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APPROXIMATION BY FUNCTIONS WITH BOUNDED DERIVATIVE ON BANACH SPACES

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Let X be a separable Banach space which admits a C^1-smooth norm, and let $G \subset X$ be an open subset. Then any real-valued, bounded and uniformly continuous map on G can be uniformly approximated on G by C^1-smooth functions with bounded derivative.

1. INTRODUCTION

In this note we consider the problem of uniformly approximating continuous, real-valued functions on Banach spaces by certain smooth functions. This question has its roots at least as far back as Weierstrass' classical theorem. For finite dimensional spaces, this problem is well understood. For infinite dimensional Banach spaces X, many results are also known, although an infinite dimensional version of the theorem of Weierstrass does not hold (see for example, [**8**]). One of the first results in this direction is due to Kurzweil, and it is shown in [**6**] that for separable spaces possessing a separating polynomial, uniform approximation of continuous maps by (real) analytic functions is possible. It is worth noting that uniform approximation of the norm of X by a C^k-smooth map enables one to construct a C^k-smooth bump function on X (a C^k-smooth function with bounded, nonempty support), and so the existence of a C^k-smooth bump function is a necessary condition for any such approximation theorems.

The next seminal paper in this area is by Bonic and Frampton [**2**], where among many other results it is shown that if X is separable and admits a C^k-smooth bump function, then any continuous function on X can be uniformly approximated by C^k-smooth maps. A version of this theorem was later shown to hold for general reflexive spaces in the key paper [**9**] by Torunczyk, and both these results were generalised to weakly compactly generated spaces in [**5**].

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Recently it has been shown that on superreflexive spaces (Banach spaces admitting a Fréchet smooth norm with uniformly continuous derivative on the unit sphere), Lipschitz functions can be uniformly approximated on bounded sets by smooth maps with uniformly continuous derivative ([**1**] and see also, [**4**]).

If we assume that the function to be approximated is convex, more is known. For example, in [**7**] it is proven that if a Banach space X has a norm with modulus of smoothness of power type 2, then any convex function bounded on bounded sets may be uniformly approximated on bounded sets by convex functions having Lipschitz derivatives. On the other hand, it should be noted that l_2 possesses an equivalent norm with Lipschitz derivative that cannot be uniformly approximated on the unit ball by smooth functions having uniformly continuous second derivative [**10**].

Despite the above mentioned results, the general question of approximation by maps with bounded derivative has remained open. We note that the uniform approximation of functions $F : X \to \mathbb{R}$ by smooth maps with bounded derivative is only possible for uniformly continuous F. In this paper we settle the problem for the separable case and bounded F, by showing that if X is separable and admits a C^1-smooth norm, then we can uniformly approximate bounded, uniformly continuous functions by C^1-smooth maps having a bounded derivative.

The notation we use is standard. X denotes a Banach space. An *open* ball with centre $p \in X$ and radius $r > 0$ is written $B_r(p)$. Smoothness here is meant in the Fréchet sense unless otherwise stated, and function shall mean real-valued function.

2. Main Results

Theorem 1. *Let X be a separable Banach space which admits a C^1-smooth norm, and $G \subset X$ open. Let $F : G \to \mathbb{R}$ be a bounded and uniformly continuous map. Then for each $\varepsilon > 0$ there exists a C^1-smooth map K on G with bounded derivative, such that for all $x \in G$,*

$$\left|K(x) - F(x)\right| < \varepsilon$$

Proof: We shall work in an equivalent C^1-smooth norm, $\|\cdot\|_1$, on X. We fix an equivalent C^1-smooth norm on c_0, $\|\cdot\|$, such that

$$\|x\|_{c_0} \leqslant \|x\| \leqslant A\,\|x\|_{c_0}, \tag{2.1}$$

for some $A > 0$, and where $\|\cdot\|_{c_0}$ is the canonical supremum norm on c_0 (see for example, [**3**, Remark V.1.6]). We let $M \geqslant 1$ be a bound for F on G. Because F is bounded, by replacing F with $F + 2M$ if necessary, we may and do suppose that $F \geqslant 1$ on G. Let $\varepsilon \in (0,1)$ also be fixed for the remainder of the proof, and by the uniform continuity of F, fix $\delta > 0$ so that

$$\|x - y\|_1 < \delta \text{ implies } \left|F(x) - F(y)\right| < \frac{\varepsilon}{A}. \tag{2.2}$$

