4

Fusion Ring and Explicit Verlinde Formula

The aim of this chapter is to prove the celebrated Verlinde formula giving an explicit expression for the dimension of the space of conformal blocks.

To facilitate this, we develop an algebraic formalism for general fusion rules and the corresponding associated ring. By a *fusion rule* on a finite set A together with an involution $*$, one means a map $F\colon \mathbb{Z}_+[A] \to \mathbb{Z}$ satisfying certain properties (cf. Definition 4.1.1). It is shown that given a nondegenerate fusion rule on A, the free $\mathbb{Z}$-module $\mathbb{Z}[A]$ with basis A acquires a natural associative and commutative ring structure together with a trace form $t\colon \mathbb{Z}[A] \to \mathbb{Z}$. This ring $\mathbb{Z}[A]$ together with the trace form is called the *fusion ring* associated to the fusion rule F on A. Further, the involution $*$ extends to a ring isomorphism of $\mathbb{Z}[A]$ and $\mathbb{Z}[A]$ is a Gorenstein ring (cf. Proposition 4.1.2). The trace form t and the involution $*$ together give rise to a natural positive-definite symmetric bilinear form $\langle\, ,\, \rangle$ on $\mathbb{Z}[A]$ (cf. Definition 4.1.4). We further show (cf. Lemma 4.1.5) that the complexified algebra $\mathbb{C}[A] := \mathbb{C} \otimes_{\mathbb{Z}} \mathbb{Z}[A]$ is a (finite-dimensional) reduced algebra (i.e., it has no nonzero nilpotent elements). Hence, we get the decomposition (as $\mathbb{C}$-algebras):

$$\varphi_A\colon \mathbb{C}[A] \simeq \mathbb{C}^{S_A}, \;\; \varphi_A(x) = (\chi(x))_{\chi \in S_A}, \;\text{for } x \in \mathbb{C}[A],$$

where S_A is the set of all the $\mathbb{C}$-algebra homomorphisms from $\mathbb{C}[A]$ to $\mathbb{C}$ (cf. Lemma 4.1.5).

The fusion rule F should be thought of as the ‘genus 0 fusion rule.’ Motivated from the Factorization Theorem, the fusion rule F is extended to a fusion map $F_g\colon \mathbb{Z}_+[A] \to \mathbb{Z}$ for any genus $g \geq 0$ (cf. Definition 4.1.6). The map F_g is explicitly determined in Proposition 4.1.7 and Corollary 4.1.8 using the trace t, a ‘Casimir element’ $\Omega \in \mathbb{Z}[A]$ defined as

$$\Omega = \sum_{\lambda \in A} \lambda \cdot \lambda^* \in \mathbb{Z}[A],$$

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and the set S_A. The matrix encoding the fusion product is diagonalized in Exercise 4.1.E.3.

Having developed the algebraic machinery of fusion rules and fusion ring in Section 4.1, the aim of Section 4.2 is to state and give a proof of the Verlinde dimension formula. For any simple Lie algebra $\mathfrak{g}$ and central charge $c > 0$, recall the set D_c consisting of dominant integral weights of $\mathfrak{g}$ with level $\leq c$. Then, the finite set D_c has a natural involution $\lambda \mapsto \lambda^* := -w_o\lambda$, where w_o is the longest element of the Weyl group of $\mathfrak{g}$. Define the function $F_c\colon \mathbb{Z}_+[D_c] \to \mathbb{Z}_+$ by $F_c(0) = 1$ and, for any $s \geq 1$ and $\lambda_i \in D_c$,

$$F_c(\lambda_1 + \cdots + \lambda_s) = \dim \mathscr{V}_{\mathbb{P}^1}(\vec{p}, \vec{\lambda}),$$

where $\vec{p} = (p_1, \ldots, p_s)$ are any distinct points in $\mathbb{P}^1$ and $\vec{\lambda} = (\lambda_1, \ldots, \lambda_s)$. (By Proposition 3.5.8 and Exercise 2.3.E.2, $F_c(\lambda_1+\cdots+\lambda_s)$ does not depend upon the choice of the points $\vec{p}$.) These F_c will be our most important examples of fusion rule (cf. Example 4.2.1). The corresponding fusion ring $\mathbb{Z}[D_c]$ is called the *fusion ring of* $\mathfrak{g}$ *at level* c and will be denoted by $R_c(\mathfrak{g})$. We denote the basis of the fusion ring $R_c(\mathfrak{g})$ by $\{[V(\lambda)]\}_{\lambda \in D_c}$ and denote the fusion product (i.e., the product in $\mathbb{Z}[D_c]$) by $\otimes^c$. Proposition 4.2.3 determines the fusion product $[V(\lambda)] \otimes^c [V(\mu)]$ in terms of the action of $sl_2(\theta)$ (sl_2 passing through the highest root space of $\mathfrak{g}$) on the components. This proposition is an easy consequence of Corollary 2.3.5. As an application of this proposition, we show that $[V(\lambda)] \otimes^c [V(\mu)]$ coincides with the usual tensor product $[V(\lambda)] \otimes [V(\mu)]$ in the case $\lambda + \mu \in D_c$. We also determine $[V(\lambda)] \otimes^c [V(\mu)]$ when $(\lambda + \mu)(\theta^\vee) = c + 1$ or $c + 2$ (cf. Corollary 4.2.4 and Exercise 4.2.E.1).

We give another definition of the fusion product $\otimes^c_F$ in terms of the $\mathfrak{g}$-equivariant Euler–Poincaré characteristic of certain vector bundles on the infinite Grassmannian $\bar{X}_G$ (cf. Definition 4.2.11). In Definition 4.2.7, we define a $\mathbb{Z}$-module map

$$\xi_c\colon R(\mathfrak{g}) \to R_c(\mathfrak{g}),$$

where $R(\mathfrak{g})$ is the representation ring of $\mathfrak{g}$. It is shown to be a (surjective) ring homomorphism if we endow $R_c(\mathfrak{g})$ with the fusion product $\otimes^c_F$ (cf. identity (8) of Subsection 4.2.18). Its kernel is determined in Lemma 4.2.8.

Then, using a result of Teleman on the Lie algebra homology of $\hat{\mathfrak{g}}_- := \mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}]$ with coefficient in the tensor product of an integrable highest-weight $\hat{\mathfrak{g}}$-module with a finite-dimensional $\mathfrak{g}$-module (cf. Theorem 4.2.16), we show that the above two fusion products $\otimes^c$ and $\otimes^c_F$ coincide (cf. Corollary 4.2.17). This identification, together with the combinatorics of the affine Weyl group W_c (cf. Definition 4.2.5), gives rise to another (closed) expression for the fusion product (cf. Lemma 4.2.12):

For $\lambda, \mu \in D_c$,

$$[V(\lambda)] \otimes^c [V(\mu)] = \sum_{\nu \in D_c} \sum_{w \in W'_c} (-1)^{\ell(w)} n^{\lambda,\mu}_{w^{-1} * \nu} [V(\nu)],$$

where W'_c is defined in Definition 4.2.7 and $n^{\lambda,\mu}_{w^{-1}*\nu} := \dim(\mathrm{Hom}_{\mathfrak{g}}(V(w^{-1}*\nu), V(\lambda) \otimes V(\mu)))$. Further, by using the equality of the two fusion products $\otimes^c$ and $\otimes^c_F$, we of course get that ξ_c is a ring homomorphism with respect to the usual fusion product $\otimes^c$ (cf. Theorem 4.2.9). The equality of $\otimes^c$ and $\otimes^c_F$ can be more easily deduced for any simple $\mathfrak{g}$ not of type $E_\bullet$ and F_4 (cf. Exercise 4.2.E.4).

The surjective ring homomorphism ξ_c allows us to determine the set of algebra homomorphisms $R_c(\mathfrak{g}) \to \mathbb{C}$ and identify them with T_c^{reg}/W (cf. Corollary 4.2.18), where W is the (finite) Weyl group of $\mathfrak{g}$ and T_c^{reg} is a certain finite subgroup of the maximal torus T of the simply-connected algebraic group G (with Lie algbera $\mathfrak{g}$) (cf. Definition 4.2.5). Now, the stage is set to state and prove the following Verlinde formula giving an explicit expression for the dimension of the space of conformal blocks (cf. Theorem 4.2.19). This is one of the most important results of the book.

Theorem Let $\mathfrak{g}$ be any simple Lie algebra and let $c > 0$ be any central charge. Let $(\Sigma, \vec{p} = (p_1, \ldots, p_s))$ be an irreducible smooth s-pointed curve of any genus $g \geq 0$ (where $s \geq 1$) and let $\vec{\lambda} = (\lambda_1, \ldots, \lambda_s)$ be a collection of weights in D_c. Then, for the space $\mathscr{V}_\Sigma(\vec{p}, \vec{\lambda})$ of covacua,

$$\dim \mathscr{V}_\Sigma(\vec{p}, \vec{\lambda}) = |T_c|^{g-1} \sum_{\mu \in D_c} \left(\left(\Pi_{i=1}^s (\mathrm{ch}_{t_\mu}([V(\lambda_i)]))\right) \cdot \Pi_{\alpha \in \Delta_+} \left(2 \sin \left(\frac{\pi}{c + h^\vee} \langle \mu + \rho, \alpha \rangle \right) \right)^{2-2g} \right),$$

where Δ_+ is the set of positive roots of $\mathfrak{g}$, $\kappa \colon \mathfrak{h}^* \to \mathfrak{h}$ is the isomorphism induced from the normalized invariant form, T_c is defined in Definition 4.2.5, $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$ and $t_\mu := \mathrm{Exp}\left(\frac{2\pi i \kappa(\mu+\rho)}{c+h^\vee}\right) \in T_c$.

In particular, if $g = 1$, then $\dim \mathscr{V}_\Sigma(p, 0) = |D_c|$.

In Exercise 4.2.E.8, $\dim \mathscr{V}_\Sigma(p, 0)$ is determined for any g if $\mathfrak{g}$ is simply-laced and $c = 1$.

4.1 General Fusion Rules and the Associated Ring

Definition 4.1.1 Let A be a finite set with an involution $*$ (i.e., a bijection of order 2). Let $\mathbb{Z}_+[A] := \bigoplus_{a\in A} \mathbb{Z}_+ a$ be the free monoid generated by A, where $\mathbb{Z}_+ := \mathbb{Z}_{\geq 0}$. The involution $*$ of A clearly extends to an involution of $\mathbb{Z}_+[A]$.

By a *fusion rule* on A, we mean a map $F\colon \mathbb{Z}_+[A] \to \mathbb{Z}$ satisfying the following conditions:

(f_1) $F(0) = 1$
(f_2) $F(a) > 0$, for some $a \in A$
(f_3) $F(x) = F(x^*)$, for $x \in \mathbb{Z}_+[A]$
(f_4) $F(x+y) = \sum_{\lambda\in A} F(x+\lambda)F(y+\lambda^*)$, for $x, y \in \mathbb{Z}_+[A]$.
The fusion rule F is said to be *nondegenerate* if it satisfies
(f_5) For any $a \in A$, there exists $\lambda_a \in A$ such that $F(a+\lambda_a) \neq 0$.

Proposition 4.1.2 *Let $F\colon \mathbb{Z}_+[A] \to \mathbb{Z}$ be a nondegenerate fusion rule. Then the abelian group $\mathbb{Z}[A] := \bigoplus_{a\in A} \mathbb{Z}$ acquires a product (defined by* (1) *below), making $\mathbb{Z}[A]$ into a commutative ring with identity.*

Moreover, it admits a unique linear form $t\colon \mathbb{Z}[A] \to \mathbb{Z}$ (called the trace) satisfying the following properties:

(a) $t(a\cdot b^*) = \delta_{a,b}$ *for* $a, b \in A$.
(b) $t\left(\Pi_{a\in A} a^{n_a}\right) = F\left(\sum_{a\in A} n_a a\right)$, *for any* $n_a \in \mathbb{Z}_+$.

Further, $*$ is a ring homomorphism of $\mathbb{Z}[A]$ and $\mathbb{Z}[A]$ is a Gorenstein ring.

The ring $\mathbb{Z}[A]$ together with the trace form t is called the *fusion ring associated to the fusion rule F*. Observe that t is $*$-invariant.

Proof Define the multiplication (called the *fusion product*)

$$a\cdot b = \sum_{\lambda\in A} F(a+b+\lambda^*)\lambda, \quad \text{for } a,b\in A, \tag{1}$$

and extend bilinearly on $\mathbb{Z}[A]$. By definition, the product is commutative. We next show that it is associative.

Take $a, b, c \in A$. Then

$$\begin{aligned}(a\cdot b)\cdot c &= \sum_{\lambda,\mu\in A} F(a+b+\lambda^*)F(\lambda+c+\mu^*)\mu \\ &= \sum_{\mu\in A} F(a+b+c+\mu^*)\mu, \quad \text{by } (f_4) \text{ of Definition 4.1.1} \qquad (2) \\ &= a\cdot(b\cdot c).\end{aligned}$$

This proves the associativity.

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We next show that $*$ is a ring homomorphism; i.e.,

$$*(a \cdot b) = (*a) \cdot (*b), \quad \text{for} \quad a, b \in A.$$

Now

$$\begin{aligned} *(a \cdot b) &= \sum_{\lambda \in A} F(a + b + \lambda^*)\lambda^* \\ &= \sum_{\lambda \in A} F(a^* + b^* + \lambda)\lambda^*, \quad \text{by } (f_3) \text{ of Definition 4.1.1} \\ &= (*a) \cdot (*b). \end{aligned}$$

We next show that there exists a unique $\mathbb{1} \in A$ such that

$$F(\mathbb{1}) = 1, \quad F(a) = 0, \quad \text{for all} \quad a \in A, a \neq \mathbb{1}. \tag{3}$$

Applying (f_1) and (f_4) of Definition 4.1.1 to $x = y = 0$, we get

$$\begin{aligned} 1 &= \sum_{\lambda \in A} F(\lambda)F(\lambda^*) \\ &= \sum_{\lambda \in A} F(\lambda)^2, \quad \text{by} \quad (f_3) \quad \text{of Definition 4.1.1.} \end{aligned}$$

From this, we get (3). (Observe that we have used (f_2) of Definition 4.1.1 here.) In particular, by (f_3) of Definition 4.1.1,

$$\mathbb{1}^* = \mathbb{1}. \tag{4}$$

Applying (f_4) of Definition 4.1.1 to $x = a \in A$, $y = a^*$, we get

$$\begin{aligned} F(a + a^*) &= \sum_{\lambda \in A} F(a + \lambda)F(a^* + \lambda^*) \\ &= \sum_{\lambda \in A} F(a + \lambda)^2, \quad \text{by } (f_3) \text{ of Definition 4.1.1.} \end{aligned} \tag{5}$$

In particular, for $a \in A$,

$$F(a + a^*) \geq F(a + a^*)^2.$$

This forces

$$F(a + a^*) = 0 \quad \text{or} \quad 1. \tag{6}$$

Further, by (5),

$$F(a + \lambda) = 0, \quad \text{for all} \quad \lambda \neq a^*, \lambda \in A. \tag{7}$$

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Since F is nondegenerate, (6) and (7) force

$$F(a + a^*) = 1. \tag{8}$$

Combining (7) and (8), we get

$$F(a + b^*) = \delta_{a,b}\,, \quad \text{for} \quad a, b \in A. \tag{9}$$

Now, define the linear form $t : \mathbb{Z}[A] \to \mathbb{Z}$ by

$$t(a) = F(a), \quad \text{for } a \in A. \tag{10}$$

Of course, this definition is forced by property (b). Now, by the definition of the product, for $a, b \in A$,

$$\begin{aligned} t(a \cdot b^*) &= \sum_{\lambda \in A} F(a + b^* + \lambda^*) F(\lambda) \\ &= F(a + b^*), \quad \text{by } (f_4) \text{ of Definition 4.1.1} \\ &= \delta_{a,b}\,, \quad \text{by (9)}. \end{aligned}$$

This proves property (a).

