, y) \in b^{2,\alpha}(B)$ for which

$$(2.1)\qquad u(y) = \langle u, R_\alpha(\cdot, y) \rangle, \qquad u \in b^{2,\alpha}(B).$$

The mapping $R_\alpha$ is called the *harmonic Bergman kernel* of $b^{2,\alpha}(B)$. Note that the projection operator $Q_\alpha$ is the integral operator with kernel $R_\alpha$:

$$(2.2)\qquad Q_\alpha[f](y) = \int_B f(x) R_\alpha(x, y) \, dV_\alpha(x),$$

for $f \in L^2(B, V_\alpha)$ and $y \in B$. The Bergman kernel is easily expressed in terms of the so-called zonal harmonics. We recall some terminology before we define these zonal harmonics.

A polynomial on $\mathbb{R}^n$ is homogeneous of degree $m$ (or *$m$-homogeneous*) if it is a finite linear combination of monomials $x_1^{k_1} \cdots x_n^{k_n}$, where $k_1, \cdots, k_n$ are non-negative integers such that $k_1 + \cdots + k_n = m$. Note that a polynomial $p$ on $\mathbb{R}^n$ is homogeneous of degree $m$ if and only if $p(tx) = t^m p(x)$ for all $x \in \mathbb{R}^n$ and all $t \in \mathbb{R}$. The space $\mathscr{H}_m(S)$ of restrictions to $S$ of harmonic homogeneous polynomials of degree $m$, the so-called *spherical harmonics* of degree $m$, is a (finite-dimensional) Hilbert space with respect to the usual inner product on $L^2(S, d\sigma)$, where $\sigma$ denotes the normalized surface-area measure on $S$. For each $\xi \in S$ the linear functional $p \mapsto p(\xi)$ on

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the space $\mathscr{H}_m(S)$ is uniquely represented by a harmonic $m$-homogeneous polynomial $Z_m(\cdot, \xi)$, called the zonal harmonic of degree $m$ at $\xi$. Extending $Z_m$ to a function on $\mathbb{R}^n \times \mathbb{R}^n$ by setting $Z_m(x, y) = |y|^m Z_m(x, y/|y|)$, and using that each zonal harmonic $Z_m(\cdot, \xi)$ is real valued (see [1, pp. 78–79]) we have

$$\int_S p(\zeta) Z_m(\zeta, y)\, d\sigma(\zeta) = p(y) \tag{2.3}$$

for every harmonic $m$-homogeneous polynomial $p$. Spherical harmonics of distinct degrees are orthogonal, that is,

$$\int_S p\, \bar{q}\, d\sigma = 0 \tag{2.4}$$

if $p$ and $q$ are harmonic homogeneous polynomials of distinct degree. If $p$ is an $m$-homogeneous harmonic polynomial, then, using integration by polar-coordinates,

$$\begin{aligned}\int_B p(x) Z_m(x, y)\, dV_\alpha(x) &= nV(B) \int_0^1 r^{n-1} \int_S p(r\zeta) Z_m(r\zeta, y)\, d\sigma(\zeta)(1-r^2)^\alpha\, dr \\ &= nV(B)p(y) \int_0^1 r^{n+2m-1}(1-r^2)^\alpha\, dr \\ &= \frac{n}{2} V(B) p(y) \frac{\Gamma(\frac{n}{2}+m)\Gamma(\alpha+1)}{\Gamma(\frac{n}{2}+m+\alpha+1)}.\end{aligned}$$

It follows that

$$R_\alpha(x, y) = \frac{2}{nV(B)} \sum_{m=0}^{\infty} \frac{\Gamma(\frac{n}{2}+m+\alpha+1)}{\Gamma(\frac{n}{2}+m)\Gamma(\alpha+1)} Z_m(x, y). \tag{2.5}$$

Using that $Z_m(y, y) = h_m|y|^{2m}$, where $h_m$ denotes the dimension of the space $\mathscr{H}_m(S)$ (see [1, page 80]), we get

$$R_\alpha(y, y) = \frac{2}{nV(B)} \sum_{m=0}^{\infty} \frac{\Gamma(\frac{n}{2}+m+\alpha+1)}{\Gamma(\frac{n}{2}+m)\Gamma(\alpha+1)} h_m|y|^{2m}.$$

