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ON A MEASURE ZERO STABILITY PROBLEM OF A CYCLIC EQUATION

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Abstract

Let G be a commutative group, Y a real Banach space and $f : G \to Y$. We prove the Ulam–Hyers stability theorem for the cyclic functional equation

$$\frac{1}{|H|}\sum_{h\in H} f(x + h\cdot y) = f(x) + f(y)$$

for all $x, y \in \Omega$, where H is a finite cyclic subgroup of $\mathrm{Aut}(G)$ and $\Omega \subset G \times G$ satisfies a certain condition. As a consequence, we consider a measure zero stability problem of the functional equation

$$\frac{1}{N}\sum_{k=1}^{N} f(z + \omega^k\zeta) = f(z) + f(\zeta)$$

for all $(z, \zeta) \in \Omega$, where $f : \mathbb{C} \to Y$, $\omega = e^{2\pi i/N}$ and $\Omega \subset \mathbb{C}^2$ has four-dimensional Lebesgue measure 0.

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1. Introduction

Throughout this paper, let G, X and Y be a commutative group, a real normed space and a real Banach space, respectively, and H be a finite cyclic subgroup of $\mathrm{Aut}(G)$ (the automorphism group of G). Denote the order of H by $|H|$. A function $f : G \to Y$ is said to be a quadratic mapping if f satisfies the equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y) \tag{1.1}$$

for all $x, y \in G$. In [16] Skof proved the Ulam–Hyers stability of the quadratic functional equation (1.1). (See also [8, 9] and [10, pages 175–179] for the history and further results on the Ulam–Hyers stability of functional equations.)

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THEOREM 1.1 [16]. *Let $\delta \geq 0$. Suppose that $f : G \to Y$ satisfies the inequality*

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \delta$$

for all $x, y \in G$. Then there exists a unique quadratic mapping $q : G \to Y$ such that

$$\|f(x) - q(x)\| \leq \tfrac{1}{2}\delta$$

for all $x \in G$.

Generalising the above result, Sibaha *et al.* [14] proved the Ulam–Hyers stability of the functional equation

$$\frac{1}{|H|}\sum_{h\in H} f(x + h\cdot y) = f(x) + f(y) \tag{1.2}$$

for all $x, y \in G$. We call $f : G \to Y$ satisfying (1.2) an *H-cyclic mapping*. (See [5] for more general results.)

THEOREM 1.2 [14]. *Let $\delta \geq 0$. Suppose that $f : G \to Y$ satisfies the inequality*

$$\left\|\frac{1}{|H|}\sum_{h\in H} f(x + h\cdot y) - f(x) - f(y)\right\| \leq \delta \tag{1.3}$$

for all $x, y \in G$. Then there exists a unique H-cyclic mapping $q : G \to \mathbb{C}$ such that

$$\|f(x) - q(x)\| \leq \delta$$

for all $x \in G$.

REMARK 1.3. In particular, if $H = \{I, -I\}$, where $I : G \to G$ is the identity, then Theorem 1.2 implies Theorem 1.1 and, if $H = \{I\}$, Theorem 1.2 implies the well-known Ulam–Hyers stability of the Cauchy functional equation [8].

It is a very interesting subject to consider functional equations or inequalities satisfied on restricted domains or satisfied under restricted conditions [1–4, 6, 11, 15]. Among the results, Jung and Rassias proved the Ulam–Hyers stability of the quadratic functional equations in a restricted domain [9, 13].

THEOREM 1.4. *Let $d > 0$. Suppose that $f : X \to Y$ satisfies the inequality*

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \delta \tag{1.4}$$

for all $x, y \in D := \{(x, y) \in X^2 : \|x\| + \|y\| \geq d\}$. Then there exists a unique quadratic mapping $q : X \to Y$ such that

$$\|f(x) - q(x)\| \leq \tfrac{7}{2}\delta \tag{1.5}$$

for all $x \in X$.