We next show that $\mathbb{1}$ is the multiplicative identify of $\mathbb{Z}[A]$. For any $x \in \mathbb{Z}_+[A]$,

$$F(x) = F(x + 0) = \sum_{\lambda \in A} F(x + \lambda) F(\lambda^*) = F(x + \mathbb{1}), \quad \text{by (3) and (4).} \tag{11}$$

Now, for any $a \in A$,

$$\begin{aligned} \mathbb{1} \cdot a &= \sum_{\lambda \in A} F(\mathbb{1} + a + \lambda^*) \lambda \\ &= \sum_{\lambda \in A} F(a + \lambda^*) \lambda, \quad \text{by (11)} \\ &= a, \quad \text{by (8) and (9)}. \end{aligned}$$

This proves that $\mathbb{1}$ is the multiplicative identity of $\mathbb{Z}[A]$.

Similar to the derivation of (2), by induction on n, it is easy to see that for any $a_1, \ldots, a_n \in A$,

$$a_1 \cdot a_2 \cdots a_n = \sum_{\mu \in A} F(a_1 + \cdots + a_n + \mu) \mu^*.$$

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Thus,

$$\begin{aligned} t(a_1 \dots a_n) &= \sum_{\mu \in A} F(a_1 + \cdots + a_n + \mu) t(\mu^*) \\ &= \sum_{\mu \in A} F(a_1 + \cdots + a_n + \mu) F(\mu^*), \quad \text{by (10)} \\ &= F(a_1 + \cdots + a_n), \quad \text{by } (f_4) \text{ of Definition 4.1.1.} \end{aligned}$$

This proves property (b).

Finally, we show that $\mathbb{Z}[A]$ is a Gorenstein ring over $\mathbb{Z}$.

Consider the $\mathbb{Z}$-linear map

$$\varphi \colon \mathbb{Z}[A] \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[A], \mathbb{Z}),$$

defined by

$$\varphi(x)(y) = t(x \cdot y), \quad \text{for} \quad x, y \in \mathbb{Z}[A],$$

where $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[A], \mathbb{Z})$ denotes the $\mathbb{Z}$-module of $\mathbb{Z}$-linear maps from $\mathbb{Z}[A]$ to $\mathbb{Z}$. By the property (a) of t, φ is a $\mathbb{Z}$-linear isomorphism. Put a $\mathbb{Z}[A]$-module structure on $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[A], \mathbb{Z})$ by

$$(x \cdot \alpha)(y) = \alpha(xy), \ \text{ for } x, y \in \mathbb{Z}[A] \text{ and } \alpha \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[A], \mathbb{Z}).$$

Then φ is a $\mathbb{Z}[A]$-module isomorphism (under the multiplication action of $\mathbb{Z}[A]$ on itself). The exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$$

gives rise to the exact sequence of $\mathbb{Z}[A]$-modules ($\mathbb{Z}[A]$ being a free $\mathbb{Z}$-module):

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[A], \mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[A], \mathbb{Q}) \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[A], \mathbb{Q}/\mathbb{Z}) \to 0.$$

Now, $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[A], \mathbb{Q})$ and $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[A], \mathbb{Q}/\mathbb{Z})$ are injective $\mathbb{Z}[A]$-modules (cf. (Lang, 1965, Chap. III, Exercise 9(d))). Thus, $\mathbb{Z}[A] \simeq \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[A], \mathbb{Z})$ has finite injective dimension. Hence, $\mathbb{Z}[A]$ is a Gorenstein ring (by one of the equivalent definitions of Gorenstein rings). This completes the proof of the proposition. □

The converse of Proposition 4.1.2 is also true. Specifically, we have the following.

Lemma 4.1.3 *Let R be a commutative ring with identity $\mathbb{1}$ endowed with a ring involution $*$ and a $*$-invariant linear form $t \colon R \to \mathbb{Z}$. Assume that R admits a finite orthonormal $\mathbb{Z}$-basis A with respect to the (symmetric) bilinear* *form* $\langle , \rangle : R \times R \to \mathbb{Z}$, $\langle x, y \rangle := t(x \cdot y^*)$. *Assume further that* $\mathbb{1} \in A$ *and* A *is stable under* $*$. *Then the map*

$$F : \mathbb{Z}_+[A] \to \mathbb{Z}, \;\; F\left(\sum_{a \in A} n_a a\right) = t\left(\Pi_{a \in A} a^{n_a}\right), \quad \textit{for} \quad n_a \in \mathbb{Z}_+, \qquad (1)$$

is a nondegenerate fusion rule such that the associated fusion ring with trace coincides with (R, t).

Proof Since $\mathbb{1} \in A$, $t(\mathbb{1}) = 1$. Thus, $F(0) = t(\mathbb{1}) = F(\mathbb{1}) = 1$. Since t is invariant under $*$, $F(x) = F(x^*)$, for any $x \in \mathbb{Z}_+[A]$. For any $a \in A$,

$$F(a + a^*) = t(a \cdot a^*) = \langle a, a \rangle = 1.$$

Finally, take $x = \sum_{a \in A} n_a a$, $y = \sum_{a \in A} m_a a \in \mathbb{Z}_+[A]$. Then

$$\begin{aligned} F(x+y) &= t\left(\left(\Pi_{a \in A} a^{n_a}\right) \cdot \left(\Pi_{a \in A} a^{m_a}\right)\right) \\ &= \left\langle \Pi_{a \in A} a^{n_a}, \; \Pi_{a \in A} (a^*)^{m_a} \right\rangle \\ &= \sum_{b \in A} \left\langle \Pi_{a \in A} a^{n_a}, b \right\rangle \left\langle b, \Pi_{a \in A} (a^*)^{m_a} \right\rangle, \\ &\quad \text{since } A \text{ is an orthonormal basis of } R \\ &= \sum_{b \in A} \left\langle \Pi_{a \in A} a^{n_a}, b \right\rangle \left\langle b^*, \Pi_{a \in A} a^{m_a} \right\rangle, \quad \text{since } t \text{ is } *\text{-invariant} \\ &= \sum_{b \in A} F(x + b^*) F(y + b). \end{aligned}$$

Thus, F satisfies all the defining properties of a nondegenerate fusion rule.

From the definition of the induced product in $\mathbb{Z}[A]$ as in identity (1) of the proof of Proposition 4.1.2 (denoted by $\cdot$), we get that, for $a, b \in A$,

$$\begin{aligned} a \cdot b &= \sum_{\lambda \in A} F(a + b + \lambda^*) \lambda \\ &= \sum_{\lambda \in A} t(ab\lambda^*) \lambda \\ &= \sum_{\lambda \in A} \langle ab, \lambda \rangle \lambda \\ &= ab, \;\; \text{since } A \text{ is an orthonormal basis.} \end{aligned}$$

This shows that the fusion product in $\mathbb{Z}[A] = R$ induced from F coincides with the original product in R.

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The induced trace form clearly coincides with t by identity (10) of the proof of Proposition 4.1.2 and (1). This proves the lemma. □

Definition 4.1.4 Let F be a nondegenerate fusion rule on a finite set A (cf. Definition 4.1.1) and let $\mathbb{Z}[A]$ be the corresponding fusion ring with trace $t\colon \mathbb{Z}[A] \to \mathbb{Z}$ (cf. Proposition 4.1.2). Define the positive-definite symmetric bilinear form $\langle , \rangle$ on $\mathbb{Z}[A]$ by

$$\langle x, y\rangle = t(xy^*). \tag{1}$$

(It is positive definite by the defining property (a) of Proposition 4.1.2.)

Let $\mathbb{R}[A]$ be the $\mathbb{R}$-algebra $\mathbb{Z}[A] \otimes_{\mathbb{Z}} \mathbb{R}$ obtained by extending the scalars. Clearly $\langle , \rangle$ extends to a positive-definite symmetric $\mathbb{R}$-bilinear form on $\mathbb{R}[A]$. Also, the involution $*$ on $\mathbb{Z}[A]$ extends to an $\mathbb{R}$-algebra involution on $\mathbb{R}[A]$.

Let $S_A := \operatorname{Spec}(\mathbb{R}[A])$ be the (finite) set of $\mathbb{R}$-algebra homomorphisms $f\colon \mathbb{R}[A] \to \mathbb{C}$, which is the same as the set of $\mathbb{C}$-algebra homomorphisms $\mathbb{C}[A] \to \mathbb{C}$, where $\mathbb{C}[A] := \mathbb{R}[A] \otimes_{\mathbb{R}} \mathbb{C}$.

Lemma 4.1.5 *With the assumptions and notation as in the above Definition 4.1.4, the $\mathbb{R}$-algebra $\mathbb{R}[A]$ is reduced. Further, the $\mathbb{C}$-algebra homomorphism $\varphi_A\colon \mathbb{C}[A] \to \mathbb{C}^{S_A}$ into the product algebra given by*

$$\varphi_A(x) = (\chi(x))_{\chi\in S_A}, \quad \textit{for} \quad x \in \mathbb{C}[A], \tag{1}$$

is an isomorphism.

Also, for any $x \in \mathbb{R}[A]$,

$$\varphi_A(x^*) = \overline{\varphi_A(x)}. \tag{2}$$

Proof We first show that $\mathbb{R}[A]$ is reduced, i.e., it has no nonzero nilpotent elements. It clearly suffices to show that for $x \in \mathbb{R}[A]$ such that $x^2 = 0$, we have $x = 0$. Now,

$$\langle xx^*, xx^*\rangle = t(x^2 {x^*}^2) = 0.$$

Thus, $xx^* = 0$, which gives $\langle x, x\rangle = 0$ and hence $x = 0$. Thus, $\mathbb{C}[A]$ is a reduced algebra as well, which implies that φ_A is an isomorphism.

We now prove (2).

Since $\mathbb{R}[A]$ is reduced, we have a canonical decomposition (as $\mathbb{R}$-algebras):

$$\mathbb{R}[A] \simeq \mathbb{R}_1 \times \cdots \times \mathbb{R}_m \times \mathbb{C}_1 \times \cdots \times \mathbb{C}_n$$

obtained from the indecomposable idempotents of $\mathbb{R}[A]$, where each $\mathbb{R}_i$ is the $\mathbb{R}$-algebra $\mathbb{R}$ and each $\mathbb{C}_j$ is the $\mathbb{R}$-algebra $\mathbb{C}$. The decomposition being canonical, for any $1 \le i \le m$ and $1 \le j \le n$,

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$$* \mathbb{R}_i = \mathbb{R}_{i'} \quad \text{and} \quad * \mathbb{C}_j = \mathbb{C}_{j'},$$

for some $1 \leq i' \leq m$ and $1 \leq j' \leq n$. We next claim that $i' = i$ and $j' = j$. For, otherwise, if $i' \neq i$ for some i, then, for any $x \in \mathbb{R}_i$, $x \cdot x^* = 0$, which gives $\langle x, x \rangle = 0$. This is a contradiction since $\langle , \rangle$ on $\mathbb{R}[A]$ is positive definite.

By the same argument, we see that $j' = j$, for all j, i.e., $*$ keeps each of the factors $\mathbb{R}_i$ and $\mathbb{C}_j$ stable. Of course, $*$ being an $\mathbb{R}$-algebra homomorphism on each factor, $*$ is the identify map on each $\mathbb{R}_i$ factor.

We next claim that $*$ is the complex conjugation on each $\mathbb{C}_j$ factor. For, if not, $*$ would be the identity map on some $\mathbb{C}_j$. This would give

$$t(x^2) = t(xx^*) = \langle x, x \rangle \geq 0, \quad \text{for all} \quad x \in \mathbb{C}_j.$$

This is a contradiction since t is a $\mathbb{R}$-linear map and $\{x^2 : x \in \mathbb{C}_j\} = \mathbb{C}_j$.

Finally, the canonical $\mathbb{C}$-algebra isomorphism $\mathbb{C}_j \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathbb{C} \times \mathbb{C}$ is given by $z \otimes w \mapsto (wz, w\bar{z})$. From this (2) follows easily. □

Definition 4.1.6 Let F be a nondegenerate fusion rule on a finite set A. Then, for any ‘genus’ $g \geq 0$, define the map (by induction on g)

$$F_g : \mathbb{Z}_+[A] \to \mathbb{Z}$$

by

$$F_0 = F, \text{ and, for any } g \geq 1, \; F_g(x) = \sum_{\lambda \in A} F_{g-1}(x + \lambda + \lambda^*), \; x \in \mathbb{Z}_+[A].$$

Also, define the ‘Casimir’ element

$$\Omega = \sum_{\lambda \in A} \lambda \cdot \lambda^* \in \mathbb{Z}[A]. \tag{1}$$

For any $x \in \mathbb{Z}[A]$, let $\mu_x : \mathbb{Z}[A] \to \mathbb{Z}[A]$ be the multiplication by $x : y \mapsto xy$. Let $\mathrm{Tr}(x)$ denote the trace of μ_x. Then, by Proposition 4.1.2(a) and Definition 4.1.4,

$$\mathrm{Tr}(x) = \sum_{\lambda \in A} \langle x \cdot \lambda, \lambda \rangle = \sum_{\lambda \in A} t(x \cdot \lambda \cdot \lambda^*) = t(x \cdot \Omega). \tag{2}$$

Proposition 4.1.7 *For $g \geq 1$ and any nondegenerate fusion rule on a finite set A and any $a_1, \ldots, a_n \in A$,*

$$F_g(a_1 + \cdots + a_n) = t(a_1 \ldots a_n \cdot \Omega^g) = Tr(a_1 \ldots a_n \cdot \Omega^{g-1}), \tag{1}$$

where t is the trace as in Proposition 4.1.2.

In fact, (1) *remains true for $g = 0$ as well by Lemma 4.1.9.*

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Proof By the definition of F_g,

$$F_g(a_1 + \cdots + a_n) = \sum_{\lambda_1,\ldots,\lambda_g \in A} F_0(a_1 + \cdots + a_n + \lambda_1 + \lambda_1^* + \cdots + \lambda_g + \lambda_g^*)$$

$$= \sum_{\lambda_1,\ldots,\lambda_g \in A} t(a_1 \ldots a_n \cdot \lambda_1 \cdot \lambda_1^* \ldots \lambda_g \cdot \lambda_g^*),$$

by the defining property 4.1.2(b)

$$= t(a_1 \ldots a_n \cdot \Omega^g).$$

This proves the first equality in (1). The second equality in (1) of course follows from the identity (2) of Definition 4.1.6. □

As a consequence of Lemma 4.1.5 and Proposition 4.1.7, we have the following.