The numbers $h_m$ can be expressed in terms of binomial coefficients:

$$h_m = \binom{n+m-2}{n-2} + \binom{n+m-3}{n-2}, \tag{2.6}$$

for $m \geq 1$ (see [1, page 94]), and it is easily shown that $h_m \sim 2m^{n-2}/(n-2)!$ as $m \to \infty$. By Stirling's formula, $\Gamma(m+c)/m! \sim m^{c-1}$ as $m \to \infty$, for all $c > 0$. Using $\approx$ to denote that the ratio of two quantities is bounded above and below by constants independent of the variable, we thus have $(\Gamma(\frac{n}{2}+m+\alpha+1)/\Gamma(\frac{n}{2}+m))h_m \approx \Gamma(m+n+\alpha)/m!$, and conclude that

$$R_\alpha(y, y) \approx \frac{1}{(1-|y|^2)^{n+\alpha}}. \tag{2.7}$$

## 3. Compact Toeplitz operators with continuous symbols

In this section we will describe compactness of Toeplitz operators on the harmonic Bergman spaces $b^{2,\alpha}(B)$ with uniformly continuous symbols. Our results generalize those recently obtained by Miao [4] for Toeplitz operators on the unweighted harmonic Bergman space ($\alpha = 0$). Miao made use of the explicit formula for the reproducing kernel of the unweighted harmonic Bergman space given in [1, Chapter 8]. For weighted harmonic Bergman spaces no such formula is available, and we will use a different approach to obtain estimates on the reproducing kernels.

PROPOSITION 3.1. *Let $-1 < \alpha < \infty$. If $f \in L^\infty(B)$ has compact support, then $T_f$ is compact on $b^{2,\alpha}(B)$.*

PROOF. If $f \in L^\infty(B)$ is supported in $\rho\bar{B}$, where $0 < \rho < 1$, then

$$\int_B \int_B |f(x)|\,|R_\alpha(x, y)|^2\, dV(x)\, dV(y) \le \|f\|_\infty \int_{\rho B} R_\alpha(y, y)\, dV(y) < \infty,$$

so $T_f$ is Hilbert-Schmidt on $b^{2,\alpha}(B)$.

PROPOSITION 3.2. *Let $-1 < \alpha < \infty$. In $b^{2,\alpha}(B)$ we have $R_\alpha(\cdot, y)/\|R_\alpha(\cdot, y)\| \to 0$ weakly as $|y| \to 1^-$.*

PROOF. Using the reproducing property and (2.7) we have

$$\|R_\alpha(\cdot, y)\|^2 = \langle R_\alpha(\cdot, y), R_\alpha(\cdot, y)\rangle = R_\alpha(y, y) \approx \frac{1}{(1 - |y|^2)^{n+\alpha}}.$$

It is easily verified that $V_\alpha(B(y, (1 - |y|)/2) \approx (1 - |y|)^{n+\alpha}$. So, if $u \in b^{2,\alpha}(B)$, then it follows with the help of (1.2) that

$$\left|\left\langle u, \frac{R_\alpha(\cdot, y)}{\|R_\alpha(\cdot, y)\|}\right\rangle\right| \le C \left(\int_{B(y,(1-|y|)/2)} |u(x)|^2\, dV_\alpha(x)\right)^{1/2},$$

and thus $\langle u, R_\alpha(\cdot, y)/\|R_\alpha(\cdot, y)\|\rangle \to 0$ as $|y| \to 1^-$.

For $-1 < \alpha < \infty$ and $f \in L^\infty(B)$ define the *Berezin transform* $\tilde{f}$ of $f$ by

$$\tilde{f}(y) = \left\langle T_f \frac{R_\alpha(\cdot, y)}{\|R_\alpha(\cdot, y)\|}, \frac{R_\alpha(\cdot, y)}{\|R_\alpha(\cdot, y)\|}\right\rangle, \quad y \in B.$$

Note that even though this is not reflected in the notation, $\tilde{f}$ does depend upon $\alpha$. Clearly, $|\tilde{f}(y)| \le \|T_f\| \le \|f\|_\infty$, for all $y \in B$.

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If $T_f$ is compact on $b^{2,\alpha}(B)$, then $T_f$ maps weakly null sequences to norm null sequences, so that by Proposition 3.2, $\|T_f(R_\alpha(\cdot, y)/\|R_\alpha(\cdot, y)\|)\| \to 0$ as $|y| \to 1^-$, and thus $\tilde{f}(y) \to 0$ as $|y| \to 1^-$. We do not know whether the converse is true in general. In the next section we will prove that the converse holds if $f$ is a radial function. Let $C_0(B)$ denote the subalgebra of $C(\bar{B})$ consisting of all continuous functions $g$ on $B$ such that $g(y) \to 0$ as $|y| \to 1^-$. For uniformly continuous symbols we have the following result.

THEOREM 3.3. *Let $-1 < \alpha < \infty$ and $f \in C(\bar{B})$. Then: $T_f$ is compact on $b^{2,\alpha}(B)$ if and only if $f \in C_0(B)$.*

In the proof we will need the following lemma.

