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THE $\mathscr{J}$-CLASSES OF AN INVERSE SEMIGROUP

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Abstract

It is shown, using the author's construction for 'labelled semilattices', that every partially ordered set, in which every two elements have a common lower bound, is isomorphic to the partially-ordered set of $\mathscr{J}$-classes of some completely semi-simple inverse semigroup.

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1. Introduction

The set of $\mathscr{J}$-classes of a semigroup forms, in a natural way, a partially-ordered set. It is easy to see that this partially-ordered set is *downward-directed*, that is, every two members have a common lower bound. Rhodes (1972) asked whether every downward-directed partially-ordered set arises in this way.

We answer this question affirmatively; in fact we prove the stronger statement that every downward-directed partially-ordered set, P, is isomorphic to the partially-ordered set of $\mathscr{J}$-classes of a completely semi-simple *inverse* semigroup.

This result was obtained for the case where P is *finite* by Hall (1973). A finite downward-directed partially-ordered set must have a least element, so a generalization of this is the case where *P has a least element*. This was established by Ash and Hall (1975). The case where P has no least element seems more involved. The proof given here depends on the existence of certain 'labelled' semilattices, which is established by Ash (1979). The terminology and basic definitions for semigroups are as in Clifford and Preston (1961 and 1967).

Preliminaries

Details of the following summary are given in Chapters II and V of Howie (1976).

For any semigroup S, Green's relations $\mathscr{L}$, $\mathscr{R}$ and $\mathscr{J}$ are the equivalence relations defined by $x\mathscr{L}y$ ($x\mathscr{R}y, x\mathscr{J}y$) if x and y generate the same left (right, two-sided) principal ideal. The equivalence relations $\mathscr{L}$ and $\mathscr{R}$ permute, that is, the composites, $\mathscr{L} \circ \mathscr{R}$ and $\mathscr{R} \circ \mathscr{L}$, are equal. Thus, the relation $\mathscr{D} = \mathscr{L} \circ \mathscr{R} = \mathscr{R} \circ \mathscr{L}$ is an equivalence relation, and $x\mathscr{D}y$ implies $x\mathscr{J}y$. The $\mathscr{L}$, $\mathscr{R}$ and $\mathscr{J}$-classes are partially-ordered by inclusion between the corresponding sorts of ideals. We let $J(x)$ denote the $\mathscr{J}$-class of $x \in S$.

Now suppose that S is an *inverse* semigroup. Let E be the semilattice of idempotents of S. For $x \in S$, $x(x^{-1}x) = x$ and so $x\mathscr{L}(x^{-1}x)$. Similarly, $x\mathscr{R}(xx^{-1})$. Now $x^{-1}x$, $x^{-1}x \in E$ and so we have the following

PROPOSITION 1.1. *Every $\mathscr{L}$, $\mathscr{R}$, $\mathscr{D}$ or $\mathscr{J}$-class of S contains an idempotent.*

The definitions of $\mathscr{D}$ and $\mathscr{J}$ are now simplified by the following.

PROPOSITION 1.2. *For $e,f \in E$, $e\mathscr{D}f$ if and only if, for some $x \in S$, $e = xx^{-1}$ and $f = x^{-1}x$.*

PROPOSITION 1.3. *For $e, f \in E$, $J(e) \leqslant J(f)$ if and only if, for some $g \in E$, $e\mathscr{D}g$ and $g \leqslant f$.*

For $e \in E$, Ee denotes $\{ge: g \in E\} = \{g \in E: g \leqslant e\}$.

PROPOSITION 1.4. *If $x \in S$, $xx^{-1} = e$ and $x^{-1}x = f$, then the map $\alpha_x: E \to E$, defined by $g \mapsto x^{-1}gx$, is a semilattice isomorphism from Ee to Ef.*

2. Labelled semilattices

Let P be a partially-ordered set.

DEFINITION 2.1. A *P-labelled semilattice* is a pair (E, l) for which E is a semilattice and $l: E \to P$ is a function (or 'labelling') with the property that, for all $e,f \in E$, if $e < f$ (in the ordering of E) then $l(e) < l(f)$ (in the ordering of P).

Let S be a completely semi-simple inverse semigroup, that is, one in which no two comparable idempotents are $\mathscr{D}$-related or, equivalently, no two comparable idempotents are $\mathscr{J}$-related. Let P denote the partially-ordered set of $\mathscr{J}$-classes of elements of S.