Corollary 4.1.8 *With the notation and assumption as in Proposition 4.1.7, for any $a_1, \ldots, a_n \in A$ and $g \geq 0$,*

$$F_g(a_1 + \cdots + a_n) = \sum_{\chi \in S_A} \chi(a_1) \ldots \chi(a_n)\chi(\Omega)^{g-1}. \tag{1}$$

Moreover, for any $\chi \in S_A$,

$$\chi(\Omega) = \sum_{\lambda \in A} |\chi(\lambda)|^2. \tag{2}$$

Proof For any $x \in \mathbb{C}[A]$, the multiplication map $\mu_x \colon \mathbb{C}[A] \to \mathbb{C}[A]$, under the identification φ_A of Lemma 4.1.5, is given by the diagonal matrix $(\chi(x))_{\chi \in S_A}$ (in the standard coordinate basis of $\mathbb{C}^{S_A}$). Thus,

$$\mathrm{Tr}(x) = \sum_{\chi \in S_A} \chi(x).$$

Combining identity (1) of Proposition 4.1.7 with the above, we get

$$F_g(a_1 + \cdots + a_n) = \mathrm{Tr}(a_1 \ldots a_n \cdot \Omega^{g-1})$$

$$= \sum_{\chi \in S_A} \chi(a_1) \ldots \chi(a_n)\chi(\Omega)^{g-1}.$$

This proves (1). To prove (2), use identity (2) of Lemma 4.1.5. □

The following lemma is due to Jiuzu Hong.

Lemma 4.1.9 *The element $\Omega \in \mathbb{Z}[A]$ (cf. identity* (1) *of Definition 4.1.6) is an invertible element in $\mathbb{C}[A]$.*

In particular, identity (1) *of Proposition 4.1.7 remains true for* $g = 0$ *as well.*

Proof By Lemma 4.1.5 it suffices to prove that, for any $\chi \in S_A$, $\chi(\Omega)$ is nonzero. Now,

$$\chi(\Omega) = \sum_{\lambda \in A} |\chi(\lambda)|^2, \text{ by identity (2) of Corollary 4.1.8}$$
$$> 0.$$

This proves the lemma. □

4.1.E Exercises

In the following Exercises 1–3, F is a nondegenerate fusion rule on a finite set A.

(1) Let $\mathbb{Q}[A] := \mathbb{Z}[A] \otimes_{\mathbb{Z}} \mathbb{Q}$ be the associated fusion algebra over $\mathbb{Q}$. Show that there is a canonical decomposition (as a $\mathbb{Q}$-algebra)

$$\mathbb{Q}[A] \simeq \Pi_{i=1}^m E_i \times \Pi_{j=1}^n F_j,$$

where each E_i is a (finite) totally real extension of $\mathbb{Q}$ and each F_j is an imaginary quadratic extension of a (finite) totally real extension F'_j of $\mathbb{Q}$.

Moreover, each E_i and F_j are stable under $*$ with $*$ acting on each E_i and F'_j via the identity map and $*$ on each F_j acts via the nontrivial automorphism of F_j over F'_j.

Hint: Follow the proof of Lemma 4.1.5.

(2) Show that, for any $g, h \in \mathbb{Z}_+$ and $x, y \in \mathbb{Z}_+[A]$,

$$F_{g+h}(x + y) = \sum_{\lambda \in A} F_g(x + \lambda) F_h(y + \lambda^*).$$

Observe that this is a higher-genus analogue of the condition (f_4) of Definition 4.1.1.

(3) By definition, A is a basis of $\mathbb{C}[A]$. For $a \in A$, let $F^a = (F^a_{b,c})_{b,c \in A}$ be the matrix of the multiplication μ_a in the A-basis. Then, by the definition of multiplication in $\mathbb{C}[A]$ (cf. identity (1) of the proof of Proposition 4.1.2),

$$F^a_{b,c} = F(a + c + b^*).$$

Also, under the identification of $\mathbb{C}[A]$ with $\mathbb{C}^{S_A}$ (as in Lemma 4.1.5), the standard coordinate basis of $\mathbb{C}^{S_A}$ gives rise to a basis of $\mathbb{C}[A]$

parameterized by S_A. For $a \in A$, let D^a be the matrix of μ_a in the S_A-basis. Then, clearly $D^a = (D^a_\chi)_{\chi \in S_A}$ is a $S_A \times S_A$ diagonal matrix with

$$D^a_\chi = \chi(a).$$

Show that there exists a unitary matrix $\Sigma = (\Sigma_{\chi,a})_{\chi \in S_A, a \in A}$ such that

$$F^a = \Sigma^{-1} \cdot D^a \cdot \Sigma, \quad \text{for all } a \in A.$$

In fact, Σ can be taken so that its entries

$$\Sigma_{\chi,a} = \frac{\chi(a)}{\left(\sum_{\lambda \in A} |\chi(\lambda)|^2\right)^{\frac{1}{2}}}.$$

Prove further that for any $a_1, \dots, a_n \in A$ and $g \geq 0$,

$$F_g(a_1 + \cdots + a_n) = \sum_{\chi \in S_A} \frac{\Sigma_{\chi,a_1} \cdots \Sigma_{\chi,a_n}}{(\Sigma_{\chi,\mathbb{1}})^{n+2g-2}},$$

where $\mathbb{1}$ is the multiplicative identity of $\mathbb{Z}[A]$ (cf. the proof of Proposition 4.1.2).

4.2 Fusion Ring of a Simple Lie Algebra and an Explicit Verlinde Dimension Formula

In this section $\mathfrak{g}$ is a simple Lie algebra over $\mathbb{C}$. We fix a level $c > 0$. Associated to the pair $(\mathfrak{g}, c)$, there is a fusion rule F_c giving rise to the fusion ring $R_c(\mathfrak{g})$. We will draw upon the general results proved in Section 4.1 to study $R_c(\mathfrak{g})$ in this section. This is our most important example of fusion rings, which leads to an explicit Verlinde dimension formula. We continue to follow the notation from Section 1.2, often without explanation.

Example 4.2.1 Let D_c be as defined in (1) of Definition 2.1.1. Define the function $F_c\colon \mathbb{Z}_+[D_c] \to \mathbb{Z}_+$ by $F_c(0) = 1$ (for the zero element of $\mathbb{Z}_+[D_c]$) and, for any $s \geq 1$ and $\lambda_i \in D_c$,

$$F_c(\lambda_1 + \cdots + \lambda_s) = \dim \mathscr{V}_{\mathbb{P}^1}(\vec{p}, \vec{\lambda}),$$

where $\vec{p} = (p_1, \dots, p_s)$ are any distinct points in $\mathbb{P}^1$, $\vec{\lambda} = (\lambda_1, \dots, \lambda_s)$ and $\mathscr{V}_{\mathbb{P}^1}(\vec{p}, \vec{\lambda})$ denotes the space of covacua on $\mathbb{P}^1$ with respect to the weights $\vec{\lambda}$ attached to the points $\vec{p}$. By Proposition 3.5.8 and Exercise 2.3.E.2, $F_c(\lambda_1 + \cdots + \lambda_s)$ does not depend upon the choice of the points $\vec{p}$.

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Define the involution $*: D_c \to D_c$ by $\lambda^* := -w_o\lambda$, where w_o is the longest element of the Weyl group of $\mathfrak{g}$. Thus, λ^* is the highest weight of the dual module $V(\lambda)^*$.

By Exercise 2.3.E.2, F_c satisfies the properties (f_2) and (f_5) of Definition 4.1.1 for $A = D_c$. The property (f_4) follows from Corollary 3.5.10(b) for $g = 0$. (If $x = 0$ as an element of $\mathbb{Z}_+[D_c]$, it follows from Exercise 2.3.E.2.) The property (f_3) follows from the following lemma. Thus, F_c is indeed a nondegenerate fusion rule.

The corresponding fusion ring $\mathbb{Z}[D_c]$ (as given by Proposition 4.1.2) is called the *fusion ring of* $\mathfrak{g}$ *at level* c. Henceforth, it will be denoted by $R_c(\mathfrak{g})$. By definition, as a $\mathbb{Z}$-module, it is freely generated by the isomorphism classes $\{[V(\lambda)]\}_{\lambda \in D_c}$ and the fusion product $\otimes^c$ (at level c) is given by (cf. identity (1) of the proof of Proposition 4.1.2):

$$[V(\lambda)] \otimes^c [V(\mu)] := \sum_{\nu \in D_c} \dim \mathscr{V}_{\mathbb{P}^1}((\lambda, \mu, \nu^*))[V(\nu)], \tag{1}$$

where $\mathscr{V}_{\mathbb{P}^1}((\lambda, \mu, \nu^*))$ denotes the space of covacua on $\mathbb{P}^1$ with respect to the weights (λ, μ, ν^*) attached to any three distinct points in $\mathbb{P}^1$.

Lemma 4.2.2 *Let $s \geq 1$. For any s-pointed curve $(\Sigma, \vec{p})$ and any $\vec{\lambda} = (\lambda_1, \dots, \lambda_s)$ with each $\lambda_i \in D_c$, there is an isomorphism*

$$\mathscr{V}_\Sigma(\vec{p}, \vec{\lambda}) \simeq \mathscr{V}_\Sigma(\vec{p}, \vec{\lambda}^*),$$

where $\vec{\lambda}^* := (\lambda_1^*, \dots, \lambda_s^*)$.

Proof Recall first that there exists an automorphism $\beta: \mathfrak{g} \to \mathfrak{g}$ such that, for any $\lambda \in D$, the $\mathfrak{g}$-module $V(\lambda)^\beta$ is isomorphic with $V(\lambda^*)$, where $V(\lambda)^\beta$ is the same vector space as $V(\lambda)$ but the $\mathfrak{g}$-module structure on $V(\lambda)^\beta$ is twisted via

$$x \odot v = \beta^{-1}(x) \cdot v, \quad \text{for } x \in \mathfrak{g} \text{ and } v \in V(\lambda) \tag{1}$$

(cf. (Bourbaki, 2005, Chap. VIII, §7, no. 6, Remark 1)).

The automorphism β clearly gives rise to an automorphism $\hat{\beta}$ of the affine Lie algebra $\hat{\mathfrak{g}}$ defined by

$$\hat{\beta}(x[f]) = \beta(x)[f], \quad \text{for} \quad x \in \mathfrak{g},\ f \in K = \mathbb{C}((t)) \text{ and } \hat{\beta}(C) = C.$$

Now, for any $\lambda \in D_c$, the twisted $\hat{\mathfrak{g}}$-module $\mathscr{H}(\lambda)^{\hat{\beta}}$ (with the same space as $\mathscr{H}(\lambda)$ and the $\hat{\mathfrak{g}}$-module structure twisted by the same formula as (1)) is isomorphic with $\mathscr{H}(\lambda^*)$. To see this, let $v_+ \in \mathscr{H}(\lambda)$ be a highest-weight vector. Then, $(\mathfrak{g} \otimes t\mathbb{C}[[t]]) \odot v_+ = 0$ in $\mathscr{H}(\lambda)^{\hat{\beta}}$. Moreover, the $\mathfrak{g}$-submodule of $\mathscr{H}(\lambda)^{\hat{\beta}}$ generated by v_+ is the same as $V(\lambda)^\beta \simeq V(\lambda^*)$. Thus, $\mathscr{H}(\lambda)^{\hat{\beta}}$ is

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an irreducible quotient of the generalized Verma module $\hat{M}(V(\lambda^*), c)$ (cf. (4) of Definition 1.2.4) and hence an irreducible quotient of the Verma module $\hat{M}(\lambda_c^*)$. Now, apply Exercise 1.2.E.6 to get that $\mathscr{H}(\lambda)^{\hat{\beta}} \simeq \mathscr{H}(\lambda^*)$.

Choose a $\hat{\mathfrak{g}}$-module isomorphism $\mathscr{H}(\lambda)^{\hat{\beta}} \simeq \mathscr{H}(\lambda^*)$ (which is unique up to a scalar multiple by Exercise 1.2.E.5) and let

$$\hat{\beta}_\lambda : \mathscr{H}(\lambda) \to \mathscr{H}(\lambda^*)$$

be the set-theoretic identity map under the above identification. Then, clearly

$$\hat{\beta}_\lambda(\hat{x} \cdot v) = \hat{\beta}(\hat{x}) \cdot \hat{\beta}_\lambda(v), \quad \text{for} \quad \hat{x} \in \hat{\mathfrak{g}}, v \in \mathscr{H}(\lambda). \tag{2}$$

Now, we are ready to prove the lemma. Let

$$\hat{\beta}_{\vec{\lambda}} : \mathscr{H}(\vec{\lambda}) := \mathscr{H}(\lambda_1) \otimes \cdots \otimes \mathscr{H}(\lambda_s) \to \mathscr{H}(\vec{\lambda}^*)$$

be the linear isomorphism $\hat{\beta}_{\vec{\lambda}} := \hat{\beta}_{\lambda_1} \otimes \cdots \otimes \hat{\beta}_{\lambda_s}$.

From property (2), it is easy to see $\hat{\beta}_{\vec{\lambda}}$ induces an isomorphism $\mathscr{V}_\Sigma(\vec{p}, \vec{\lambda}) \simeq \mathscr{V}_\Sigma(\vec{p}, \vec{\lambda}^*)$. This proves the lemma. □

As a consequence of Corollary 2.3.5 and the definition of the fusion product $\otimes^c$ (as in (1) of Example 4.2.1), we get the following result.

Proposition 4.2.3 *For any $\lambda, \mu \in D_c$, $[V(\lambda)] \otimes^c [V(\mu)]$ is the isomorphism class of the quotient $Q_{\lambda,\mu}$ of $V(\lambda) \otimes V(\mu)$ by the $\mathfrak{g}$-submodule $K_{\lambda,\mu}$ generated by*

$$\bigoplus_{p+q+r>2c} \left(V(\lambda)_{(p)} \otimes V(\mu)_{(q)}\right)_{(r)},$$

where $\left(V(\lambda)_{(p)} \otimes V(\mu)_{(q)}\right)_{(r)}$ denotes the isotypic component of $V(\lambda)_{(p)} \otimes V(\mu)_{(q)}$ corresponding to the irreducible representation indexed by r of $sl_2(\theta)$ (cf. Definition 2.3.1).

In particular, $Q_{\lambda,\mu}$ has no components $V(\nu)$ with $\nu \notin D_c$.