LEMMA 3.4. *Let $-1 < \alpha < \infty$. If $0 < \delta < 1$, then there exists a finite positive number $C_\delta$ such that $|R_\alpha(x, y)| \le C_\delta$ whenever $x, y \in B$ are such that $|x - y| \ge \delta$.*

PROOF. If $k$ is an integer, then it follows from [2, Lemma 3.2] that there exists a finite number $C$ such that $|R_k(x, y)| \le C/(1 - 2x \cdot y + |x|^2|y|^2)^{(n+k)/2}$, for all $x, y \in B$. Since $1 - 2x \cdot y + |x|^2|y|^2 = |x - y|^2 + (1 - |x|^2)(1 - |y|^2) > |x - y|^2$, we obtain

$$|R_k(x, y)| \le \frac{C}{|x - y|^{n+k}}, \tag{3.5}$$

for all $x, y \in B$, and the stated result follows with $C_\delta = C/\delta^{n+k}$.

Suppose $k - 1 < \alpha < k$, where $k \ge 0$ is an integer. Using (2.5) it is easy to verify that

$$R_\alpha(x, y) = \frac{k!}{\Gamma(\alpha + 1)\Gamma(k - \alpha)} \int_0^1 2t^{n+2\alpha+1} R_k(tx, ty)(1 - t^2)^{k-\alpha-1}\, dt. \tag{3.6}$$

If $x, y \in B$ are such that $|x - y| \ge \delta$ it follows from (3.6) and (3.5) that

$$\begin{aligned} |R_\alpha(x, y)| &\le \frac{k!}{\Gamma(\alpha + 1)\Gamma(k - \alpha)} \int_0^1 2t^{n+2\alpha+1} |R_k(tx, ty)|(1 - t^2)^{k-\alpha-1}\, dt \\ &\le \frac{k!}{\Gamma(\alpha + 1)\Gamma(k - \alpha)} \frac{2C}{\delta^{n+k}} \int_0^1 t^{2\alpha-k+1}(1 - t^2)^{k-\alpha-1}\, dt. \end{aligned}$$

So the constant $C_\delta = C\Gamma(\alpha - \frac{k}{2} + 1)k!/(\delta^{n+k}\Gamma(\alpha + 1)\Gamma(\frac{k}{2} + 1))$ works.

PROOF OF THEOREM 3.3. The implication using $f \in C_0(B)$ follows easily from Proposition 3.1.

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To prove the other implication we need only show that $\tilde{f} - f \in C_0(B)$. Using Lemma 3.4 and the observation that

$$\tilde{f}(y) - f(y) = \frac{1}{R_\alpha(y, y)} \int_B (f(x) - f(y)) R_\alpha(x, y)^2 \, dV_\alpha(x)$$

we get

$$\begin{aligned} |\tilde{f}(y) - f(y)| &\le \frac{1}{R_\alpha(y, y)} \int_B |f(x) - f(y)| \, R_\alpha(x, y)^2 \, dV_\alpha(x) \\ &\le \omega(\delta) \frac{1}{R_\alpha(y, y)} \int_{B(y,\delta)} R_\alpha(x, y)^2 \, dV_\alpha(x) + \frac{2C_\delta^2 \|f\|_\infty}{R_\alpha(y, y)} \\ &\le \omega(\delta) + C'_\delta (1 - |y|^2)^{n+\alpha} \|f\|_\infty, \end{aligned}$$

where $\omega(\delta) = \sup\{|f(x) - f(z)| : x, z \in B, \ |x - z| < \delta\}$. Letting $|y| \to 1^-$ we obtain

$$\limsup_{|y| \to 1^-} |\tilde{f}(y) - f(y)| \le \omega(\delta),$$

for each $0 < \delta < 1$. Since $f \in C(\bar{B})$, $\omega(\delta) \to 0$ as $\delta \to 0^+$ we conclude that $\tilde{f}(y) - f(y) \to 0$ as $|y| \to 1^-$.

If $\mathscr{L}(b^{2,\alpha}(B))$ denotes the Banach algebra of all bounded linear operators on $b^{2,\alpha}(B)$, and $\mathscr{K}$ denotes the ideal of compact operators in $\mathscr{L}(b^{2,\alpha}(B))$, then the essential spectrum of an operator $T$ in $\mathscr{L}(b^{2,\alpha}(B))$, denoted by $\sigma_e(T)$, is the spectrum of the operator $T + \mathscr{K}$ in the Calkin algebra $\mathscr{L}(b^{2,\alpha}(B))/\mathscr{K}$.