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By separability, let $\{x_j\}_{j=1}^{\infty}$ be a dense subset of G, and define an open covering of G by

$$C_j^1 = \{B_{\delta/4}(x_j)\}_{j=1}^{\infty}.$$

We write $C_j^2 = B_{\delta/2}(x_j)$ and $C_j^3 = B_\delta(x_j)$ to give two further open covers $\{C_j^2\}$ and $\{C_j^3\}$ for G. Also, define a cover $\{D_j\}$ of G by

$$D_1 = C_1^2$$
$$D_j = C_j^2 \backslash \Big(\bigcup_{i<j} C_i^2\Big).$$

Let $\zeta_1 \in C^1(\mathbb{R}, [0,1])$ be Lipschitz such that, $\zeta_1(t) = 0$ if and only if $t \leqslant \delta/2$, and $\zeta_1(t) = 1$ if and only if $t \geqslant \delta$. With this notation, we define for each j a C^1-smooth map on X by

$$f_j(x) = \zeta_1(\|x - x_j\|_1). \tag{2.3}$$

Next, let $\zeta_2 \in C^1(\mathbb{R}, [0,1])$ be Lipschitz such that, $\zeta_2(t) = 0$ if and only if $t \geqslant \delta/2$, and $\zeta_2(t) = 1$ if and only if $t \leqslant \delta/4$. We set $g_0 = 0$ and for each $j \geqslant 1$ define a C^1-smooth map on G by

$$g_j(x) = \zeta_2(\|x - x_j\|_1). \tag{2.4}$$

We observe that because both ζ_1 and ζ_2 are Lipschitz as is $\|\cdot\|_1$, it follows from (2.3) and (2.4) that both f_j and g_j are Lipschitz with constants independent of j.

Next, let e_k be the canonical basis vectors in c_0, and set $c_{00} = \operatorname{span}\{e_k\} \subset c_0$. For each $j \geqslant 1$, define a C^1-smooth map $\phi_j : X \to c_{00}$ by

$$\phi_j(x) = \{g_1(x), g_2(x), \dots, g_{j-1}(x), 0, 0, \dots\}.$$

Note that the maps $x \to \|\phi_j(x)\|^2$ are C^1-smooth, where $\|\cdot\|$ is the C^1-smooth norm on c_0 chosen above. Moreover, these maps are Lipschitz with constant independent of j, since the same is true of the g_j and the g_j are all bounded above by 1.

Put,

$$\psi_j(x) = f_j(x) + \|\phi_j(x)\|^2.$$

As noted above, because the functions f_j and $\|\phi_j(x)\|^2$ are Lipschitz with constant independent of j, we have that the maps ψ_j are Lipschitz with constant independent of j.

We collect some of the properties of the functions ψ_j in the next lemma.

Lemma 1. *For the functions ψ_j defined above, we have*

(i) $\psi_j(x) \geqslant 1$ *for all* $x \in G\backslash C_j^3$

(ii) *For each* $x \in G$, *there exists a* j_0 *such that*

$$\psi_{j_0}(x) = 0.$$

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(iii) *For each $x \in G$, there exists a j_0 and an $\eta > 0$ such that*

$$\psi_j(y) \geqslant 1$$

for all y with $\|y - x\|_1 < \eta$ and $j > j_0$.

PROOF: (i) For any $x \in G$ we have, since $\|\phi_j(x)\|^2 \geqslant 0$,

$$\psi_j(x) \geqslant f_j(x),$$

while for $x \in G \backslash C_j^3$ we have by construction that $f_j(x) = 1$, and the result follows.