Proof Observe first that for any $\nu \notin D_c$, $V(\nu)$ does not occur in $Q_{\lambda,\mu}$:

If $V(\nu)$ does not occur in $V(\lambda) \otimes V(\mu)$, there is nothing to prove. So, assume that $\nu(\theta^\vee) \geq c + 1$ and there is a copy $V(\nu) \subset V(\lambda) \otimes V(\mu)$. Consider the $sl_2(\theta)$-submodule V_1 of $V(\nu)$ passing through the highest-weight vector of $V(\nu)$. Then,

$$V_1 \subset \bigoplus_{p,q \geq 0} \left(V(\lambda)_{(p)} \otimes V(\mu)_{(q)}\right)_{(\nu(\theta^\vee))}.$$

But, for $\left(V(\lambda)_{(p)} \otimes V(\mu)_{(q)}\right)_{(\nu(\theta^\vee))}$ to be nonzero, we must have $p + q \geq \nu(\theta^\vee)$. Thus, from the definition of $K_{\lambda,\mu}$, $V_1 \subset K_{\lambda,\mu}$ hence so is $V(\nu) \subset K_{\lambda,\mu}$. This proves that $V(\nu)$ does not occur in $Q_{\lambda,\mu}$.

We next show that, for any $\nu \in D_c$, the multiplicity $m^{\nu}_{\lambda,\mu} := \dim \mathscr{V}_{\mathbb{P}^1}(\lambda, \mu, \nu^*)$ of $[V(\nu)]$ in $[V(\lambda)] \otimes^c [V(\mu)]$ coincides with the multiplicity $n^{\nu}_{\lambda,\mu}$ of $V(\nu)$ in $Q_{\lambda,\mu}$.

By Corollary 2.3.5,

$$\begin{aligned} m^{\nu}_{\lambda,\mu} = \dim \Big\{ &\mathfrak{g}\text{-module maps } f\colon V(\lambda) \otimes V(\mu) \otimes V(\nu^*) \to \mathbb{C} \\ &\text{such that } f \text{ vanishes on } \bigoplus_{p+q+r>2c} V(\lambda)_{(p)} \otimes V(\mu)_{(q)} \otimes V(\nu^*)_{(r)} \Big\} \\ = \dim \{ &\mathfrak{g}\text{-module maps } \bar{f}\colon V(\lambda) \otimes V(\mu) \to V(\nu) \text{ such that } \bar{f} \\ &\text{vanishes on } [V(\lambda)_{(p)} \otimes V(\mu)_{(q)}]_{(r)} \text{ with } p+q+r > 2c \} \\ = n^{\nu}_{\lambda,\mu}. & \end{aligned}$$

This proves the proposition. □

Corollary 4.2.4 *With the notation as in Proposition 4.2.3,*

(a) $[V(\lambda)] \otimes^c [V(\mu)] = [V(\lambda) \otimes V(\mu)]$, *if* $\lambda + \mu \in D_c$.

(b) If $(\lambda+\mu)(\theta^\vee) = c+1$, *then* $[V(\lambda)] \otimes^c [V(\mu)]$ *is obtained from* $V(\lambda) \otimes V(\mu)$ *by removing all the components* $V(\nu)$ *with* $\nu(\theta^\vee) \geq c+1$. *(In fact, in this case* $V(\lambda) \otimes V(\mu)$ *cannot have any component* $V(\nu)$ *with* $\nu(\theta^\vee) > c+1$, *since* $(\lambda+\mu)(\theta^\vee) = c+1$.*)*

Proof (a) In this case, clearly $K_{\lambda,\mu} = 0$, where $K_{\lambda,\mu}$ is as defined in Proposition 4.2.3. This proves (a) by Proposition 4.2.3.

(b) Let $V(\nu) \subset V(\lambda) \otimes V(\mu)$ be a component with $\nu(\theta^\vee) = c+1$. Then

$$V(\nu)_{(\nu(\theta^\vee))} \subset V(\lambda)_{(\lambda(\theta^\vee))} \otimes V(\mu)_{(\mu(\theta^\vee))}.$$

Thus

$$V(\nu)_{(\nu(\theta^\vee))} \subset K_{\lambda,\mu} \quad \text{and hence} \quad V(\nu) \subset K_{\lambda,\mu}.$$

Conversely, take p, q, r such that $p+q+r \geq 2c+1$ and $(V(\lambda)_{(p)} \otimes V(\mu)_{(q)})_{(r)} \neq 0$. Then $p = \lambda(\theta^\vee)$, $q = \mu(\theta^\vee)$ and $r = (\lambda+\mu)(\theta^\vee)$. But any $\mathfrak{g}$-component $V(\nu)$ of the $\mathfrak{g}$-submodule of $V(\lambda) \otimes V(\mu)$ generated by $\big(V(\lambda)_{(\lambda(\theta^\vee))} \otimes V(\mu)_{(\mu(\theta^\vee))}\big)_{((\lambda+\mu)(\theta^\vee))}$ clearly satisfies $\nu(\theta^\vee) \geq (\lambda+\mu)(\theta^\vee) = c+1$. Thus, $K_{\lambda,\mu}$ does not contain any component $V(\nu)$ with $\nu(\theta^\vee) \leq c$. This proves (b) and hence the corollary is proved. □

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Definition 4.2.5 Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ and let W be its Weyl group. Then W acts naturally on the weight lattice $P \subset \mathfrak{h}^*$, where

$$P := \{\lambda \in \mathfrak{h}^* : \lambda(\alpha_i^\vee) \in \mathbb{Z} \text{ for all the simple coroots } \alpha_i^\vee\},$$

and hence also on $P_{\mathbb{R}} := P \otimes_{\mathbb{Z}} \mathbb{R}$.

Let $Q \subset P$ be the root lattice and let $Q_{lg} \subset Q$ be the sublattice generated by the long roots (if all the root lengths are equal, we call them long roots). Let $h^\vee := 1 + \rho(\theta^\vee)$ be the *dual Coxeter number*, where ρ is the half sum of positive roots of $\mathfrak{g}$ and θ is the highest root.

Let G be the connected simply-connected complex algebraic group with Lie algebra $\mathfrak{g}$ and let $T \subset G$ be the maximal torus with Lie algebra $\mathfrak{h}$. (Here we have deviated from our usual convention to denote the Lie algebra of a group by the corresponding Gothic character.)

Let W_c be the *affine Weyl group* of $\mathfrak{g}$ at level c, which is, by definition, the group of affine transformations of $P_{\mathbb{R}}$ generated by W and the translation $\lambda \mapsto \lambda + (c + h^\vee)\theta$. Since each long root is W-conjugate to θ, W_c is the semi-direct product of W by the lattice $(c + h^\vee)Q_{lg}$. For any $\alpha \in \Delta$, where Δ is the set of all the roots of $\mathfrak{g}$, and $n \in \mathbb{Z}$, define the *affine wall*

$$H_{\alpha,n} = \{\lambda \in P_{\mathbb{R}} : \langle \lambda, \alpha \rangle = n(c + h^\vee)\}.$$

Let

$$H := \bigcup_{\alpha \in \Delta,\, n \in \mathbb{Z}} H_{\alpha,n}.$$

The connected components of $P_{\mathbb{R}} \backslash H$ are called *alcoves*. Then, the closure of any alcove is a fundamental domain for the action of W_c on $P_{\mathbb{R}}$ and W_c acts simply transitively on the set of alcoves (cf. (Bourbaki, 2002, Chap. VI, §2, no. 1)). In particular, W_c acts freely on $P_{\mathbb{R}} \backslash H$. Moreover, W_c is a Coxeter group (cf. (Bourbaki, 2002, Chap. V, §3, no. 2)). The *fundamental alcove* is defined by

$$A^\circ := \{\lambda \in P_{\mathbb{R}} : \lambda(\alpha_i^\vee) > 0 \text{ for all the simple coroots } \alpha_i^\vee \text{ and } \lambda(\theta^\vee) < c + h^\vee\}.$$

Its closure in $P_{\mathbb{R}}$ is clearly given by

$$A = \{\lambda \in P_{\mathbb{R}} : \lambda(\alpha_i^\vee) \geq 0 \text{ for all the simple coroots } \alpha_i^\vee \text{ and } \lambda(\theta^\vee) \leq c + h^\vee\}.$$

Then, it is easy to see that

$$A^\circ = A \backslash H. \tag{1}$$

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Define the shifted action of W_c on $P_{\mathbb{R}}$ by

$$w * \lambda = w(\lambda + \rho) - \rho, \quad \text{for } w \in W_c \text{ and } \lambda \in P_{\mathbb{R}}. \tag{2}$$

It is easy to see that the map

$$D_c \to A^\circ \cap P, \quad \mu \mapsto \mu + \rho, \tag{3}$$

is a bijection. Since W_c keeps P stable and, moreover, it acts simply transitively on the set of alcoves, for any $\lambda \in P\backslash H$, there exists a unique $w \in W_c$ and $\mu \in D_c$ such that

$$\lambda - \rho = w * \mu. \tag{4}$$

Conversely, for any $w \in W_c$ and $\mu \in D_c$,

$$(w * \mu) + \rho \in P\backslash H. \tag{5}$$

Further, for any $\lambda \in H_{\alpha,n}$ with $\alpha \in \Delta$ and $n \in \mathbb{Z}$,

$$\tau_{\frac{2n(c+h^\vee)\alpha}{\langle\alpha,\alpha\rangle}} \cdot s_\alpha(\lambda) = s_\alpha \cdot \tau_{\frac{-2n(c+h^\vee)\alpha}{\langle\alpha,\alpha\rangle}}(\lambda) = \lambda, \tag{6}$$

where τ_β denotes the translation by β and $s_\alpha(\lambda) := \lambda - \lambda(\alpha^\vee)\alpha \in W$ is the reflection corresponding to the root α. (Observe that for any $\alpha \in \Delta$,

$$\frac{2\alpha}{\langle\alpha,\alpha\rangle} \in Q_{lg}, \tag{7}$$

which can easily be seen from the tables in (Bourbaki, 2002, Plates II–IX).)

Denote by $T_c \subset T$ the subgroup

$$T_c = \{t \in T : e^\beta(t) = 1, \text{ for all } \beta \in (c + h^\vee)Q_{lg}\},$$

and let T_c^{reg} be the subset of T_c consisting of regular elements, i.e.,

$$\begin{aligned} T_c^{\text{reg}} &:= \{t \in T_c : w \cdot t \neq t \text{ for any } w \neq 1 \in W\} \\ &= \{t \in T_c : e^\alpha(t) \neq 1 \text{ for any root } \alpha\}. \end{aligned}$$

Let $\kappa: \mathfrak{h}^* \to \mathfrak{h}$ be the isomorphism induced from the invariant form normalized by $\langle\theta,\theta\rangle = 2$.

Lemma 4.2.6 *(a) The map* $\varphi: \lambda \mapsto \operatorname{Exp}\left(\frac{2\pi i\kappa(\lambda)}{c+h^\vee}\right)$ *induces an isomorphism of groups:*

$$\bar{\varphi}: P/(c+h^\vee)Q_{lg} \xrightarrow{\sim} T_c.$$

(b) The map $\lambda \mapsto \operatorname{Exp}\left(\frac{2\pi i\kappa(\lambda+\rho)}{c+h^\vee}\right)$ *induces a bijection:*

$$D_c \xrightarrow{\sim} T_c^{reg}/W.$$

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Proof (a) For any long root α and $\lambda \in P$,

$$e^{(c+h^\vee)\alpha}\left(\operatorname{Exp}\left(\frac{2\pi i\kappa(\lambda)}{c+h^\vee}\right)\right) = e^{2\pi i\langle\lambda,\alpha\rangle}$$
$$= e^{2\pi i\lambda(\alpha^\vee)}, \quad \text{since } \alpha \text{ is a long root}$$
$$= 1.$$

Thus, $\operatorname{Im}\varphi \subset T_c$.

Let $\lambda = (c+h^\vee)\alpha$, for a long root α. Then

$$\operatorname{Exp}\left(\frac{2\pi i\kappa(\lambda)}{c+h^\vee}\right) = \operatorname{Exp}(2\pi i\kappa(\alpha)) = 1,$$

since

$$\kappa(\Delta_{lg}) \subset Q^\vee \tag{1}$$

as the following calculation shows, where Δ_{lg} denotes the set of long roots. For $\alpha \in \Delta_{lg}$,

$$\omega_i(\kappa(\alpha)) = \langle\omega_i,\alpha\rangle = \langle\omega_i,\alpha^\vee\rangle \in \mathbb{Z},$$

where $\{\omega_1,\ldots,\omega_\ell\} \subset \mathfrak{h}^*$ are the fundamental weights. Thus, φ factors through $(c+h^\vee)Q_{lg}$. We next show that $\overline{\varphi}$ is injective.

Take $\lambda \in P$ such that $\operatorname{Exp}\left(\frac{2\pi i\kappa(\lambda)}{c+h^\vee}\right) = 1$. Then, $\frac{\kappa(\lambda)}{c+h^\vee} \in Q^\vee$, which gives

$$\lambda \in (c+h^\vee)\kappa^{-1}(Q^\vee). \tag{2}$$

For any simple root α_i,

$$\kappa^{-1}(\alpha_i^\vee) = \frac{2\alpha_i}{\langle\alpha_i,\alpha_i\rangle}.$$

Taking a simple root α_k in the W-orbit of non-long α_i such that $\langle\alpha_k,\alpha_j\rangle \neq 0$ for a long root α_j and considering $s_k(\alpha_j)$, we see that

$$\kappa^{-1}(Q^\vee) \subset Q_{lg}.$$

Combining this with (1), we get

$$\kappa^{-1}(Q^\vee) = Q_{lg}. \tag{3}$$

In particular, W_c is canonically isomorphic with the affine Weyl group defined in Definition 1.2.12. Combining (2) and (3), we get that $\overline{\varphi}$ is injective.

Take $t \in T_c$ and choose $\lambda \in \mathfrak{h}^*$ such that $\operatorname{Exp}\left(\frac{2\pi i\kappa(\lambda)}{c+h^\vee}\right) = t$. Then

$$e^\beta\left(\operatorname{Exp}\left(\frac{2\pi i\kappa(\lambda)}{c+h^\vee}\right)\right) = 1 \quad \text{for all} \quad \beta \in (c+h^\vee)Q_{lg}.$$

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This gives

$$\lambda(\kappa(\alpha)) \in \mathbb{Z}, \quad \text{for all} \quad \alpha \in Q_{lg}.$$

Hence, by (3), we get

$$\lambda(Q^\vee) \subset \mathbb{Z} \Rightarrow \lambda \in P.$$

This proves the surjectivity of $\overline{\varphi}$ and hence (a) is proved.