THEOREM 3.7. *Let $-1 < \alpha < \infty$ and $f \in C(\bar{B})$. The essential spectrum of the operator $T_f$ on $b^{2,\alpha}(B)$ is $\sigma_e(T_f) = f(S)$.*

In the proof of the above theorem we will make use of Hankel operators. For $f \in L^\infty(B)$ the *Hankel operator* $H_f : b^{2,\alpha}(B) \to L^2(B, dV_\alpha)$ is defined by $H_f u = (I - Q_\alpha)[fu]$, $u \in b^{2,\alpha}(B)$. In [5] it was shown that for every $f \in C(\bar{B})$ the Hankel operator $H_f$ is compact on $b^{2,\alpha}(B)$. The following identity gives a simple relationship between Toeplitz and Hankel operators:

$$T_{fg} - T_f T_g = H_{\bar{f}}^* H_g, \tag{3.8}$$

for $f, g \in L^\infty(B)$.

PROOF OF THEOREM 3.7. We first show that $f(S) \subset \sigma_e(T_f)$. Suppose $\xi = f(\zeta)$, with $\zeta \in S$. We claim that $T_{f-\xi}$ cannot be left-invertible in the Calkin algebra, so that $\xi \in \sigma_e(T_f)$. To prove this claim we first observe that the argument in the proof of

Theorem 3.3 shows that $\|(f-f(y))R_\alpha(\cdot,y)/\|R_\alpha(\cdot,y)\|\| \to 0$ as $|y|\to 1^-$, and thus $\|T_{f-\xi}R_\alpha(\cdot,y)/\|R_\alpha(\cdot,y)\|\| \to 0$ as $y\to\zeta$ in $B$. If $T$ is a bounded linear operator on $b^{2,\alpha}(B)$, then

$$\left\langle (I-TT_{f-\xi})\frac{R_\alpha(\cdot,y)}{\|R_\alpha(\cdot,y)\|}, \frac{R_\alpha(\cdot,y)}{\|R_\alpha(\cdot,y)\|}\right\rangle = 1-\left\langle T_{f-\xi}\frac{R_\alpha(\cdot,y)}{\|R_\alpha(\cdot,y)\|}, \frac{T^*R_\alpha(\cdot,y)}{\|R_\alpha(\cdot,y)\|}\right\rangle \to 1$$

as $y\to\zeta$ in $B$. It follows that $TT_{f-\xi}-I$ cannot be compact on $b^{2,\alpha}(B)$, proving our claim.

To complete the proof we show that if $\xi\in\mathbb{C}\setminus f(S)$, then $T_{f-\xi}$ is invertible in the Calkin algebra. Suppose $\xi\notin f(S)$. Then there are $0<r<1$ and $g\in C(\bar{B})$ such that $(f-\xi)g=1$ on $\bar{B}\setminus B(0,r)$. The function $h=1-(f-\xi)g$ is compactly supported, so by Proposition 3.1, $T_h$ is compact on $b^{2,\alpha}(B)$. Using (3.8) we have

$$T_{f-\xi}T_g = T_{(f-\xi)g} - H^*_{\bar{f}-\bar{\xi}}H_g = I - T_h - H^*_{\bar{f}}H_g.$$

By [5, Theorem 4.3], the operator $H^*_{\bar{f}}H_g$ is compact, thus $T_h + H^*_{\bar{f}}H_g\in\mathscr{K}$, and consequently $T_{f-\xi}$ is right-invertible in the Calkin algebra. That $T_{f-\xi}$ is also left-invertible in the Calkin algebra is proved similarly.

COROLLARY 3.9. *Let $-1<\alpha<\infty$. If $f\in C(\bar{B})$, then the essential norm of $T_f$ on $b^{2,\alpha}(B)$ is given by $\|T_f\|_e = \sup_{\zeta\in S}|f(\zeta)|$.*

PROOF. By (3.8), $T_f^*T_f - T_fT_f^* = H^*_{\bar{f}}H_{\bar{f}} - H^*_fH_f$ is compact on $b^{2,\alpha}(B)$. Thus $T_f+\mathscr{K}$ is normal in the Calkin algebra, so that its norm is equal to its spectral radius, and the stated result follows from Theorem 3.7.

## 4. Compact Toeplitz operators with radial symbols

Korenblum and Zhu [3] proved that for a radial symbol the Toeplitz operator on the Bergman space of analytic functions on the unit disk is compact precisely when its Berezin transform vanishes near the unit circle. In [6] the author generalized Korenblum and Zhu's result to the setting of weighted Bergman spaces of analytic functions on the unit ball in $\mathbb{C}^n$. The following theorem shows that the analogous result holds for Toeplitz operators on weighted harmonic Bergman spaces.