The function $J: E \to P$ given by $e \mapsto J(e)$ has the property that if $e,f \in E$, $e < f$, then $J(e) \leqslant J(f)$ and, by assumption, $J(e) < J(f)$. We thus have the following

THEOREM 2.2. *(E, J) is a P-labelled semilattice.*

DEFINITION 2.3. A P-labelled semilattice (E, l) is *full* if
(i) for all $p \in P$, there exists $e \in E$ for which $l(e) = p$, and
(ii) for all $p, q \in P$ with $p < q$, there exist $e, f \in E$ with $e < f$, $l(e) = p$ and $l(f) = q$.

Let S, E, J and P be as above. Each $p \in P$ is $J(e)$ for some $e \in E$, by Proposition 1.1. If $p, q \in P$ and $p < q$, then there are $g, f \in E$ with $p = J(g) < J(f) = q$. Thus, by Proposition 1.3, there exists $e \in E$ with $e < f$ and $g \mathscr{D} e$, so that $J(e) = J(g) = p$. This establishes the following

THEOREM 2.4. *(E, J) is a full P-labelled semilattice.*

DEFINITION 2.5. A P-labelled semilattice (E, l) is *uniform*, if, for all $e, f \in E$ with $l(e) = l(f)$, there exists a semilattice isomorphism α: $Ee \cong Ef$ such that, for all $g \in Ee$, $l(\alpha(g)) = l(g)$.

THEOREM 2.6. *If S, E, J and P are as above, then (E, J) is a uniform P-labelled semilattice.*

PROOF. If $e, f \in E$ and $J(e) = J(f)$, then by Proposition 1.3, $e \mathscr{D} g$ for some $g \in E$ with $g \leqslant f$. By assumption on S, $g = f$. So $e \mathscr{D} f$ and, by Proposition 1.2, $e = xx^{-1}$ and $f = x^{-1}x$ for some $x \in S$. By Proposition 1.4, the function α_x: $Ee \to Ef$ is a semilattice isomorphism. Now take any $g \in Ee$. Then

$$\alpha_x(g) = x^{-1}gx \quad \text{and} \quad x(x^{-1}gx)x^{-1} = ege = g,$$

so $\alpha_x(g) \mathscr{D} g$, and therefore $J(\alpha_x(g)) = J(g)$.

3. The existence of uniform labelled semilattices

Theorems 2.2, 2.4, 2.6 show that, if P is the partially-ordered set of $\mathscr{J}$-classes of a completely semi-simple inverse semigroup, then there exists a full, uniform P-labelled semilattice. Any two $\mathscr{J}$-classes, $J(e)$ and $J(f)$, have a common lower bound, $J(ef)$, so such a P is downward-directed.

The Main Theorem of Ash (1979) establishes the converse.

PROPOSITION 3.1. *For any downward-directed partially-ordered set, P, there exists a full, uniform, P-labelled semilattice.*

From this we may deduce the desired result, as follows.

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4. Inverse semigroups from labelled semilattices

Let P be any partially-ordered set, (E, l) any P-labelled semilattice.

DEFINITION 4.1. $T_{E,l}$ denotes the set of semilattice isomorphisms $\alpha: Ee \cong Ef$ between principal ideals Ee, Ef, of E such that, for each $g \in Ee$, $l(\alpha(g)) = l(g)$. Clearly, $T_{E,l}$ forms an inverse semigroup under composition.

$T_{E,l}$ is an inverse subsemigroup of T_E, defined in Munn (1966). The definition of $T_{E,l}$ is clearly, in a sense, a generalization of that of T_E. For $e \in E$, let $[e]$ denote the identity function on Ee. As usual, for $\alpha \in T_{E,l}$, $J(\alpha)$ denotes the $\mathcal{J}$-class of α.

THEOREM 4.2. (i) *The idempotents of* $T_{E,l}$ *are exactly the elements* $[e]$ *for* $e \in E$ *and* $[e] \leqslant [f]$ *if and only if* $e \leqslant f$.

(ii) *If* $e, f \in E$, *and* $J([e]) \leqslant J([f])$ *in* $T_{E,l}$, *then* $l(e) \leqslant l(f)$.

PROOF. (i) By the properties of inverse semigroups, every idempotent of $T_{E,l}$ is $\alpha\alpha^{-1}$ for some $\alpha \in T_{E,l}$. If $\alpha: Ee \to Eg$, then $\alpha\alpha^{-1} = [e]$. Conversely, clearly $[e][f] = [ef]$, so each $[e]$ is idempotent and $[e] \leqslant [f]$ if and only if $e \leqslant f$.