(b) As in Definition 4.2.5 (see (3) of Definition 4.2.5), the map

$$D_c \xrightarrow{\sim} A^\circ \cap P, \; \lambda \mapsto \lambda + \rho, \tag{4}$$

is a bijection. Moreover, by the (a) part, the W-equivariant map

$$\overline{\varphi}\colon\; P/(c+h^\vee)Q_{lg} \xrightarrow{\sim} T_c$$

is an isomorphism. Thus, $\overline{\varphi}$ induces a bijection

$$P/W_c \xrightarrow{\sim} T_c/W. \tag{5}$$

Now, for any $\lambda \in H_{\alpha,n}$ (with $\alpha \in \Delta$ and $n \in \mathbb{Z}$), by identity (6) of Definition 4.2.5,

$$s_\alpha \cdot \tau_{\frac{-2n(c+h^\vee)\alpha}{\langle\alpha,\alpha\rangle}}(\lambda) = \lambda.$$

Of course, by (7) of Definition 4.2.5, for any $\lambda \in P$,

$$\overline{\varphi}(\lambda) = \overline{\varphi}(\tau_{\frac{-2n(c+h^\vee)\alpha}{\langle\alpha,\alpha\rangle}}\lambda).$$

Thus, for $\lambda \in H_{\alpha,n}$,

$$s_\alpha(\overline{\varphi}(\lambda)) = \overline{\varphi}(\lambda); \quad \text{in particular,} \quad \overline{\varphi}(\lambda) \in T_c \backslash T_c^{\mathrm{reg}}. \tag{6}$$

Further, since W_c acts freely on $P\backslash H$, by (5) and (6),

$$(P\backslash H)/W_c \xrightarrow{\sim} T_c^{\mathrm{reg}}/W. \tag{7}$$

Finally, since A° is an alcove and W_c acts simply transitively on the set of alcoves (cf. Definition 4.2.5), the canonical inclusion induces a bijection:

$$A^\circ \cap P \xrightarrow{\sim} (P\backslash H)/W_c. \tag{8}$$

Combining the bijections (4), (8) and (7), we get the (b)-part of the lemma. □

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Definition 4.2.7 Let $R(\mathfrak{g})$ be the representation ring of $\mathfrak{g}$. Define the $\mathbb{Z}$-linear map

$$\xi_c \colon R(\mathfrak{g}) \to R_c(\mathfrak{g})$$

as follows. For $\lambda \in D$ (where D is as in Definition 1.2.6) such that $\lambda + \rho$ lies on an affine wall $H_{\alpha,n}$ (for some $\alpha \in \Delta$ and $n \in \mathbb{Z}$), define

$$\xi_c([V(\lambda)]) = 0.$$

Otherwise, there is a unique $\mu \in D_c$ and $w \in W'_c$ such that $\lambda = w^{-1} * \mu$, where W'_c is the set of minimal-length coset representatives in W_c/W (cf. (4) of Definition 4.2.5). (Observe that, for $w \in W_c$ and $\mu \in D_c$, $w^{-1} * \mu \in D$ if and only if $w \in W'_c$, see Kostant (2004, Remark 1.3).) In this case, we define

$$\xi_c([V(\lambda)]) = \epsilon(w)[V(\mu)],$$

where $\epsilon(w)$ is the sign of the Coxeter group element w.

The following lemma follows easily from the definition of the map ξ_c since $R(\mathfrak{g}) = \bigoplus_{\lambda \in D} \mathbb{Z}[V(\lambda)]$.

Lemma 4.2.8 *The kernel of $\xi_c \colon R(\mathfrak{g}) \to R_c(\mathfrak{g})$ is the $\mathbb{Z}$-submodule of $R(\mathfrak{g})$ spanned by*

(a) $[V(\lambda)]$, *$\lambda \in D$ such that $\lambda + \rho \in H$, and*
(b) $[V(w^{-1} * \mu)] - \epsilon(w)[V(\mu)]$, *for $\mu \in D_c$ and $w \in W'_c$.*

Proof Take $\lambda \in D$ such that $\lambda + \rho \notin H$. Then, W_c acts freely on $\lambda + \rho$ (cf. Definition 4.2.5). From this the lemma follows easily. □

The following is a crucial result used in the proof of an explicit Verlinde dimension formula (cf. Theorem 4.2.19).

Theorem 4.2.9 *For any simple Lie algebra $\mathfrak{g}$ and any level $c > 0$, the map*

$$\xi_c \colon R(\mathfrak{g}) \to R_c(\mathfrak{g})$$

is a surjective ring homomorphism.

In particular, the fundamental representations $\{[V(\omega_i)]\}_{1 \le i \le \ell}$ generate the fusion ring $R_c(\mathfrak{g})$.

Before we come to the proof of this theorem, we give a different geometric definition of a product in $R_c(\mathfrak{g})$ which (as shown below in Corollary 4.2.17) coincides with the product $\otimes^c$. The proof of Theorem 4.2.9 is given in Subsection 4.2.18.

As earlier, we fix a level $c > 0$. Let G be the (simple) simply-connected complex algebraic group with Lie algebra $\mathfrak{g}$ and let $\bar{\hat{G}} = \bar{\hat{G}}_{0_1}$ be the corresponding affine Kac–Moody group (corresponding to the basic weight $0_1 \in \hat{D}$), which is a $\mathbb{G}_m$ central extension of the formal loop group $\bar{G}((t))$ (cf. Proposition 1.4.12). Let $\hat{\mathcal{P}} \subset \bar{\hat{G}}$ be the parahoric subgroup, which is the inverse image of $\bar{G}[[t]]$ under the central extension $\bar{p}\colon \bar{\hat{G}} \to \bar{G}((t))$. The central extension uniquely splits over $\bar{G}[[t]]$ (cf. Remark 1.4.13) and hence

$$\hat{\mathcal{P}} \simeq \mathbb{G}_m \times \bar{G}[[t]].$$

Recall the infinite Grassmannian $\bar{X}_G$, which is an ind-projective variety (cf. Propositions 1.3.18 and 1.3.24) with $\mathbb{C}$-points

$$\bar{X}_G(\mathbb{C}) = G((t))/G[[t]].$$

Then, $\bar{\hat{G}}$ acts on $\bar{X}_G$ through $\bar{p}$ via the action of $\bar{G}((t))$ on $\bar{X}_G$ (cf. Proposition 1.3.18(c)).

Lemma 4.2.10 *(a) Let V be a finite-dimensional representation of G. Then there exists a $\bar{\hat{G}}$-equivariant vector bundle $\mathscr{L}_c(V)$ over $\bar{X}_G$ such that the fiber of $\mathscr{L}_c(V)$, over the base point $\bar{o} \in \bar{X}_G$, which is a module for $(\bar{p})^{-1}(\bar{G}[[t]]) \simeq \mathbb{G}_m \times \bar{G}[[t]]$ is acted by the $\mathbb{G}_m$-factor via the character $z \mapsto z^{-c}$ and $\bar{G}[[t]]$ acts through the representation V^* of G under the evaluation map $\bar{G}[[t]] \to G, t \mapsto 0$.*

Moreover,

(b) For any $\lambda \in D$ such that $\lambda + \rho \in H$, $H^i(\bar{X}_G, \mathscr{L}_c(V(\lambda))) = 0$ for all $i \geq 0$, where H is defined in Definition 4.2.5.

And,

(c) For $\mu \in D_c$ and $w \in W'_c$, $H^i(\bar{X}_G, \mathscr{L}_c(V(w^{-1} * \mu))) = 0$ unless $i = \ell(w)$, and

$$H^{\ell(w)}(\bar{X}_G, \mathscr{L}_c(V(w^{-1} * \mu))) \simeq \mathscr{H}(\mu_c)^*$$

as a module of Lie $\bar{\hat{G}} = \hat{\mathfrak{g}}$ (cf. Definition B.22 and Lemma 1.4.6).

Thus,

$$H^{\ell(w)}(\bar{X}_G, \mathscr{L}_c(V(w^{-1} * \mu)))^{\hat{\mathfrak{g}}_-} \simeq V(\mu)^*,$$

where, as in (4) *of Definition 1.2.2, $\hat{\mathfrak{g}}_- := \mathfrak{g} \otimes (t^{-1}\mathbb{C}[t^{-1}])$.*

Proof To prove the existence of $\mathscr{L}_c(V)$, it clearly suffices to assume that $V = V(\lambda)$ for $\lambda \in D$. Consider the affine full-flag ind-variety $\bar{X}_G(B)$ defined

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in Exercise 1.3.E.11, where $B \subset G$ is the Borel subgroup with Lie algebra $\mathfrak{b}$ (as at the beginning of Section 1.2). Then, parallel to $\bar{X}_G$, $\hat{\bar{G}}$ acts on $\bar{X}_G(B)$ through $\bar{p}\colon \hat{\bar{G}} \to \bar{G}((t))$. Let $\hat{\mathcal{B}}$ be the (affine) Borel subgroup of $\hat{\bar{G}}$, which is the inverse image of $\mathcal{B} \subset \bar{G}((t))$ defined in Exercise 1.3.E.11. Recall that $\mathcal{B} := ev_0^{-1}(B)$ is the closed subgroup scheme under the evaluation map $ev_0\colon \bar{G}[[t]]) \to G, t \mapsto 0$.

Now, for any $\lambda \in D$, there exists a $\hat{\bar{G}}$-equivariant line bundle $\mathscr{L}_c^B(\lambda)$ over $\bar{X}_G(B)$ such that the fiber of $\mathscr{L}_c^B(\lambda)$ over the base point $\bar{o}_B \in \bar{X}_G(B)$ (which is a module for $\hat{\mathcal{B}} \simeq \mathbb{G}_m \times \mathcal{B}$) is acted by the $\mathbb{G}_m$-factor via the character $z \mapsto z^{-c}$ and $\mathcal{B}$ acts through the character $e^{-\lambda}$ of B under the evaluation map $\mathcal{B} \to B, t \mapsto 0$. More generally, as we will need in Subsection 4.2.18, for any finite-dimensional B-module M, there exists a $\hat{\bar{G}}$-equivariant vector bundle $\mathscr{L}_c^B(M)$ over $\bar{X}_G(B)$ such that the fiber of $\mathscr{L}_c^B(M)$ over the base point $\bar{o}_B \in \bar{X}_G(B)$ is acted by the $\mathbb{G}_m$-factor via the character $z \mapsto z^{-c}$ and $\mathcal{B}$ acts through the representation M^* of B. We denote this representation of $\hat{\mathcal{B}}$ by M_c^*. We now show the existence of such $\mathscr{L}_c^B(M)$.

Since $\bar{G}((t)) \to \bar{X}_G(B)$ is a locally trivial principal $\mathcal{B}$-bundle (cf. Exercise 1.3.E.11) and $\hat{\bar{G}} \xrightarrow{\bar{p}} \bar{G}((t))$ is a locally trivial principal $\mathbb{G}_m$-bundle trivial over $\bar{G}[t^{-1}]^- \times \bar{G}[[t]]$ (cf. Proposition 1.4.12), we get that the composite $\beta\colon \hat{\bar{G}} \to \bar{X}_G(B)$ is a locally trivial principal $\hat{\mathcal{B}}$-bundle. Consider the $\hat{\mathcal{B}}$-equivariant vector bundle $\theta\colon \hat{\bar{G}} \times M_c^* \to \hat{\bar{G}}$ under the projection θ, where $\hat{\mathcal{B}}$ acts on $\hat{\bar{G}} \times M_c^*$ via

$$b \cdot (g, v) = (gb^{-1}, b \cdot v), \quad \text{for } b \in \hat{\mathcal{B}}, g \in \hat{\bar{G}} \text{ and } v \in M_c^*.$$

Thus, by Theorem C.17, θ descends to give the vector bundle $\mathscr{L}_c^B(M)$ over $\bar{X}_G(B)$ as above. Recall from the proof of Theorem C.17 that, thought of as a $\mathbb{C}$-space functor, $\mathscr{L}_c^B(M)$ is the sheafification of the functor

$$S \rightsquigarrow \left(\hat{\bar{G}}(S) \times (M_c^*)(S)\right) / \hat{\mathcal{B}}(S)$$

(also see Kumar (2002, Corollary 8.2.5) for another construction of $\mathscr{L}_c^B(M)$).

Now, define

$$\mathscr{L}_c(V(\lambda)) := \pi_* \left(\mathscr{L}_c^B(\lambda)\right)$$

for the locally trivial G/B-fibration $\pi : \bar{X}_G(B) \to \bar{X}_G$ (cf. Exercise 1.3.E.11). By the classical BWB theorem, $\mathscr{L}_c(V(\lambda))$ satisfies the property (a). Further, from the degenerate Leray spectral sequence applied to the fibration π with respect to the line bundle $\mathscr{L}_c^B(\lambda)$, for any $i \geq 0$ and $\lambda \in D$,

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$$H^i\left(\bar{X}_G(B), \mathscr{L}_c^B(\lambda)\right) \simeq H^i\left(\bar{X}_G, \mathscr{L}_c(V(\lambda))\right).$$

Thus, (b) and (c)-parts of the lemma follow from the affine analogue of the BWB theorem (cf. (Kumar, 2002, Corollary 8.3.12)). (Observe that we have used here Proposition 1.3.24.) □

Definition 4.2.11 For any $\lambda, \mu \in D_c$, define the following (*a priori* different from $\otimes^c$) product $\otimes_F^c$:

$$[V(\lambda)] \otimes_F^c [V(\mu)] = \chi_{\mathfrak{g}}\left(\bar{X}_G, \mathscr{L}_c(V(\lambda) \otimes V(\mu))\right)^*, \tag{1}$$

where for any finite-dimensional G-module V, we define the virtual G-module

$$\chi_{\mathfrak{g}}(\bar{X}_G, \mathscr{L}_c(V)) := \sum_{i \geq 0} (-1)^i \left[H^i(\bar{X}_G, \mathscr{L}_c(V))^{\hat{\mathfrak{g}}_-}\right] \in R_c(\mathfrak{g}). \tag{2}$$

As shown below in the following Lemma 4.2.12, the above sum is a finite sum and it is determined there. In particular, it lies in $R_c(\mathfrak{g})$.

We can rewrite (1) as follows. Let

$$H^i\left(\bar{X}_G, \mathscr{L}_c(V(\lambda) \otimes V(\mu))\right) \simeq \bigoplus_{\nu \in D_c} d_i^{\lambda,\mu}(\nu)\, \mathscr{H}(\nu)^*,$$

as $\hat{\mathfrak{g}}$-modules, for some (unique) $d_i^{\lambda,\mu}(\nu) \in \mathbb{Z}_+$ (cf. Lemma 4.2.10). Then

$$[V(\lambda)] \otimes_F^c [V(\mu)] = \sum_{\nu \in D_c} \left(\sum_{i \geq 0} (-1)^i d_i^{\lambda,\mu}(\nu)\right) [V(\nu)]. \tag{3}$$

Lemma 4.2.12 *For any* $\lambda, \mu \in D_c$ *,*

$$[V(\lambda)] \otimes_F^c [V(\mu)] = \sum_{\nu \in D_c} \sum_{w \in W_c'} (-1)^{\ell(w)} n_{w^{-1} * \nu}^{\lambda,\mu} [V(\nu)], \tag{1}$$

where

$$n_{w^{-1} * \nu}^{\lambda,\mu} := \dim\left(\operatorname{Hom}_{\mathfrak{g}}\left(V(w^{-1} * \nu), V(\lambda) \otimes V(\mu)\right)\right).$$

Proof Decompose as $\mathfrak{g}$-modules (cf. Definition 4.2.7):

$$V(\lambda) \otimes V(\mu) \simeq \left(\bigoplus_{\nu \in D_c} \bigoplus_{w \in W_c'} n_{w^{-1} * \nu}^{\lambda,\mu} V(w^{-1} * \nu)\right) \bigoplus \bigoplus_{\gamma \in D \cap (H - \rho)} d_\gamma^{\lambda,\mu} V(\gamma), \tag{2}$$

for some $d_\gamma^{\lambda,\mu} \in \mathbb{Z}_+$.