THEOREM 4.1. *Let $-1<\alpha<\infty$ and let $f$ be a bounded measurable radial function on $B$. Then: $T_f$ is compact on $b^{2,\alpha}(B)$ if and only if $\tilde{f}\in C_0(B)$.*

The proof of Theorem 4.1 makes use of the following lemma.

LEMMA 4.2. *If $f$ is a bounded measurable radial function on $B$, then each homogeneous harmonic polynomial of degree $m$ is an eigenvector of $T_f$ with eigenvalue given by*

$$\lambda_m = \frac{\Gamma(\frac{n}{2}+m+\alpha+1)}{\Gamma(\frac{n}{2}+m)\Gamma(\alpha+1)} \int_0^1 2r^{n+2m-1}\varphi(r)(1-r^2)^\alpha\, dr, \tag{4.3}$$

*where $\varphi$ is a bounded measurable function on* $[0, 1)$, *for which $f(x) = \varphi(|x|)$, for all $x \in B$.*

PROOF. If $p$ is an $m$-homogeneous harmonic polynomial, then, using (2.3), (2.4) and (2.5),

$$\int_S p(\zeta)R_\alpha(r\zeta, y)\, d\sigma(\zeta) = \frac{2\Gamma(\frac{n}{2}+m+\alpha+1)}{nV(B)\Gamma(\frac{n}{2}+m)\Gamma(\alpha+1)} r^m p(y)$$

and thus we have

$$\begin{aligned}(T_f p)(y) &= \int_B f(x)p(x)R_\alpha(x, y)\, dV_\alpha(x)\\ &= nV(B)\int_0^1 r^{n+m-1}\varphi(r)\int_S p(\zeta)R_\alpha(r\zeta, y)\, d\sigma(\zeta)(1-r^2)^\alpha\, dr\\ &= p(y)\frac{\Gamma(\frac{n}{2}+m+\alpha+1)}{\Gamma(\frac{n}{2}+m)\Gamma(\alpha+1)}\int_0^1 2r^{n+2m-1}\varphi(r)(1-r^2)^\alpha\, dr,\end{aligned}$$

establishing the proof.

As in Korenblum and Zhu's argument, we will need a Tauberian theorem. The following lemma follows from a classical result of Hardy and Littlewood (see [6]).

LEMMA 4.4. *Let $0 < \nu < \infty$, and let $(b_m)$ be a sequence of complex numbers for which* $\sup\{|(m+1)b_m - mb_{m-1}| : m > 1\} < \infty$. *Then:*

$$(1-t)^{\nu+1}\sum_{m=0}^{\infty}\frac{\Gamma(m+\nu+1)}{m!\,\Gamma(\nu+1)} b_m t^m \to 0 \quad as \quad t \to 1^-$$

*if and only if $b_m \to 0$ as $m \to \infty$.*

PROOF OF THEOREM 4.1. If $f(x) = \varphi(|x|)$, for all $x \in B$, where $\varphi$ is a bounded measurable function on $[0, 1)$, then integrating by polar coordinates, using (2.5) and (2.4), it is readily verified that

$$\tilde{f}(y) = \frac{1}{R_\alpha(y, y)}\sum_{m=0}^{\infty}\frac{\Gamma(\frac{n}{2}+m+\alpha+1)}{\Gamma(\frac{n}{2}+m)\Gamma(\alpha+1)} h_m \lambda_m |y|^{2m},$$

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where $\lambda_m$ is as in (4.3).

Now, suppose $\tilde{f}(y) \to 0$ as $|y| \to 1^-$. Using (2.7) we get

$$(4.5) \qquad (1-|y|^2)^{n+\alpha} \sum_{m=0}^{\infty} \frac{\Gamma(\frac{n}{2}+m+\alpha+1)}{\Gamma(\frac{n}{2}+m)\Gamma(\alpha+1)} h_m \lambda_m |y|^{2m} \to 0$$

as $|y| \to 1^-$. We will use Lemma 4.4 to prove that $\lambda_m \to 0$ as $m \to \infty$. We first show that $\sup\{m|\lambda_m - \lambda_{m-1}| : m \geq 1\} < \infty$. Rewriting the integrand in the integral in (4.3) using $r^2\varphi(r) = \varphi(r) - \varphi(r)(1-r^2)$, it is easily seen that

$$\lambda_m = \frac{n+2m+2\alpha}{n+2m-2}\lambda_{m-1} + \frac{\Gamma(\frac{n}{2}+m+\alpha+1)}{\Gamma(\frac{n}{2}+m)\Gamma(\alpha+1)} \int_0^1 2r^{n+2m-3}\varphi(r)\,(1-r^2)^{\alpha+1}\,dr,$$