(ii) Fix $x \in G$. From the construction of the cover $\{D_j\}$, we have that there exists a j_0 with $x \in C_{j_0}^2$ while $x \in G \backslash C_i^2$ for $i = 1, \ldots, j_0 - 1$. It now follows from the construction of the f_j and g_j that $f_{j_0}(x) = 0$, and $\|\phi_{j_0}(x)\| = 0$. Hence,

$$\psi_{j_0}(x) = f_{j_0}(x) + \|\phi_{j_0}(x)\|^2 = 0.$$

(iii) Fix $x \in G$, and note that for any j, since $f_j \geqslant 0$,

$$\psi_j(x) \geqslant \|\phi_j(x)\|^2.$$

Now, using the fact that the $\{x_j\}$ are dense, for some $j_0 > 1$, $x \in C_{j_0}^1$, and by openness of $C_{j_0}^1$, we choose $\eta > 0$ so that $B_\eta(x) \subset C_{j_0}^1$. By the construction of the g_j, for $y \in B_\eta(x)$ we have $g_{j_0}(y) = 1$, and so for $j > j_0$ and such y,

$$\|\phi_j(y)\| \geqslant \|\phi_j(y)\|_{c_0} = \max_{1 \leqslant k < j} |g_k(y)| \geqslant 1.$$

It follows that for $j > j_0$ and $y \in B_\eta(x)$, $\|\phi_j(y)\|^2 = 1$, and the result follows.

Let $h \in C^1(\mathbb{R}, [0,1])$ be Lipschitz such that $h' \leqslant 0$, $h(t) = 1$ if and only if $t \leqslant 1/4$, and $h(t) = 0$ for $t \geqslant 1$. It follows from the fact that h is Lipschitz and our observations above, that the maps $h(\psi_j(x))$ are Lipschitz with constant independent of j. ∎

Observe that by Lemma 1 (iii), for each $x \in G$ there are a neighbourhood N_x and a j_x so that $y \in N_x$ and $j > j_x$ implies $\psi_j(y) \geqslant 1$. Therefore we have by the definition of h that for $y \in N_x$ and $j > j_x$, $h(\psi_j(y)) = 0$, and it follows from this that the map $x \to \{h(\psi_j(x))\}$ depends locally on only finitely many coordinates. We next define the function $\psi : G \to \mathbb{R}$ by,

$$\psi(x) = \left\| \left\{ h(\psi_j(x) \right\} \right\|,$$

and note that below in (2.5) we show that on G, $\psi \geqslant 1$, from which it will follow that ψ is C^1-smooth on G.

Finally, we set

$$K(x) = \frac{\|\{F(x_j)h(\psi_j(x))\}\|}{\psi(x)}.$$

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We now show that $\psi \geqslant 1$ on G, and since $F \geqslant 1$, this will show that both ψ and K are C^1-smooth on G. Note that by Lemma 1 (ii), for each x there exists j_0 such that $\psi_{j_0}(x) = 0$, and so $h(\psi_{j_0}(x)) = 1$. Hence,

$$\psi(x) = \left\|\left\{h(\psi_j(x))\right\}\right\| \geqslant \left\|\left\{h(\psi_j(x))\right\}\right\|_{c_0} \geqslant h(\psi_{j_0}(x)) = 1. \tag{2.5}$$

We next show that K is Lipschitz on G, from which the boundedness of K' will follow. As pointed out above, the functions $h(\psi_j(x))$ are Lipschitz with constant independent of j, and so are the functions $F(x_j)h(\psi_j(x))$, since F is bounded. It now follows that the numerator of K is a bounded, coordinatewise Lipschitz map into c_0, and as such is bounded and Lipschitz. Finally, the denominator of K is Lipschitz and bounded below by 1 from (2.5), and so the quotient function K is Lipschitz.