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Then, from decomposition (2) and Lemma 4.2.10, we get

$$
\begin{aligned}
&H^i(\bar{X}_G, \mathscr{L}_c(V(\lambda) \otimes V(\mu))) \\
&\simeq \left(\bigoplus_{\nu \in D_c} \bigoplus_{w \in W'_c} n^{\lambda,\mu}_{w^{-1}*\nu} H^i\left(\bar{X}_G, \mathscr{L}_c(V(w^{-1} * \nu))\right)\right) \\
&\quad \bigoplus \bigoplus_{\gamma \in D \cap (H-\rho)} d^{\lambda,\mu}_{\gamma} H^i\left(\bar{X}_G, \mathscr{L}_c(V(\gamma))\right) \\
&\simeq \bigoplus_{\nu \in D_c} \left(\bigoplus_{\substack{w \in W'_c \\ \ell(w)=i}} n^{\lambda,\mu}_{w^{-1}*\nu} \right) \mathscr{H}(\nu)^*, \text{ by Lemma 4.2.10.} \qquad (3)
\end{aligned}
$$

Thus,

$$\chi_{\mathfrak{g}}\left(\bar{X}_G, \mathscr{L}_c(V(\lambda) \otimes V(\mu))\right) = \sum_{\nu \in D_c} \sum_{w \in W'_c} (-1)^{\ell(w)} n^{\lambda,\mu}_{w^{-1}*\nu} [V(\nu)^*].$$

From the definition of $[V(\lambda)] \otimes^c_F [V(\mu)]$ as in (1) of Definition 4.2.11, we get (1). This proves the lemma. □

Definition 4.2.13 For any $\mu \in D$ and nonzero $z \in \mathbb{C}$, realize $V(\mu)$ as a module for the affine Lie algebra $\tilde{\mathfrak{g}} := (\mathfrak{g} \otimes \mathcal{A}) \oplus \mathbb{C}C$ (cf. (1) of Definition 1.2.1) via the evaluation at z:

$$ev_z : (\mathfrak{g} \otimes \mathcal{A}) \oplus \mathbb{C}C \to \mathfrak{g}, \ \ C \mapsto 0 \ \text{ and } x[f] \mapsto f(z)x, \ \text{ for } x \in \mathfrak{g} \ \text{ and } f \in \mathcal{A}.$$

We denote this $\tilde{\mathfrak{g}}$-module by $V_z(\mu)$.

For any $\nu \in D_c$ and $\mu \in D$, we give a resolution of $\mathscr{H}(\nu) \otimes V_z(\mu)$.

First, recall the BGG resolution consisting of $\hat{\mathfrak{g}}$-modules and $\hat{\mathfrak{g}}$-module maps (cf. (Kumar, 2002, Theorem 9.1.3)):

$$\cdots \xrightarrow{\delta_2} F_1 \xrightarrow{\delta_1} F_0 \xrightarrow{\epsilon} \mathscr{H}(\nu) \to 0,$$

where

$$F_p := \bigoplus_{\substack{w \in W'_c \\ \ell(w)=p}} \hat{M}(V(w^{-1} * \nu), c).$$

Tensoring with $V_z(\mu)$, we get the resolution

$$\cdots \to F_1 \otimes V_z(\mu) \to F_0 \otimes V_z(\mu) \to \mathscr{H}(\nu) \otimes V_z(\mu) \to 0.$$

Recall from Definition 1.2.2 that $\hat{\mathfrak{g}}_- := \mathfrak{g} \otimes (t^{-1}\mathbb{C}[t^{-1}])$.

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Lemma 4.2.14 *For any $\nu \in D_c$, $\mu \in D$ and $z \in \mathbb{C}$, the Lie algebra homology $H_*(\hat{\mathfrak{g}}_-, \mathscr{H}(\nu) \otimes V_z(\mu))$ as a $\mathfrak{g}$-module is given by the homology of the following complex consisting of $\mathfrak{g}$-modules and $\mathfrak{g}$-module maps:*

$$\cdots \to \hat{F}_p \to \cdots \xrightarrow{\hat{\delta}_2} \hat{F}_1 \xrightarrow{\hat{\delta}_1} \hat{F}_0 \to 0,$$

where

$$\hat{F}_p := \bigoplus_{\substack{w \in W'_c \\ \ell(w)=p}} \Big(V(w^{-1} * \nu) \otimes V(\mu) \Big).$$

Proof By the proof of Lemma 1.2.5, for any $\mathfrak{g}$-module V, as $U(\hat{\mathfrak{g}}_-)$-modules,

$$\hat{M}(V,c) \simeq U(\hat{\mathfrak{g}}_-) \otimes_{\mathbb{C}} V,$$

where $U(\hat{\mathfrak{g}}_-)$ acts on the right-hand side via the left multiplication on the first factor. Thus, by the Hopf principle (cf. (Kumar, 2002, Proposition 3.1.10)), as $U(\hat{\mathfrak{g}})$-modules,

$$\hat{M}(V,c) \otimes V_z(\mu) \simeq U(\hat{\mathfrak{g}}_-) \otimes_{\mathbb{C}} (V \otimes V_z(\mu)). \tag{1}$$

In particular, $\hat{M}(V,c) \otimes V_z(\mu)$ is free as a $U(\hat{\mathfrak{g}}_-)$-module. Thus, the homology $H_*(\hat{\mathfrak{g}}_-, \mathscr{H}(\nu) \otimes V_z(\mu))$ is given by the homology of the complex:

$$\cdots \to \mathbb{C} \otimes_{U(\hat{\mathfrak{g}}_-)} \Big(F_1 \otimes V_z(\mu) \Big) \to \mathbb{C} \otimes_{U(\hat{\mathfrak{g}}_-)} \Big(F_0 \otimes V_z(\mu) \Big) \to 0. \tag{2}$$

By (1), there exists a $\mathfrak{g}$-module isomorphism (for any $\mathfrak{g}$-module V):

$$\mathbb{C} \otimes_{U(\hat{\mathfrak{g}}_-)} \Big(\hat{M}(V,c) \otimes V_z(\mu) \Big) \simeq V \otimes V_z(\mu) \simeq V \otimes V(\mu). \tag{3}$$

Combining (2) and (3), we get the lemma. □

Proposition 4.2.15 *The products $\otimes^c$ and $\otimes^c_F$ in $R_c(\mathfrak{g})$ coincide if and only if for all λ, μ, $\nu \in D_c$,*

$$\overline{\chi}_{\mathfrak{g}}(\lambda, \mu, \nu) = 0,$$

where

$$\overline{\chi}_{\mathfrak{g}}(\lambda, \mu, \nu) := \sum_{i \geq 1} (-1)^i \dim \big(\mathrm{Hom}_{\mathfrak{g}}(V(\lambda), H_i(\hat{\mathfrak{g}}_-, \mathscr{H}(\nu) \otimes V_1(\mu))) \big).$$

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Proof By Lemma 4.2.12, for $\lambda, \mu \in D_c$,

$$\begin{aligned}[V(\lambda)] \otimes_F^c [V(\mu)] &= \sum_{\nu\in D_c} \sum_{w\in W_c'} (-1)^{\ell(w)} n^{\lambda,\mu}_{w^{-1}*\nu}[V(\nu)] \\ &= \sum_{\nu\in D_c} \sum_{w\in W_c'} (-1)^{\ell(w)} \dim \Big(\mathrm{Hom}_{\mathfrak{g}}(V(\lambda), V(w^{-1} * \nu) \\ &\quad \otimes V(\mu^*))\Big) [V(\nu)] \\ &= \sum_{\nu\in D_c} \sum_{i\geq 0}(-1)^{i} \dim \big(\mathrm{Hom}_{\mathfrak{g}}(V(\lambda), H_i(\hat{\mathfrak{g}}_-, \mathscr{H}(\nu) \\ &\quad \otimes V_1(\mu^*))\big) [V(\nu)], \qquad (1)\end{aligned}$$

by Lemma 4.2.14 using the Euler–Poincaré principle. Further, by the definition of $\otimes^c$ (cf. (1) of Example 4.2.1),

$$[V(\lambda)] \otimes^c [V(\mu)] = \sum_{\nu\in D_c} \dim \mathscr{V}_{\mathbb{P}^1}(\lambda, \mu, \nu^*)[V(\nu)], \qquad (2)$$

where (λ, μ, ν^*) are attached to the points $(\infty, 1, 0)$ on $\mathbb{P}^1$, respectively.

By Lemma 4.2.2,

$$\begin{aligned}\mathscr{V}_{\mathbb{P}^1}(\lambda, \mu, \nu^*) &\simeq \mathscr{V}_{\mathbb{P}^1}(\lambda^*, \mu^*, \nu) \\ &\simeq [\mathscr{H}(\nu) \otimes V_\infty(\lambda^*) \otimes V_1(\mu^*)]_{\mathfrak{g}\otimes\mathbb{C}[t^{-1}]}, \text{ by Theorem 2.2.2.} \\ &\simeq \mathrm{Hom}_{\mathfrak{g}}(V(\lambda), H_0(\hat{\mathfrak{g}}_-, \mathscr{H}(\nu) \otimes V_1(\mu^*))). \qquad (3)\end{aligned}$$

Combining (1)–(3) and replacing μ by μ^*, we get the proposition. □

We now recall the following result from Teleman (1995, Theorem 0), the proof of which is omitted due to its length. Actually, he proves a more general result, but the following version is sufficient for our purposes.

Theorem 4.2.16 *For any λ, μ, $\nu \in D_c$ and any $i \geq 1$, $V(\lambda)$ does not occur in $H_i(\hat{\mathfrak{g}}_-, \mathscr{H}(\nu) \otimes V_1(\mu))$ as a $\mathfrak{g}$-module.* □

From the above theorem, one can completely determine $H_i(\hat{\mathfrak{g}}_-, \mathscr{H}(\nu) \otimes V_1(\mu))$ as a $\mathfrak{g}$-module, provided one knows the $\mathfrak{g}$-module $H_0(\hat{\mathfrak{g}}_-, \mathscr{H}(\nu) \otimes V_1(\mu))$ (cf. Exercise 4.2.E.6).

As an immediate consequence of Lemma 4.2.12, Proposition 4.2.15 and Theorem 4.2.16, we get the following.

Corollary 4.2.17 *The two products $\otimes^c$ and $\otimes_F^c$ in $R_c(\mathfrak{g})$ coincide for any simple Lie algebra $\mathfrak{g}$ and any central charge $c > 0$.*

Hence, for any $\lambda, \mu \in D_c$,

$$[V(\lambda)] \otimes^c [V(\mu)] = \sum_{\nu \in D_c} \sum_{w \in W'_c} (-1)^{\ell(w)} \times \dim \left(\mathrm{Hom}_{\mathfrak{g}}(V(w^{-1} * \nu), V(\lambda) \otimes V(\mu))\right) [V(\nu)].$$

Now, we are ready to prove Theorem 4.2.9.

4.2.18 Proof of Theorem 4.2.9

In view of Corollary 4.2.17, it suffices to show that the map $\xi_c \colon R(\mathfrak{g}) \to R_c(\mathfrak{g})$ is a ring homomorphism with respect to the product $\otimes_F^c$ in $R_c(\mathfrak{g})$. As an immediate consequence of ξ_c being a ring homomorphism with respect to the product $\otimes_F^c$ in $R_c(\mathfrak{g})$, we get that $\otimes_F^c$ is associative (since ξ_c is surjective).

For any finite-dimensional G-module V, define the virtual $\hat{\mathfrak{g}}$-module

$$\chi\left(\bar{X}_G, \mathscr{L}_c(V)\right) := \sum_{i \geq 0} (-1)^i H^i\left(\bar{X}_G, \mathscr{L}_c(V)\right).$$

By the definition of ξ_c and Lemma 4.2.10, for any $\lambda \in D$,

$$\xi_c([V(\lambda)]) = \chi_{\mathfrak{g}}\left(\bar{X}_G, \mathscr{L}_c(V(\lambda))\right)^*, \tag{1}$$

where $\chi_{\mathfrak{g}}$ is defined by (2) of Definition 4.2.11.

We next show that for any $\lambda \in D$, $w \in W_c$ with $w^{-1} * \lambda \in D$ and finite-dimensional G-module V,

$$\chi\left(\bar{X}_G, \mathscr{L}_c(V(\lambda) \otimes V)\right) = (-1)^{\ell(w)} \chi\left(\bar{X}_G, \mathscr{L}_c\left(V(w^{-1} * \lambda) \otimes V\right)\right). \tag{2}$$

In particular,

$$\chi_{\mathfrak{g}}\left(\bar{X}_G, \mathscr{L}_c(V(\lambda) \otimes V)\right) = (-1)^{\ell(w)} \chi_{\mathfrak{g}}\left(\bar{X}_G, \mathscr{L}_c\left(V(w^{-1} * \lambda) \otimes V\right)\right). \tag{3}$$

Since V is a G-module, from the Leray spectral sequence applied to the locally trivial G/B-fibration $\bar{X}_G(B) \to \bar{X}_G$ (cf. Exercise 1.3.E.11) with respect to the vector bundle $\mathscr{L}_c^B(\mathbb{C}_\lambda \otimes V)$ and the classical BWB theorem, we get

$$\begin{aligned}\chi\left(\bar{X}_G, \mathscr{L}_c(V(\lambda) \otimes V)\right) &= \chi\left(\bar{X}_G(B), \mathscr{L}_c^B(\mathbb{C}_\lambda \otimes V)\right) \\ &= \sum_{\beta \in P_V} n_\beta\, \chi\left(\bar{X}_G(B), \mathscr{L}_c^B(\lambda + \beta)\right),\end{aligned} \tag{4}$$

where the vector bundle $\mathscr{L}_c^B(M)$ over $\bar{X}_G(B)$ for any finite-dimensional B-module M is as in the proof of Lemma 4.2.10, $\mathbb{C}_\lambda$ denotes the 1-dimensional representation of B corresponding to the character e^λ, P_V is the set of weights of V and the character

$$\mathrm{ch}(V) = \sum_{\beta \in P_V} n_\beta e^\beta.$$

Similarly,

$$\begin{aligned}
&\chi(\bar{X}_G, \mathscr{L}_c(V(w^{-1} * \lambda) \otimes V)) \\
&= \sum_{\beta \in P_V} n_\beta \, \chi\left(\bar{X}_G(B), \mathscr{L}_c^B(w^{-1} * \lambda + \beta)\right) \\
&= \sum_{\beta \in P_V} n_\beta \, \chi\left(\bar{X}_G(B), \mathscr{L}_c^B(w^{-1} * (\lambda + \beta)\right), \text{ since } n_\beta = n_{v\beta}, \text{ for any } v \in W \\
&= (-1)^{\ell(w)} \sum_{\beta \in P_V} n_\beta \, \chi\left(\bar{X}_G(B), \mathscr{L}_c^B(\lambda + \beta)\right),
\end{aligned}$$

by Kumar (2002, Corollary 8.3.12). (5)

Combining (4) and (5), we get (2).