thus

$$\lambda_m - \lambda_{m-1} = \frac{2(1+\alpha)}{n+2m-2}\lambda_{m-1} + \frac{\Gamma(\frac{n}{2}+m+\alpha+1)}{\Gamma(\frac{n}{2}+m)\Gamma(\alpha+1)} \int_0^1 2r^{n+2m-3}\varphi(r)\,(1-r^2)^{\alpha+1}\,dr,$$

and the claim follows from the estimate

$$\left| \int_0^1 2r^{n+2m-3}\varphi(r)\,(1-r^2)^{\alpha+1}\,dr \right| \leq \|\varphi\|_\infty \int_0^1 2r^{n+2m-3}\,(1-r^2)^{\alpha+1}\,dr = \|f\|_\infty \frac{\Gamma(\frac{n}{2}+m-1)\Gamma(\alpha+2)}{\Gamma(\frac{n}{2}+m+\alpha+1)}.$$

Write

$$\frac{\Gamma(\frac{n}{2}+m+\alpha+1)}{\Gamma(\frac{n}{2}+m)\Gamma(\alpha+1)} h_m \lambda_m = \frac{\Gamma(m+n+\alpha)}{m!\,\Gamma(n+\alpha)} b_m.$$

Then

$$(m+1)b_m - mb_{m-1} = \{(m+1)a_m - ma_{m-1}\}\,\lambda_m + m(\lambda_m - \lambda_{m-1})a_{m-1},$$

where the $a_m$ are given by

$$a_m = \frac{m!\,\Gamma(n+\alpha)}{\Gamma(m+n+\alpha)} \frac{\Gamma(\frac{n}{2}+m+\alpha+1)}{\Gamma(\frac{n}{2}+m)\Gamma(\alpha+1)} h_m.$$

It follows that the $b_m$ satisfy the condition of Lemma 4.4 once we show that the $(m+1)a_m - ma_{m-1}$ are bounded. A calculation shows that

$$(m+1)a_m - ma_{m-1} = \frac{(m+1)!\,\Gamma(n+\alpha)\Gamma(\frac{n}{2}+m+\alpha+1)}{\Gamma(m+n+\alpha)\Gamma(\frac{n}{2}+m)\Gamma(\alpha+1)}(h_m - h_{m-1}) + \frac{(3-n)m+(n+\alpha)(2-\frac{n}{2})-1}{(m+n+\alpha-1)(\frac{n}{2}+m-1)} \frac{m!\,\Gamma(n+\alpha)\Gamma(\frac{n}{2}+m+\alpha)}{\Gamma(m+n+\alpha-1)\Gamma(\frac{n}{2}+m-1)\Gamma(\alpha+1)} h_{m-1}.$$

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It follows easily from (2.6) that

$$h_m - h_{m-1} = \frac{(n+2m-2)(n-1)}{m(m-1)} \frac{(n+m-4)!}{(m-2)!(n-2)!},$$

for $m \geq 2$, thus $h_m - h_{m-1} \approx m^{n-3}$. Recalling that $h_m \approx m^{m-2}$, our claim that the $(m+1)a_m - ma_{m-1}$ are bounded follows with the help of Stirling's formula. Applying Lemma 4.4 we conclude that $b_m \to 0$ as $m \to \infty$. Since the $a_m$ have a non-zero limit, we conclude that $\lambda_m \to 0$ as $m \to \infty$, and thus $T_f$ is compact on $b^{2,\alpha}(B)$.

## 5. Compact Toeplitz operators with discontinuous symbols

We write $b^2(B)$ for the unweighted harmonic Bergman space $b^{2,0}(B)$. If $T_f$ is compact on $b^2(B)$, then $T_{f^2}$ need not be compact. In fact, there are functions $f$ on $B$ for which $T_f$ is compact, and $T_{f^2}$ is the identity operator on $b^2(B)$. We will show this by considering a class of Toeplitz operators whose symbols are radial functions taking only the values 1 and $-1$.

Let $(r_k)$ and $(s_k)$ be sequences of positive numbers converging to 1 with $0 = r_1 < s_1 < r_2 < s_2 < \cdots$, and define $f$ on $B$ by

$$f(x) = \begin{cases} -1, & \text{if } r_k \leq |x| < s_k, \\ 1, & \text{if } s_k \leq |x| < r_{k+1}. \end{cases} \tag{5.1}$$

By Lemma 4.2 the eigenvalues of $T_f$ are given by

$$\lambda_m = \sum_{k=1}^{\infty} \left( r_{k+1}^{n+2m} - 2s_k^{n+2m} + r_k^{n+2m} \right). \tag{5.2}$$

THEOREM 5.3. *Let $f$ be the function given by* (5.1) *with $r_k = 1 - 1/k^c$, where $c > 0$, and $s_k = (r_k + r_{k+1})/2$. Then: $T_f$ is trace-class on $b^2(B)$ if $c < 1/(n-1)$.*