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It is very natural to ask if the restricted domain D in Theorem 1.4 can be replaced by a smaller subset $\Omega \subset D$ (for example a subset of measure 0 if X is a measure space). In [7], the stability of (1.4) is considered in a set $\Omega \subset \{(x, y) \in \mathbb{R}^2 : |x| + |y| \geq d\}$ of measure $m(\Omega) = 0$ when $f : \mathbb{R} \to Y$. As a result, it is proved that if $f : \mathbb{R} \to Y$ satisfies (1.4) for all $(x, y) \in \Omega$, then there exists a unique quadratic mapping $q : \mathbb{R} \to Y$ satisfying (1.5). As a consequence, it is also proved that if f satisfies the asymptotic condition

$$\|f(x + y) + f(x - y) - 2f(x) - 2f(y)\| \to 0$$

as $|x| + |y| \to \infty$ in Ω, then f is a quadratic mapping.

In this paper, generalising the results in [7, 9, 13, 14], we consider the Ulam–Hyers stability of the functional equation (1.2) in restricted domains. Firstly, we assume that $\Omega \subset G \times G$ satisfies the following condition: for given $x, y \in G$ there exists $t \in G$ such that

$$\{(x + h \cdot y,\ t), (x + h \cdot t,\ y), (x, t)\ :\ h \in H\} \subset \Omega. \tag{1.6}$$

As an abstract approach, we first prove that if $f : G \to Y$ satisfies (1.3) for all $(x, y) \in \Omega$, then there exists a unique H-cyclic mapping q such that

$$\|f(x) - q(x)\| \leq 3\delta$$

for all $x \in G$. In particular, if $G = X$ and $d \geq 0$, then $\Omega = \{(x, y) \in X \times X : \|x\| + \|y\| \geq d\}$ satisfies (1.6).

Secondly, when $G = \mathbb{C}$, by constructing a subset $\Omega \subset \mathbb{C}^2$ of measure 0 satisfying (1.6), we consider a measure zero stability problem of (1.2): that is, we consider the inequality

$$\left\| \frac{1}{N} \sum_{k=1}^{N} f(z + \omega^k \zeta) - f(z) - f(\zeta) \right\| \leq \delta$$

for all $(z, \zeta) \in \Omega$, where $\omega = e^{2\pi i/N}$. Finally, we refine the results in [7, 9, 13] and prove that if f satisfies (1.4) for all $(x, y) \in \Omega$, then there exists a unique quadratic mapping $q : \mathbb{R} \to Y$ such that

$$\|f(x) - q(x)\| \leq \tfrac{3}{2}\delta$$

for all $x \in \mathbb{R}$, where $\Omega \subset \mathbb{R}^2$ has two-dimensional Lebesgue measure 0.

2. Stability of the equation in restricted domains

Throughout this section we assume that $\Omega \subset G \times G$ satisfies (1.6). We prove the Ulam–Hyers stability of (1.2) in Ω.

THEOREM 2.1. *Let H be a finite cyclic subgroup of the group of automorphisms of G and $\delta \geq 0$. Suppose that $f : G \to Y$ satisfies the inequality*

$$\left\| \frac{1}{|H|} \sum_{h \in H} f(x + h \cdot y) - f(x) - f(y) \right\| \leq \delta \tag{2.1}$$

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for all $(x, y) \in \Omega$. *Then there exists a unique* H*-cyclic mapping* $q : G \to Y$ *such that*

$$\|f(x) - q(x)\| \le 3\delta$$

for all $x \in G$.

PROOF. Let

$$Q(f)(x, y) = \frac{1}{|H|} \sum_{h \in H} f(x + h \cdot y) - f(x) - f(y).$$

Then

$$\sum_{k \in H} Q(f)(x + k \cdot y, t) = \frac{1}{|H|} \sum_{k \in H} \sum_{h \in H} f(x + k \cdot y + h \cdot t) - \sum_{k \in H} f(x + k \cdot y) - |H| f(t) \quad (2.2)$$

and

$$\sum_{k \in H} Q(f)(x + k \cdot t, y) = \frac{1}{|H|} \sum_{k \in H} \sum_{h \in H} f(x + k \cdot t + h \cdot y) - \sum_{k \in H} f(x + k \cdot t) - |H| f(y). \quad (2.3)$$