Specializing (2) to the case when $\lambda \in D$ is such that $\lambda + \rho \in H_{\alpha,n}$ (with $\alpha \in \Delta$ and $n \in \mathbb{Z}$), we get from (6) of Definition 4.2.5 (since $\tau_{\frac{2n(c+h^\vee)\alpha}{\langle\alpha,\alpha\rangle}}$ has even length by Kumar (2002, Exercise 13.1.E.3) and $(s_\alpha \cdot \tau_{\frac{-2n(c+h^\vee)\alpha}{\langle\alpha,\alpha\rangle}})(\lambda+\rho) = \lambda+\rho$ by (6) of Definition 4.2.5),

$$\chi(\bar{X}_G, \mathscr{L}_c(V(\lambda) \otimes V)) = -\chi(\bar{X}_G, \mathscr{L}_c(V(\lambda) \otimes V))$$

and hence

$$\chi(\bar{X}_G, \mathscr{L}_c(V(\lambda) \otimes V)) = 0. \tag{6}$$

In particular,

$$\chi_{\mathfrak{g}}(\bar{X}_G, \mathscr{L}_c(V(\lambda) \otimes V)) = 0. \tag{7}$$

We are now ready to prove that ξ_c is a ring homomorphism with respect to the product $\otimes_F^c$ in $R_c(\mathfrak{g})$, i.e., for $\lambda, \mu \in D$,

$$\xi_c([V(\lambda) \otimes V(\mu)]) = \xi_c([V(\lambda)]) \otimes_F^c \xi_c([V(\mu)]). \tag{8}$$

If at least one of λ or μ (say λ) is such that $\lambda + \rho \in H$, then by (1) and (7), both sides of (8) are zero. So, assume that both of $\lambda + \rho$ and $\mu + \rho$ lie in $P \backslash H$, i.e., there exists $v, w \in W'_c$ and $\lambda_o, \mu_o \in D_c$ such that $\lambda = v^{-1} * \lambda_o$ and $\mu = w^{-1} * \mu_o$.

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Then, by (1) and (3),

$$\begin{aligned}\xi_c([V(\lambda)\otimes V(\mu)]) &= \chi_{\mathfrak{g}}\Big(\bar{X}_G, \mathscr{L}_c\,(V(\lambda)\otimes V(\mu))\Big)^* \\ &= (-1)^{\ell(v)+\ell(w)}\chi_{\mathfrak{g}}\left(\bar{X}_G, \mathscr{L}_c(V(\lambda_o)\otimes V(\mu_o))\right)^* \\ &= (-1)^{\ell(v)+\ell(w)}\xi_c\left([V(\lambda_o)]\right)\otimes^c_F \xi_c\left([V(\mu_o)]\right), \\ &\quad \text{by (1) of Definition 4.2.11} \\ &= \xi_c\left([V(\lambda)]\right)\otimes^c_F \xi_c\left([V(\mu)]\right).\end{aligned}$$

This completes the proof of (8) and hence Theorem 4.2.9 is proved. □

For any $t \in T$, we get an algebra homomorphism

$$\mathrm{ch}_t\colon R(\mathfrak{g}) \to \mathbb{C},$$

where for any representation δ of G in a finite-dimensional vector space V,

$$\mathrm{ch}_t([V]) := \mathrm{trace}_V\,\delta(t).$$

As a consequence of the Weyl character formula, Lemmas 4.1.5, 4.2.6, 4.2.8 and Theorem 4.2.9, we get the following.

Corollary 4.2.18 *For any $t \in T_c^{reg}$ (cf. Definition 4.2.5), the character $ch_t\colon R(\mathfrak{g}) \to \mathbb{C}$ factors through $R_c(\mathfrak{g})$ via ξ_c (cf. Definition 4.2.7) to give an algebra homomorphism*

$$ch_{t,c}\colon R_c(\mathfrak{g}) \to \mathbb{C}.$$

Moreover, $\{ch_{t,c}\}_{t\in T_c^{reg}/W}$ bijectively parameterizes the set S_{D_c} of algebra homomorphisms of $R_c(\mathfrak{g})$ to $\mathbb{C}$.

Proof To prove that, for any $t \in T_c^{\mathrm{reg}}$, ch_t factors through $R_c(\mathfrak{g})$, by Lemma 4.2.6, we can assume that

$$t = \mathrm{Exp}\left(\frac{2\pi i\kappa(\mu+\rho)}{c+h^\vee}\right), \quad \text{for some } \mu \in D_c.$$

By the Weyl character formula, for any $\lambda \in D$,

$$\mathrm{ch}_t([V(\lambda)]) = \frac{\sum\limits_{w\in W}\epsilon(w)e^{w(\lambda+\rho)}(t)}{\sum\limits_{w\in W}\epsilon(w)e^{w\rho}(t)}.$$

Since $t \in T$ is regular, the denominator is nonzero.

So, by Lemma 4.2.8, it suffices to prove that

$$\sum_{w\in W}\epsilon(w)e^{\frac{2\pi i}{c+h^\vee}\langle w(\lambda+\rho),\mu+\rho\rangle} = \epsilon(v\tau_\beta)\sum_{w\in W}\epsilon(w)e^{\frac{2\pi i}{c+h^\vee}\langle wv(\lambda+\rho+\beta),\mu+\rho\rangle}, \quad (1)$$

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for any $v \in W$, $\lambda, \mu \in D_c$ and $\beta \in (c + h^\vee)Q_{lg}$, and

$$\sum_{w \in W} \epsilon(w) e^{\frac{2\pi i}{c+h^\vee}\langle w(\lambda+\rho), \mu+\rho\rangle} = 0, \text{ for } \lambda \in D \text{ with } \lambda + \rho \in H \text{ and any } \mu \in D_c. \tag{2}$$

For any $\beta \in (c + h^\vee)Q_{lg}$, $\epsilon(\tau_\beta) = 1$, by (3) of the proof of Lemma 4.2.6 and Kumar (2002, Exercise 13.1.E.3). Moreover, for any $\beta \in (c + h^\vee)Q_{lg}$ (and hence $w \cdot \beta \in (c + h^\vee)Q_{lg}$),

$$\langle \beta, \mu + \rho\rangle \in (c + h^\vee)\mathbb{Z}, \text{ by (3) of Lemma 4.2.6.} \tag{3}$$

This proves (1).

To prove (2), let $\lambda + \rho \in H_{\alpha,n}$ for some $\alpha \in \Delta$ and $n \in \mathbb{Z}$. Then, by identity (6) of Definition 4.2.5, $s_\alpha \cdot \tau_{\frac{-2n(c+h^\vee)\alpha}{\langle\alpha,\alpha\rangle}}(\lambda + \rho) = \lambda + \rho$. Thus,

$$\begin{aligned}\sum_{w \in W} \epsilon(w) e^{\frac{2\pi i}{c+h^\vee}\langle w(\lambda+\rho), \mu+\rho\rangle} &= \sum_{w \in W} \epsilon(w) e^{\frac{2\pi i}{c+h^\vee}\langle ws_\alpha(\lambda+\rho-\frac{2n(c+h^\vee)\alpha}{\langle\alpha,\alpha\rangle}),\ \mu+\rho\rangle} \\ &= -\sum_{w \in W} \epsilon(w) e^{\frac{2\pi i}{c+h^\vee}\langle w(\lambda+\rho-\frac{2n(c+h^\vee)\alpha}{\langle\alpha,\alpha\rangle}),\ \mu+\rho\rangle} \\ &= -\sum_{w \in W} \epsilon(w) e^{\frac{2\pi i}{c+h^\vee}\langle w(\lambda+\rho), \mu+\rho\rangle}\end{aligned}$$

and hence

$$\sum_{w \in W} \epsilon(w) e^{\frac{2\pi i}{c+h^\vee}\langle w(\lambda+\rho), \mu+\rho\rangle} = 0,$$

proving (2).

Since $\xi_c : R(\mathfrak{g}) \to R_c(\mathfrak{g})$ is a surjective algebra homomorphism and $\mathrm{ch}_t : R(\mathfrak{g}) \to \mathbb{C}$ is an algebra homomorphism, we get that $\mathrm{ch}_{t,c} : R_c(\mathfrak{g}) \to \mathbb{C}$ is an algebra homomorphism.

By Lemma 4.1.5,

$$|S_{D_c}| = |D_c|. \tag{4}$$

Further, by Lemma 4.2.6,

$$\left|T_c^{\mathrm{reg}}/W\right| = |D_c|. \tag{5}$$

Since

$$\gamma : R(\mathfrak{g}) \otimes_{\mathbb{Z}} \mathbb{C} \xrightarrow{\sim} \mathbb{C}[T/W]$$

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is an isomorphism (cf. (Bröcker and tom Dieck, 1985, Chap. VI, Proposition 2.1)), where

$$\gamma([V])(t) = \mathrm{ch}_t([V]), \quad \text{for} \quad t \in T/W,$$

we get that $\{\mathrm{ch}_{t,c}\}_{t\in T_c^{\mathrm{reg}}/W}$ are all distinct. Thus, by (4) and (5), $\{\mathrm{ch}_{t,c}\}_{t\in T_c^{\mathrm{reg}}/W}$ bijectively parameterizes S_{D_c}. This proves the corollary. □

We now come to the following *Verlinde formula*, which is one of the most important results of the book.

Theorem 4.2.19 *Let $\mathfrak{g}$ be any simple Lie algebra and let $c > 0$ be any central charge. Let $(\Sigma, \vec{p} = (p_1, \ldots, p_s))$ be an irreducible smooth s-pointed curve of any genus $g \geq 0$ (where $s \geq 1$) and let $\vec{\lambda} = (\lambda_1, \ldots, \lambda_s)$ be a collection of weights in D_c. Then*

$$\dim \mathscr{V}_\Sigma(\vec{p},\vec{\lambda}) = |T_c|^{g-1} \sum_{\mu\in D_c} \left(\left(\Pi_{i=1}^s (\mathrm{ch}_{t_\mu}([V(\lambda_i)]))\right) \cdot \Pi_{\alpha\in\Delta_+} \left(2\sin\left(\frac{\pi}{c+h^\vee}\langle \mu+\rho, \alpha\rangle\right)\right)^{2-2g} \right), \tag{1}$$

where Δ_+ is the set of positive roots of $\mathfrak{g}$, $h^\vee$ is the dual Coxeter number, T_c is as in Definition 4.2.5, $\kappa : \mathfrak{h}^* \to \mathfrak{h}$ is the isomorphism induced from the normalized invariant form and $t_\mu := \mathrm{Exp}\left(\frac{2\pi i\kappa(\mu+\rho)}{c+h^\vee}\right) \in T_c$.

Moreover,

$$|T_c| = (c+h^\vee)^\ell |P/Q|\,|Q/Q_{lg}|, \tag{2}$$

where ℓ is the rank of $\mathfrak{g}$, P (resp. Q) is the weight (resp. root) lattice and Q_{lg} is the sublattice of Q generated by the long roots.

In particular, for $g = 1, s = 1$ and $\vec{\lambda} = (0)$,

$$\dim \mathscr{V}_\Sigma(\vec{p},\vec{\lambda}) = |D_c|.$$

Proof Let F_c be the fusion rule as in Example 4.2.1. By Corollaries 3.5.10(a) and 4.1.8,

$$\dim \mathscr{V}_\Sigma(\vec{p},\vec{\lambda}) = \sum_{\chi\in S_{D_c}} \chi([V(\lambda_1)])\ldots\chi([V(\lambda_s)])\chi(\Omega)^{g-1}, \tag{3}$$

where S_{D_c} is the set of algebra homomorphisms $R_c(\mathfrak{g}) \to \mathbb{C}$ and

$$\chi(\Omega) = \sum_{\nu\in D_c} |\chi([V(\nu)])|^2. \tag{4}$$

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By Corollary 4.2.18 and Lemma 4.2.6,

$$S_{D_c} = \{\mathrm{ch}_{t_\mu,c}\}_{\mu\in D_c}. \tag{5}$$

We now determine $\chi(\Omega)$ for any $\chi = \mathrm{ch}_{t_\mu,c}$.

Let $L^2(T_c)$ be the space of $\mathbb{C}$-valued functions on T_c with inner product

$$\langle f, g\rangle = \frac{1}{|T_c|}\sum_{t\in T_c} f(t)\overline{g(t)}.$$

For any $\mu \in D_c$, consider the function on T_c:

$$J_\mu(t) = \sum_{w\in W} \epsilon(w) e^{w(\mu+\rho)}(t).$$

Since J_μ is W-anti-invariant, i.e.,

$$J_\mu(vt) = \epsilon(v) J_\mu(t), \quad \text{for any } v \in W \text{ and } t \in T_c,$$

J_μ vanishes on $T_c\backslash T_c^{\mathrm{reg}}$. Of course, $|J_\mu|^2$ is W-invariant. Thus, by Lemma 4.2.6(b),

$$\sum_{\nu\in D_c} |J_\mu(t_\nu)|^2 = \frac{|T_c|}{|W|}||J_\mu||^2. \tag{6}$$

Now, we claim that for any $\mu \in D_c$, $\{e^{w(\mu+\rho)}\}_{w\in W}$ are distinct characters of T_c, i.e., for $w \neq 1$,

$$e^{w(\mu+\rho)}{}_{|T_c} \neq e^{\mu+\rho}{}_{|T_c}.$$

By Lemma 4.2.6(a), it is equivalent to the assertion that

$$\langle w(\mu+\rho) - (\mu+\rho), \lambda\rangle \notin (c+h^\vee)\mathbb{Z}, \quad \text{for some} \quad \lambda \in P.$$

If not, assuming $\langle w(\mu+\rho) - (\mu+\rho), \lambda\rangle \in (c+h^\vee)\mathbb{Z}$, for all $\lambda \in P$, we get from (3) of the proof of Lemma 4.2.6 that

$$w(\mu+\rho) - (\mu+\rho) = \beta, \quad \text{for some} \quad \beta \in (c+h^\vee)Q_{lg}.$$

Thus,

$$\tau_{-\beta}\cdot w(\mu+\rho) = \mu+\rho,$$

which is a contradiction, since $\mu+\rho \in A^\circ$ by (3) of Definition 4.2.5. This proves that $\{e^{w(\mu+\rho)}{}_{|T_c}\}_{w\in W}$ are distinct characters.