PROOF. Note that each of the terms of the series in (5.2) is positive. Using that the multiplicity of $\lambda_m$ is $h_m$ and the fact that $h_m \approx (m+n-2)!/m!$, we conclude that

$$\begin{aligned} \operatorname{tr}(T_f) = \sum_{m=0}^{\infty} \lambda_m h_m &= \sum_{k=1}^{\infty}\sum_{m=0}^{\infty} \left\{ r_{k+1}^{n+2m} + r_k^{n+2m} - 2s_k^{n+2m} \right\} h_m \\ &\leq C \sum_{k=1}^{\infty}\sum_{m=0}^{\infty} \left\{ r_{k+1}^{n+2m} + r_k^{n+2m} - 2s_k^{n+2m} \right\} \frac{(m+n-2)!}{m!\,(n-2)!} \\ &= C \sum_{k=1}^{\infty} \left\{ \frac{r_{k+1}^n}{(1-r_{k+1}^2)^{n-1}} + \frac{r_k^n}{(1-r_k^2)^{n-1}} - \frac{2s_k^n}{(1-s_k^2)^{n-1}} \right\}. \\ &\leq C' \sum_{k=1}^{\infty} \frac{\delta_k^2}{(1-r_k)^{n+1}}, \end{aligned}$$

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where $\delta_k = r_{k+1} - s_k = s_k - r_k$. So, if $\sum_{k=1}^{\infty} \delta_k^2/(1-r_k)^{n+1} < \infty$ then $T_f$ is trace-class on $b^2(B)$. For $r_k = 1 - 1/k^c$ and $s_k = (r_k + r_{k+1})/2$ we have $2\delta_k = 1/k^c - 1/(k+1)^c \approx 1/k^{c+1}$, thus $\delta_k^2/(1-r_k)^{n+1} \approx 1/k^{2-(n-1)c}$, and we see that $T_f$ is trace-class on $b^2(B)$ if $(n-1)c < 1$, that is, if $c < 1/(n-1)$.

THEOREM 5.4. *Let $f$ be the function given by (5.1) with $s_k = (r_k + r_{k+1})/2$ and put $\delta_k = r_{k+1} - s_k = s_k - r_k$. Then: $T_f$ is compact on $b^2(B)$ if and only if*

$$\delta_k/(1-r_k) \to 0 \quad as \quad k \to \infty.$$

In the proof of the above theorem we will need the following lemma.

LEMMA 5.5. *Let $m$ be a positive integer, and let $(r_k)$ be an increasing sequence of positive numbers converging to 1 and $2\delta_k = r_{k+1} - r_k$, for all $k$, then*

$$\sum_{k=1}^{\infty} \frac{2\delta_k}{(1-r_k^2 t)^{m+1}} \le \frac{1}{(1-t)^m} \tag{5.6}$$

*for all $0 < t < 1$.*

PROOF. Since $r_{k+1} > r_k$, we have

$$r_{k+1}^{j+1} - r_k^{j+1} \ge (r_{k+1} - r_k)(j+1)r_k^j \ge 2(j+1)\delta_k r_k^{2j},$$

and consequently

$$\sum_{k=1}^{\infty} 2\delta_k r_k^{2j} \le \frac{1}{j+1} \sum_{k=1}^{\infty} (r_{k+1}^{j+1} - r_k^{j+1}) = \frac{1}{j+1}(1 - r_1^{j+1}) \le \frac{1}{j+1}.$$

Thus

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{2\delta_k}{(1-r_k^2 t)^{m+1}} &= \sum_{j=0}^{\infty} \frac{(j+m)!}{j!\,m!} \left\{ \sum_{k=1}^{\infty} 2\delta_k r_k^{2j} \right\} t^j \le \sum_{j=0}^{\infty} \frac{(j+m)!}{j!\,m!} \frac{1}{j+1} t^j \\ &= \frac{1}{m} \left\{ \frac{1}{(1-|t|^2)^m} - 1 \right\} \Big/ t \le \frac{1}{(1-t)^m},\end{aligned}$$

proving the stated result.