Finally we show $|K(x) - F(x)| < \varepsilon$ for $x \in G$. We use below the fact that $F \geqslant 0$. Fix $x \in G$. Then,

$$\begin{aligned}
|K(x) - F(x)| &= \left|\frac{\|\{F(x_j)h(\psi_j(x))\}\|}{\psi(x)} - F(x)\right| \\
&= \left|\frac{\|\{F(x_j)h(\psi_j(x))\}\|}{\psi(x)} - \frac{F(x)\psi(x)}{\psi(x)}\right| \\
&= \frac{1}{\psi(x)}\left|\left\|\left\{F(x_j)h(\psi_j(x))\right\}\right\| - F(x)\left\|\left\{h(\psi_j(x))\right\}\right\|\right| \\
&= \frac{1}{\psi(x)}\left|\left\|\left\{F(x_j)h(\psi_j(x))\right\}\right\| - \left\|\left\{F(x)h(\psi_j(x))\right\}\right\|\right| \\
&\leqslant \frac{1}{\psi(x)}\left\|\left\{F(x_j)h(\psi_j(x))\right\} - \left\{F(x)h(\psi_j(x))\right\}\right\| \\
&= \frac{1}{\psi(x)}\left\|\left\{h(\psi_j(x))\left(F(x_j) - F(x)\right)\right\}\right\|.
\end{aligned}$$

Now, using (2.1),

$$\begin{aligned}
\left\|\left\{h(\psi_j(x))\left(F(x_j) - F(x)\right)\right\}\right\| &\leqslant A\left\|\left\{h(\psi_j(x))\left(F(x_j) - F(x)\right)\right\}\right\|_{c_0} \\
&= A\max_j\left\{\left|h(\psi_j(x))\left(F(x_j) - F(x)\right)\right|\right\} \\
&= A\max_j\left\{h(\psi_j(x))\left|\left(F(x_j) - F(x)\right)\right|\right\}.
\end{aligned}$$

Set $J = \{j : x \in C_j^3\}$, and note that for all $j \in J$ we have, since $x, x_j \in C_j^3 = B_\delta(x_j)$ and using (2.2),

$$h(\psi_j(x))\left|\left(F(x_j) - F(x)\right)\right| < h(\psi_j(x))\frac{\varepsilon}{A} \leqslant \frac{\varepsilon}{A}.$$

On the other hand, for $j \notin J$ we have by Lemma 1 (i),

$$h(\psi_j(x))\left|\left(F(x_j) - F(x)\right)\right| \leqslant h(\psi_j(x))2M = 0.$$

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Therefore, since $\psi(x) \geqslant 1$ by (2.5), for any j we have

$$\frac{h(\psi_j(x))|(F(x_j) - F(x))|}{\psi(x)} < \frac{\varepsilon}{A}.$$

It follows that

$$|K(x) - F(x)| < \varepsilon.$$

∎

The conditions on F in Theorem 1 cannot be relaxed very much, as the following indicates.

Corollary 1. *Let X be a separable Banach space which admits a C^1-smooth norm. Then a function F can be uniformly approximated on bounded sets by C^1-smooth functions with bounded derivative if and only if F is uniformly continuous and bounded on bounded sets.*

Proof: Sufficiency is by Theorem 1. For necessity, note that since functions with bounded derivative are Lipschitz, it follows easily from the hypothesis that on bounded sets, any such F is uniformly continuous and bounded. ∎

Remarks.

(i) The hypothesis that X be separable and admit a C^1-smooth norm is equivalent to X^* being separable (see for example, [**3**, Corollary II.3.3]).

(ii) Because separable spaces admit Gâteaux smooth norms (see for example, [**3**, Theorem II.3.1(ii)]), a Gâteaux smooth version of Theorem 1 holds for any separable space X without additional assumptions on X.

(iii) Since c_0 does not admit a norm with bounded second derivative, the method of proof here will not work for approximation by maps with bounded second derivative.

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