From (2.2) and (2.3)

$$\begin{aligned} \sum_{k \in H} Q(f)(x + k \cdot t, y) - \sum_{k \in H} Q(f)(x + k \cdot y, t) \quad (2.4) \\ = \sum_{k \in H} f(x + k \cdot y) + |H| f(t) - \sum_{k \in H} f(x + k \cdot t) - |H| f(y) \\ = |H| Q(f)(x, y) - |H| Q(f)(x, t). \end{aligned}$$

Thus, from (2.4)

$$Q(f)(x, y) = \frac{1}{|H|} \sum_{k \in H} Q(f)(x + k \cdot t, y) - \frac{1}{|H|} \sum_{k \in H} Q(f)(x + k \cdot y, t) + Q(f)(x, t). \quad (2.5)$$

In view of (1.6) and (2.1), for given $x, y \in G$ we can choose $t \in G$ such that

$$\|Q(f)(x + k \cdot t, y)\| \le \delta, \quad \|Q(f)(x + k \cdot y, t)\| \le \delta \quad \text{and} \quad \|Q(f)(x, t)\| \le \delta \quad (2.6)$$

for all $k \in H$. Now, it follows from (2.5) and (2.6) that

$$\begin{aligned} \|Q(f)(x, y)\| \le \frac{1}{|H|} \sum_{k \in H} \|Q(f)(x + k \cdot t, y)\| \\ + \frac{1}{|H|} \sum_{k \in H} \|Q(f)(x + k \cdot y, t)\| + \|Q(f)(x, t)\| \le 3\delta \end{aligned}$$

for all $x, y \in G$. Thus, by Theorem 1.2, there exists a unique H-cyclic mapping $q : G \to Y$ such that

$$\|f(x) - q(x)\| \le 3\delta$$

for all $x \in G$. This completes the proof. □

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Now, let G be a real normed space with norm $\|\cdot\|$ and $\Omega = \{(x, y) : \|x\| + \|y\| \geq d\}$ with $d > 0$. Then Ω satisfies (1.6). Thus, as a direct consequence of Theorem 2.1 we obtain the following corollary.

Corollary 2.2. *Let $d, \delta \geq 0$. Suppose that $f : X \to Y$ satisfies the inequality*

$$\left\| \frac{1}{|H|} \sum_{h \in H} f(x + h \cdot y) - f(x) - f(y) \right\| \leq \delta$$

for all $x, y \in X$ such that $\|x\| + \|y\| \geq d$. Then there exists a unique H-cyclic mapping $q : X \to Y$ such that

$$|f(x) - q(x)| \leq 3\delta$$

for all $x \in X$.

As a consequence of the Corollary 2.2, we obtain the asymptotic behaviour of f satisfying

$$\left\| \frac{1}{|H|} \sum_{h \in H} f(x + h \cdot y) - f(x) - f(y) \right\| \to 0 \tag{2.7}$$

as $\|x\| + \|y\| \to \infty$. We need the following lemma.

Lemma 2.3. *Let $f : X \to Y$ be a bounded H-cyclic mapping. Then $f = 0$.*

Proof. Assume that $\|f(x)\| \leq M$ for all $x \in X$ with $M > 0$. Letting $y = x$ in (1.2) and using the triangle inequality we have

$$\|2f(x)\| = \left\| \frac{1}{|H|} \sum_{h \in H} f(x + h \cdot x) \right\| \leq \frac{1}{|H|} \sum_{h \in H} \|f(x + h \cdot x)\| \leq M$$

for all $x \in X$. Thus, we have $\|f(x)\| \leq \frac{1}{2}M$ for all $x \in X$. Continuing this process we obtain $\|f(x)\| \leq 2^{-n}M$ for all $x \in X$ and $n \in \mathbb{N}$, which implies that $f(x) = 0$ for all $x \in X$. This completes the proof. □

Corollary 2.4. *Suppose that $f : X \to Y$ satisfies (2.7). Then f is an H-cyclic mapping.*