PROOF OF THEOREM 5.4. Write $R$ for the reproducing kernel of $b^2(B)$.
*Step 1.* Using equation (2.5) it is easy to show that for $0 < r < 1$ we have

$$\int_{rB} R(x, y)^2 \, dV(x) = \sum_{m=0}^{\infty} (n+2m) r^{n+2m} h_m |y|^{2m}.$$

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Consequently,

$$\tilde{f}(y) = \frac{1}{R(y,y)} \sum_{k=1}^{\infty} \sum_{m=0}^{\infty} (n+2m)(r_{k+1}^{n+2m} - 2s_k^{n+2m} + r_k^{n+2m}) h_m |y|^{2m}. \tag{5.7}$$

*Step 2*. Similar to the proof of Theorem 5.3: there exists a finite positive constant $C$ such that

$$\tilde{f}(y) \leq C(1-|y|^2)^n \sum_{k=1}^{\infty} \left\{ \frac{r_{k+1}^n}{(1-r_{k+1}^2|y|^2)^n} + \frac{r_k^n}{(1-r_k^2|y|^2)^n} - \frac{2s_k^n}{(1-s_k^2|y|^2)^n} \right\},$$

for all $y \in B$.

*Step 3*. There is a finite positive constant $M$ for which we have the inequality

$$\frac{r_{k+1}^n}{(1-r_{k+1}^2|y|^2)^n} + \frac{r_k^n}{(1-r_k^2|y|^2)^n} - \frac{2s_k^n}{(1-s_k^2|y|^2)^n} \leq \frac{M\delta_k^2}{(1-r_k^2|y|^2)^{n+2}},$$

for all $k \geq 1$ and all $y \in B$.

*Step 4*. It follows from steps 2 and 3 and Lemma 5.5 that

$$\begin{aligned}
\tilde{f}(y) \leq\ & C(1-|y|^2)^n \sum_{k=1}^{K} \left\{ \frac{r_{k+1}^n}{(1-r_{k+1}^2|y|^2)^n} + \frac{r_k^n}{(1-r_k^2|y|^2)^n} - \frac{2s_k^n}{(1-s_k^2|y|^2)^n} \right\} \\
& + C'(1-|y|^2)^n \sum_{k=K}^{\infty} \frac{\delta_k^2}{(1-r_k^2|y|^2)^{n+2}} \\
\leq\ & C(1-|y|^2)^n \sum_{k=1}^{K} \left\{ \frac{r_{k+1}^n}{(1-r_{k+1}^2|y|^2)^n} + \frac{r_k^n}{(1-r_k^2|y|^2)^n} - \frac{2s_k^n}{(1-s_k^2|y|^2)^n} \right\} \\
& + C'(1-|y|^2)^n \sup_{k\geq K} \left( \frac{\delta_k}{1-r_k} \right) \sum_{k=K}^{\infty} \frac{\delta_k}{(1-r_k^2|y|^2)^{n+1}} \\
\leq\ & C(1-|y|^2)^n \sum_{k=1}^{K} \left\{ \frac{r_{k+1}^n}{(1-r_{k+1}^2|y|^2)^n} + \frac{r_k^n}{(1-r_k^2|y|^2)^n} - \frac{2s_k^n}{(1-s_k^2|y|^2)^n} \right\} \\
& + C' \sup_{k\geq K} \left( \frac{\delta_k}{1-r_k} \right),
\end{aligned}$$

for all $y \in B$ and every integer $K \geq 1$. Because each of the terms in the series (5.7) is positive,

$$\limsup_{|y|\to 1^-} |\tilde{f}(y)| \leq C' \sup_{k\geq K} \left( \frac{\delta_k}{1-r_k} \right),$$

for every integer $K \geq 1$. So, if $\delta_k/(1-r_k) \to 0$ as $k \to \infty$, then $\tilde{f}(y) \to 0$ as $|y| \to 1^-$, and by Theorem 4.1, $T_f$ is compact on $b^2(B)$.

To prove the converse, note that the inequalities in steps 2 and 3 can be reversed to obtain

$$\tilde{f}(y) \geq c(1-|y|^2)^n \sum_{k=1}^{\infty} \frac{\delta_k^2}{(1-r_k^2|y|^2)^{n+2}},$$

for all $y \in B$. In particular, for $\zeta \in S$ we have $\tilde{f}(r_k\zeta) \geq c(1-r_k^2)^n\delta_k^2/(1-r_k^4)^{n+2}$, which easily implies that $(\delta_k/(1-r_k))^2 \leq C\tilde{f}(r_k\zeta)$, for all $k \geq 1$. If $T_f$ is compact on $b^2(B)$, then $\tilde{f}(r_k\zeta) \to 0$, and hence $\delta_k/(1-r_k) \to 0$ as $k \to \infty$.

COROLLARY 5.8. *Let $f$ be the function given by* (5.1) *with $r_k = 1 - 1/k^c$, where $c > 0$ and $s_k = (r_k + r_{k+1})/2$. Then $T_f$ is compact on $b^2(B)$.*

PROOF. Since $\delta_k \approx 1/k^{c+1}$, we have $\delta_k/(1-r_k) \approx 1/k$.

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