(ii) If $J([e]) \leqslant J([f])$ then, by Propositions 1.2 and 1.3, there exist $g \in E$ and $\alpha \in T_{E,l}$ for which $[g] \leqslant [f]$, $\alpha\alpha^{-1} = [e]$ and $\alpha^{-1}\alpha = [g]$. Thus, $g \leqslant f$ and $\alpha: Ee \cong Eg$. By definition of $T_{E,l}$, $l(e) = l(g)$ and, by Definition 2.1, $l(g) \leqslant l(f)$, so $l(e) \leqslant l(f)$.

When (E, l) is full and uniform, we have the following converse of Theorem 4.2(ii).

THEOREM 4.3. *Let* (E, l) *be a full, uniform* P*-labelled semilattice. If* $e, f \in E$ *and* $l(e) \leqslant l(f)$, *then* $J([e]) \leqslant J([f])$.

PROOF. Suppose $e, f \in E$ and $l(e) \leqslant l(f)$. By Definition 2.3, there exists $g \leqslant f$ with $l(e) = l(g)$. By Definition 2.5, there exists $\alpha: Ee \cong Eg$ with $a \in T_{E,l}$. Thus $\alpha\alpha^{-1} = [e]$ and $\alpha^{-1}\alpha = [g]$, so, by Propositions 1.2 and 1.3, $J([e]) \leqslant J([f])$.

From Theorems 4.2 and 4.3, we have the following

COROLLARY 4.4. *If* (E, l) *is a full, uniform* P*-labelled semilattice, then* $T_{E,l}$ *is a completely semi-simple inverse semigroup whose partially ordered set of* $\mathcal{J}$*-classes is isomorphic to* P.

PROOF. Suppose that $T_{E,l}$ does contain two comparable $\mathcal{D}$-related idempotents. By Theorem 4.2(i), these are of the form $[e]$, $[f]$ with $e, f \in E$ and $e < f$. By

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Definition 2.1, $l(e) < l(f)$. But now $[e]\mathscr{D}[f]$, so $J([f]) = J([e])$. Thus, by Theorem 4.2(ii), $l(f) \leqslant l(e)$, which is contradictory.

By Proposition 1.1, the $\mathscr{J}$-classes of $T_{E,l}$ are exactly $J([e])$ for some $e \in E$. By Theorems 4.2(ii) and 4.3, for $e, f \in E$, $J([e]) \leqslant J([f])$ if and only if $l(e) \leqslant l(f)$. Also, by Definition 2.3, each element of P is $l(e)$ for some $e \in E$. Thus, the map $J([e]) \mapsto l(e)$ provides an isomorphism between the partially-ordered set of $\mathscr{J}$-classes of $T_{E,l}$ and P.

Proposition 3.1, Definition 4.1 and Corollary 4.4 now yield our Main Theorem.

THEOREM 4.5. *Let P be any downward-directed partially-ordered set. Then there exists a completely semi-simple inverse semigroup whose partially-ordered set of $\mathscr{J}$-classes is isomorphic to P.*

5. Conclusion

The problem of proving Theorem 4.5 led the author, eventually, to the rather elaborate construction which establishes Proposition 3.1 and which is given in Ash (1979). Perhaps a simpler proof of Theorem 4.5 is possible, in which case Theorems 2.2, 2.4 and 2.6 would give a simpler proof of Proposition 3.1. Otherwise, Proposition 3.1 establishes the existence of inverse semigroups for which no other construction is known, and it is hoped that this may be useful in other contexts.

It would, perhaps, be of interest to characterize other aspects of Green's relations on semigroups. The most general question would be as follows. For any semigroup, let H be its set of $\mathscr{H}$-classes and let $\leqslant_L$, $\leqslant_R$ be the partial-orderings induced on H by inclusion between left, right ideals respectively. Then the relations induced on H by $\mathscr{L}$, $\mathscr{R}$, $\mathscr{D}$ and $\mathscr{J}$ and by inclusion between two-sided ideals are all determined by $\leqslant_L$ and $\leqslant_R$. Which structures $(H, \leqslant_L, \leqslant_R)$ arise in this way?

ADDED IN PROOF: Theorem 4.5 has recently been obtained independently by J. Meakin. His paper, "The partially-ordered set of $\mathscr{J}$-classes ot an inverse semigroup" is to appear in the Journal of the London Mathematical Society.

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