Proof. The condition (2.7) implies that for each $n \in \mathbb{N}$, there exists $d_n > 0$ such that

$$\left\| \frac{1}{|H|} \sum_{h \in H} f(x + h \cdot y) - f(x) - f(y) \right\| \leq \frac{1}{n}$$

for all $(x, y) \in X^2$ such that $\|x\| + \|y\| \geq d_n$. By Corollary 2.2, there exists a unique H-cyclic mapping $q_n : X \to Y$ such that

$$|f(x) - q_n(x)| \leq \frac{3}{n} \tag{2.8}$$

for all $x \in X$. Replacing n by m in (2.8) and using the triangle inequality,

$$|q_m(x) - q_n(x)| \leq \frac{3}{n} + \frac{3}{m} \leq 6 \tag{2.9}$$

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for all $x \in X$. Let $q_{m,n}(x) = q_m(x) - q_n(x)$ for all $x \in X$. Then, by (2.9), $q_{m,n}$ is a bounded H-cyclic mapping. By Lemma 2.3 we have $q_{m,n} = 0$ and hence $q_m = q_n := q$ for all $m, n \in \mathbb{N}$. Letting $n \to \infty$ in (2.8) we have $f(x) = q(x)$ for all $x \in X$. This completes the proof. □

As an interesting example, let $G = \mathbb{C}$, $f : \mathbb{C} \to Y$ and $H = \{\omega^k : k = 1, 2, \ldots, N\}$ for a fixed positive integer N, where $\omega = e^{2\pi i/N}$. Then, as a direct consequence of Theorem 2.1, we obtain the following stability result for the functional equation

$$\frac{1}{N}\sum_{k=1}^{N} p(z + \omega^k\zeta) = p(z) + p(\zeta) \tag{2.10}$$

for all $z, \zeta \in \mathbb{C}$.

Corollary 2.5. *Let $d, \delta \ge 0$. Suppose that $f : \mathbb{C} \to Y$ satisfies the inequality*

$$\left\| \frac{1}{N}\sum_{k=1}^{N} f(z + \omega^k\zeta) - f(z) - f(\zeta) \right\| \le \delta$$

for all $z, \zeta \in \mathbb{C}$ such that $|z| + |\zeta| \ge d$. Then there exists a unique mapping $q : \mathbb{C} \to Y$ satisfying (2.10) such that

$$\|f(z) - p(z)\| \le 3\delta$$

for all $z \in \mathbb{C}$.

Remark 2.6. Let $A_n : \mathbb{C}^n \to Y$ be an n-additive function: that is, for each $1 \le i \le n$,

$$A(z_1, \ldots, z_i + \eta_i, \ldots, z_n) = A(z_1, \ldots, z_i, \ldots, z_n) + A(z_1, \ldots, \eta_i, \ldots, z_n)$$

for all $z_1, \ldots, z_n, \eta_i \in \mathbb{C}$. Then it is easy to see that

$$p(z) = A_n(z, \ldots, z) \tag{2.11}$$

is a solution of (2.10). Is it true that all general solutions of (2.10) are given by (2.11)?

Let $I : G \to G$ be the identity mapping and $H = \{I, -I\}$. Then (1.6) is reduced to the following: for given $x, y \in G$ there exists $t \in G$ such that

$$\{(x + y, t), (x - y, t), (x + t, y), (x - t, y), (x, t)\} \subset \Omega.$$

As a direct consequence of Theorem 2.1 we obtain the following corollary.

Corollary 2.7. *Let $d, \delta \ge 0$. Suppose that $f : G \to Y$ satisfies the inequality*

$$\|f(x + y) + f(x - y) - 2f(x) - 2f(y)\| \le \delta$$

for all $x, y \in \Omega$. Then there exists a unique quadratic mapping $q : G \to Y$ such that

$$\|f(x) - q(x)\| \le \tfrac{3}{2}\delta$$

for all $x \in G$.

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In particular, if G is a real normed space, as a direct consequence of Corollary 2.7, we obtain the following, which refines Theorem 1.4.

COROLLARY 2.8. *Let $d, \delta \geq 0$. Suppose that $f : X \to Y$ satisfies the inequality*

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \delta$$

for all $x, y \in X$ such that $\|x\| + \|y\| \geq d$. Then there exists a unique quadratic mapping $q : X \to Y$ such that

$$\|f(x) - q(x)\| \leq \tfrac{3}{2}\delta$$

for all $x \in X$.

In particular, letting $H = \{I\}$ we obtain the following as a direct consequence of Corollary 2.2.