Thus, by the orthogonality relation for the finite group T_c, we get

$$||J_\mu||^2 = |W|. \tag{7}$$

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Taking $\chi = \mathrm{ch}_{t_\mu, c}$ in (4), we get

$$\begin{aligned}\chi(\Omega) &= \sum_{\nu \in D_c} |\operatorname{ch}_{t_\mu,c}([V(\nu)])|^2 \\ &= \frac{\sum_{\nu \in D_c} |J_\nu(t_\mu)|^2}{\Pi_{\alpha\in\Delta}(e^\alpha(t_\mu)-1)}, \quad \text{by the Weyl character formula} \\ &= \frac{\sum_{\nu \in D_c} |J_\mu(t_\nu)|^2}{\Pi_{\alpha\in\Delta}(e^\alpha(t_\mu)-1)}, \quad \text{by the definition of } J_\mu \\ &= \frac{|T_c|}{\Pi_{\alpha\in\Delta_+}\left(2\sin\frac{\pi}{c+h^\vee}\langle\mu+\rho,\alpha\rangle\right)^2}, \quad \text{by (6) and (7).} \qquad (8)\end{aligned}$$

Combining (3), (5) and (8), we get (1).

By Lemma 4.2.6(a),

$$|T_c| = (c+h^\vee)^\ell |P/Q|\,|Q/Q_{lg}|.$$

This proves (2) and hence the theorem is proved. □

Remarks 4.2.20 (a) The expression for dim $\mathscr{V}_\Sigma(\vec{p},\vec{\lambda})$ as in (1) of Theorem 4.2.19 remains valid for any s-pointed stable curve Σ by Theorem 3.5.9 and Lemma 3.3.3.

(b) The number $|P/Q|$ is called the *index of connection*. It is the order of the fundamental group π_1 of the corresponding adjoint group. Its values are given by (cf. (Bourbaki, 2002, Plates I–IX)):

- $A_\ell (\ell \geq 1) : \ell + 1$
- $B_\ell(\ell \geq 2)$, $C_\ell(\ell \geq 2)$, $E_7 : 2$
- $D_\ell(\ell \geq 4) : 4$
- $E_6 : 3$
- G_2, F_4, $E_8 : 1$

(c) The order $|Q/Q_{lg}|$ of course is 1 for simply-laced $\mathfrak{g}$. This order for non-simply-laced $\mathfrak{g}$ is given as follows:

- $B_\ell(\ell \geq 2) : 2$
- $C_\ell(\ell \geq 2) : 2^{\ell-1}$
- $F_4 : 4$
- $G_2 : 6$

The above values can be read off from Bourbaki (2002, Plates I–IX).

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4.2.E Exercises

(1) Show that for any $\lambda, \mu \in D_c$ such that $(\lambda + \mu)(\theta^\vee) = c + 2$, $[V(\lambda)] \otimes^c [V(\mu)]$ is obtained from $V(\lambda) \otimes V(\mu)$ by removing all the components $V(\nu)$ with $\nu(\theta^\vee) = c + 2$ or $c + 1$ along with those components $V(\nu)$ with $\nu(\theta^\vee) = c$ that intersect $V(\lambda)_{(\lambda(\theta^\vee))} \otimes V(\mu)_{(\mu(\theta^\vee))}$ nontrivially.

Hint: Use Proposition 4.2.3.

(2) Following the notation of Lemma 4.2.14, it is easy to see that

$$\hat{F}_1 = V(\nu + m\theta) \otimes V(\mu), \quad \text{where} \quad m := c + 1 - \nu(\theta^\vee).$$

Show that the differential $\hat{\delta}_1 \colon \hat{F}_1 \to \hat{F}_0 = V(\nu) \otimes V(\mu)$ is the composite map $\eta \circ (j \otimes I)$ given as follows:

$$V(\nu + m\theta) \otimes V(\mu) \xhookrightarrow{j \otimes I} (V(\nu) \otimes V(\theta)^{\otimes m}) \otimes V(\mu)$$

and

$$\eta \colon (V(\nu) \otimes V(\theta)^{\otimes m}) \otimes V(\mu) \to V(\nu) \otimes V(\mu)$$

is given by

$$\begin{aligned} &\eta(v \otimes (x_1 \otimes \cdots \otimes x_m) \otimes w) \\ &\qquad = v \otimes (x_m \ldots x_1 \cdot w), \quad \text{for } v \in V(\nu), x_i \in V(\theta) = \mathfrak{g}, w \in V(\mu). \end{aligned}$$

Hint: Use the definition of $\mathscr{H}(\nu)$ as in Definition 1.2.6 to describe the $\hat{\mathfrak{g}}$-module map $\delta_1 \colon F_1 \to F_0$ as in Definition 4.2.13. Now, use the explicit identification (1) of the proof of Lemma 4.2.14.

(3) Simply using Corollary 4.2.4 (and not using Theorem 4.2.9), show that for any $c > 0$, $R_c(\mathfrak{g})$ under the product $\otimes^c$ is generated by $\{[V(\omega_i)] : \omega_i \in D_c\}$, where $\{\omega_i\}_{1 \le i \le \ell}$ are the fundamental weights.

Hint: Choose an element $H \in Q^\vee$ such that $\alpha_i(H) > 0$ for each simple root α_i. Now, use the induction on $\lambda(H)$ to show that $[V(\lambda)]$ lies in the ring generated by $\{[V(\omega_i)] : \omega_i \in D_c\}$.

Thus, to show that $\otimes^c = \otimes^c_F$ in $R_c(\mathfrak{g})$, it suffices to prove that

$$[V(\lambda)] \otimes^c [V(\omega_i)] = [V(\lambda)] \otimes^c_F [V(\omega_i)], \quad \text{for any } \lambda, \omega_i \in D_c.$$

(4) Show that for any $\mathfrak{g}$ of type A_ℓ, B_ℓ, C_ℓ, D_ℓ or G_2 and any central charge $c > 0$, $\overline{\chi}_{\mathfrak{g}}(\lambda, \omega_i, \nu) = 0$, for any $\lambda, \nu, \omega_i \in D_c$, where $\overline{\chi}_g$ is defined in Proposition 4.2.15.

Thus, in view of the above Exercise 3 and Proposition 4.2.15, this gives an alternative (and much simpler) proof of the equality $\otimes^c = \otimes^c_F$ in $R_c(\mathfrak{g})$ for $\mathfrak{g}$ of any type other that $E_\bullet$ and F_4 (cf. Corollary 4.2.17 for any $\mathfrak{g}$).

Hint: Use some 'partial' determination of $H_*(\hat{\mathfrak{g}}_-, \mathscr{H}(\nu) \otimes V_1(\omega_i))$ for those ω_i such that $\omega_i(\theta^\vee) \leq 2$ by observing that any irreducible $\mathfrak{g}$-submodule of the tensor product $V(\lambda) \otimes V(\mu)$ has highest weight of the form $\lambda + \beta$ for some weight β of $V(\mu)$ and using the following Exercise (7). Further, any fundamental weight ω_i of level 1 cannot belong to Q_{lg}

(5) Show that for any $\mu, \nu \in D_c$, if a $\mathfrak{g}$-module $V(\lambda)$ occurs in $H_0(\hat{\mathfrak{g}}_-, \mathscr{H}(\nu) \otimes V_1(\mu))$, then $\lambda \in D_c$.

(6) For any $\mu, \nu \in D_c$, consider the decomposition as $\mathfrak{g}$-modules (cf. the above Exercise 5):

$$H_0(\hat{\mathfrak{g}}_-, \mathscr{H}(\nu) \otimes V_1(\mu)) \simeq \bigoplus_{\lambda \in D_c} m_\lambda^{\mu,\nu} V(\lambda).$$

Then, assuming the validity of Theorem 4.2.16, show that, for any $p \geq 0$, as $\mathfrak{g}$-modules,

$$H_p\left(\hat{\mathfrak{g}}_-, \mathscr{H}(\nu) \otimes V_1(\mu)\right) \simeq \bigoplus_{\lambda \in D_c} m_\lambda^{\mu,\nu} \Bigg(\bigoplus_{\substack{w \in W'_c \\ \ell(w)=p}} V(w^{-1} * \lambda) \Bigg).$$

Hint: Use the Hochschild–Serre spectral sequence for Lie algebra homology.

For $\mu = 0$, this is a result due to Garland and Lepowsky (1976).

(7) Show that

$$W'_c = \{v\tau_\alpha : v \text{ is the shortest coset representative in } W/W_\alpha \text{ and } \\ \alpha \in (c + h^\vee)Q_{lg} \text{ is anti-dominant weight}\},$$

where $W_\alpha \subset W$ is the stabilizer of α.

(8) Show that for any simply-laced $\mathfrak{g}$ and $c = 1$,

$$\dim \mathscr{V}_\Sigma(p, 0) = |Z(G)|^g,$$

where g is the genus of the smooth irreducible projective curve Σ, p is any point of Σ and $Z(G)$ is the center of simply-connected G with Lie algebra $\mathfrak{g}$.

4.C Comments

We refer to the Bourbaki talk (Sorger, 1994a) for a brief survey of the Verlinde formula and its proof.

The content of Section 4.1 (including Exercises 4.1.E) is largely taken from Beauville (1996) (barring Lemma 4.1.9 which is a personal communication due to Jiuzu Hong). Also, Exercise 4.1.E.3 is well known (see, e.g., (Verlinde, 1988), (Moore and Seiberg, 1989), (Kac, 1990, Exercise 13.34) and (Beauville, 1996, §6.4)). We also refer to Szenes (1995) and Ueno (2008, §5.5) for the contents of this section.

Section 4.2 is largely taken from Faltings (1994, §§5, 6) and Beauville (1996). Proposition 4.2.3 is contained in Tsuchiya, Ueno and Yamada (1989, Example 2.2.8). Corollary 4.2.18 (equivalently Theorem 4.2.9) was conjectured in Faltings (1994, Conjecture 5.1), wherein it was proved for all the classical $\mathfrak{g}$ and $\mathfrak{g}$ of type G_2. The uniform proof (of Theorem 4.2.9) given in Subsection 4.2.18 uses another definition of the fusion product $\otimes_F^c$ given in terms of the $\mathfrak{g}$-equivariant Euler–Poincaré characteristic of certain vector bundles on the infinite Grassmannian $\bar{X}_G$ (cf. Definition 4.2.11) and its coincidence with the usual fusion product $\otimes^c$ defined by (1) of Example 4.2.1. The definition of the fusion product $\otimes_F^c$ (as in Definition 4.2.11) and the result that $\xi_c\colon R(\mathfrak{g}) \to R_c(\mathfrak{g})$ is a ring homomorphism with respect to the fusion product $\otimes_F^c$ (cf. Section 4.2.18), as well as (then) conjectural equality of $\otimes_F^c$ with $\otimes^c$ is due to Kumar (1997b) (apparently the equality of $\otimes_F^c$ with $\otimes^c$ was also conjectured by Bott, as mentioned in Teleman (1995)). Lemma 4.2.14 and Proposition 4.2.15 are also taken from Kumar (1997b). Now, as in Corollary 4.2.17, the equality of $\otimes_F^c$ with $\otimes^c$ follows from a result due to Teleman (1995) (cf. Theorem 4.2.16). In fact, Teleman (1995) determines the Lie algebra cohomology of the pair $(\mathfrak{g}[t^{-1}], \mathfrak{g})$ with coefficients in the tensor product $\mathscr{H}(\lambda_c) \otimes \vec{V}(\vec{\lambda})$ of an integrable highest-weight $\hat{\mathfrak{g}}$-module with finite-dimensional evaluation modules. More generally, Teleman has determined the Lie algebra cohomology of the pair $(\mathfrak{g} \otimes \mathbb{C}[\Sigma], \mathfrak{g})$ with coefficients in $\mathscr{H}(\lambda_c) \otimes \vec{V}(\vec{\lambda})$ and proved its rigidity under nodal deformations of Σ (cf. (Teleman, 1996)).

The Verlinde formula (Theorem 4.2.19) was, in some form, conjectured by Verlinde (1988). **A very significant part of the proof of the formula (in the precise form of Theorem 4.2.19) was done by** Tsuchiya, Ueno and Yamada (1989).

Exercises 4.2.E.1 and 4.2.E.4 are essentially due to Faltings (1994). Exercises 4.2.E.2 and 4.2.E.6 are due to Kumar (1997b) and Exercise 4.2.E.8 is due to Faltings (2009) (also see (Zhu, 2017, Corollary 4.2.5)).

There are several (geometric) proofs of the Verlinde formula for the dimension of the space of generalized theta functions for $G = \mathrm{SL}_2$ (or more generally for GL_2 with fixed determinant) as in Kirwan (1992), Szenes (1993), Bertram (1993), Bertram and Szenes (1993), Narasimhan and Ramadas (1993), Daskalopoulos and Wentworth (1993, 1996), Thaddeus (1994) and Zagier (1995) (and possibly more). Jeffrey and Kirwan (1998) contains a proof of the Verlinde formula for GL_N using Witten's formula (Witten, 1991) for the symplectic volume of the moduli space. For the volume computation of moduli spaces, we refer, in addition, to the papers Pantev (1994), Beauville (1997), Boysal and Vergne (2010), Oprea (2011), Krepski and Meinrenken (2013) and Baldoni, Boysal and Vergne (2015). Alekseev, Meinrenken and Woodward (2000) gave a generalization of the Verlinde formula for some non-simply-connected groups. Bismut and Labourie (1999) gave a symplectic geometry proof of the Verlinde formula for $c \gg 0$ and any G. As proved later in Chapter 8, the space of generalized theta functions is isomorphic with the space of conformal blocks.

Fuchs and Schweigert (1997) gave a proof of the Verlinde formula for genus $g = 0$ using Theorem 4.2.9 but without invoking the Factorization Theorem in $g = 0$ case.

Following the works of Moore and Seiberg (1988, 1989), Huang formulated and proved a generalization of the Verlinde conjecture in the framework of the theory of vertex operator algebras using the results on the duality and modular invariance of genus 0 and 1 correlation functions (cf. (Huang, 2008)). For more general conformal blocks arising from vertex operator algebras, under some natural assumptions, factorization, local freeness and computation of Chern classes has been done in Damiolini, Gibney and Tarasca (2019, 2020).

An explicit residue formula for $\dim \mathscr{V}_\Sigma(p, 0_c)$ for $G = \mathrm{SL}_N$ is given in Szenes (1995). Zagier (1996) contains several number-theoretical and combinatorial aspects of the Verlinde formula for GL_n (especially for $n = 2, 3$). Further, an explicit formula for $\dim \mathscr{V}_\Sigma(p, 0_c)$ for the classical groups can be found in Oxbury and Wilson (1996).

Fakhruddin (2012) gave a formula for the Chern classes of the Verlinde bundle (i.e., bundle of the conformal blocks) over the moduli stack $\bar{\mathcal{M}}_{0,s}$ of stable s-pointed curves of genus 0 as well as the first Chern class of the Verlinde bundle over the moduli stack $\bar{\mathcal{M}}_{g,s}$ of stable s-pointed curves of any genus g (see also (Mukhopadhyay, 2016c)). Marian et al. (2017) extended this work by giving an explicit formula in terms of the tautological classes for the total Chern character of the Verlinde bundle over the moduli stack $\bar{\mathcal{M}}_{g,s}$.