COROLLARY 2.9. *Let $d, \delta \geq 0$. Suppose that $f : X \to Y$ satisfies the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \delta$$

for all $x, y \in X$ such that $\|x\| + \|y\| \geq d$. Then there exists a unique mapping $a : X \to Y$ satisfying

$$a(x+y) = a(x) + a(y)$$

such that

$$\|f(x) - a(x)\| \leq 3\delta$$

for all $x \in X$.

3. The Ulam–Hyers stability in a set of Lebesgue measure zero

In this section, we consider the functional inequality

$$\left\| \frac{1}{N} \sum_{k=1}^{N} f(z + \omega^k \zeta) - f(z) - f(\zeta) \right\| \leq \delta \tag{3.1}$$

for all $(z, \zeta) \in \Omega$, where $f : \mathbb{C} \to Y$ and $\Omega \subset \mathbb{C}^2$ is of four-dimensional Lebesgue measure zero, and the quadratic functional inequality

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \delta \tag{3.2}$$

for all $(x, y) \in \Omega \subset \mathbb{R}^2$, where $f : \mathbb{R} \to Y$ and Ω is of two-dimensional Lebesgue measure zero, which refines the result in [7].

We first consider (3.1). As we see in the Corollary 2.5, the inequality (3.1) is a particular case of (2.1) when $G = \mathbb{C}$ and $H = \{\omega^k : k = 1, 2, \ldots, N\}$ for a fixed positive integer N, where $\omega = e^{2\pi k i/N}$. Now, (1.6) reduces to the following: for given $z, \zeta \in \mathbb{C}$ there exists $\eta \in \mathbb{C}$ such that

$$\{(z + \omega^k \zeta, \eta), (z + \omega^k \eta, \zeta), (z, \eta)\} \subset \Omega \tag{3.3}$$

for all $k = 1, 2, \ldots, N$. By virtue of Theorem 2.1 it suffices to construct a set $\Omega \subset \mathbb{C}^2$ of measure zero satisfying (3.3).

It is known from [12, Theorem 1.6] that there exists a set $K \subset \mathbb{R}$ of Lebesgue measure 0 such that $\mathbb{R} \setminus K$ is of first Baire category: that is, $\mathbb{R} \setminus K$ is a countable union of nowhere dense subsets of $\mathbb{R}$.

Lemma 3.1 [7, Lemma 2.4]. *Let K be a subset of $\mathbb{R}$ of measure 0 such that $\mathbb{R} \setminus K$ is of first Baire category. Then, for any countable subsets $U \subset \mathbb{R}$, $V \subset \mathbb{R} \setminus \{0\}$ and $M > 0$, there exists $\lambda \geq M$ such that*

$$U + \lambda V = \{u + \lambda v : u \in U, v \in V\} \subset K. \tag{3.4}$$

From now on we identify $\mathbb{C}$ with $\mathbb{R}^2$.

Theorem 3.2. *Let K be the set defined in Lemma 3.1, R be the rotation*

$$R = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} & 0 \\ 0 & \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} & 0 \\ 0 & \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{bmatrix}$$

and $\Omega = R^{-1}(K \times K \times K \times K)$. Then Ω satisfies (3.3) and has four-dimensional Lebesgue measure 0.

Proof. Let $z = x + iy$, $\zeta = u + iv$, $\eta = t + is \in \mathbb{C}$, $k = 1, 2, \ldots, N$, and let

$$\begin{aligned} P_{z,\zeta,\eta,k} = &\Big\{\Big(x + u\cos\frac{2\pi k}{N} - v\sin\frac{2\pi k}{N}, y + u\sin\frac{2\pi k}{N} + v\cos\frac{2\pi k}{N}, t, s\Big)\Big\} \\ &\cup \Big\{\Big(x + t\cos\frac{2\pi k}{N} - s\sin\frac{2\pi k}{N}, y + t\sin\frac{2\pi k}{N} + s\cos\frac{2\pi k}{N}, u, v\Big), (x, y, t, s)\Big\}. \end{aligned}$$

Then Ω satisfies (3.3) if and only if, for every $z = x + iy, \zeta = u + iv \in \mathbb{C}$, there exists $\eta = t + is \in \mathbb{C}$ such that

$$R\Big(\bigcup_{k=1}^{N} P_{z,\zeta,\eta,k}\Big) \subset K \times K \times K \times K. \tag{3.5}$$

The inclusion (3.5) is equivalent to

$$S_{z,\zeta,\eta} := \bigcup_{k=1}^{N} \Big\{\frac{\sqrt{2}}{2}(p_1 \pm p_3), \frac{\sqrt{2}}{2}(p_2 \pm p_4) : (p_1, p_2, p_3, p_4) \in P_{z,\zeta,\eta,k}\Big\} \subset K.$$

Now, we can choose $\alpha \in \mathbb{R}$ ($\alpha \neq 0$) such that

$$\cos\frac{2\pi k}{N} - \alpha\sin\frac{2\pi k}{N} \neq 0, \quad \sin\frac{2\pi k}{N} + \alpha\cos\frac{2\pi k}{N} \neq 0$$

for all $k = 1, 2, \ldots, N$. Then it is easy to check that the set $S_{z,\zeta,t+\alpha ti}$ is contained in the set of form $U + tV$, where

$$U = \bigcup_{k=1}^{N} \left\{ \frac{\sqrt{2}}{2}\left(x + u\cos\frac{2\pi k}{N} - v\sin\frac{2\pi k}{N}\right), \frac{\sqrt{2}}{2}\left(y + u\sin\frac{2\pi k}{N} + v\cos\frac{2\pi k}{N}\right) \right\}$$
$$\cup \left\{ \frac{\sqrt{2}}{2}(x-u), \frac{\sqrt{2}}{2}(y-v), \frac{\sqrt{2}}{2}(x+u), \frac{\sqrt{2}}{2}(y+v), \frac{\sqrt{2}x}{2}, \frac{\sqrt{2}y}{2} \right\},$$
$$V = \bigcup_{k=1}^{N} \left\{ \pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}\alpha}{2}, \frac{\sqrt{2}}{2}\left(\cos\frac{2\pi k}{N} - \alpha\sin\frac{2\pi k}{N}\right), \frac{\sqrt{2}}{2}\left(\sin\frac{2\pi k}{N} + \alpha\cos\frac{2\pi k}{N}\right) \right\}.$$

By (3.4) in Lemma 3.1, for given $z = x + iy, \zeta = u + iv \in \mathbb{C}$ and $M > 0$ there exists $t \geq M$ such that

$$S_{z,\zeta,t+\alpha ti} \subset U + tV \subset K.$$

Thus, Ω satisfies (3.3). This completes the proof. □

Theorem 3.3. *There exists a set $\Omega \subset \mathbb{R}^4$ of Lebesgue measure zero such that if $f : \mathbb{R}^4 \to Y$ satisfies the inequality*

$$\left\| \frac{1}{N}\sum_{k=1}^{N} f(z + \omega^k\zeta) - f(z) - f(\zeta) \right\| \leq \delta$$

for all $(z, \zeta) \in \Omega$, then there exists a unique mapping $q : \mathbb{R}^4 \to Y$ satisfying

$$\frac{1}{N}\sum_{k=1}^{N} q(z + \omega^k\zeta) = q(z) + q(\zeta)$$

for all $z, \zeta \in \mathbb{C}$ such that

$$\|f(z) - q(z)\| \leq 3\delta$$

for all $z \in \mathbb{C}$.

Remark 3.4. It is easy to see that the set $\Omega_d := \{(z, \zeta) \in \Omega : |z| + |\zeta| \geq d\}$ also satisfies (3.3). Thus, the result of Theorem 3.3 holds true when Ω is replaced by Ω_d. Thus, as a consequence of Theorem 3.3 with the above remark, we obtain the strong version of asymptotic behaviour of f satisfying

$$\left\| \frac{1}{N}\sum_{k=1}^{N} f(z + \omega^k\zeta) - f(z) - f(\zeta) \right\| \to 0 \tag{3.6}$$

as $|z| + |\zeta| \to \infty$, $(z, \zeta) \in \Omega$.

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COROLLARY 3.5. *Suppose that* $f : \mathbb{R}^4 \to Y$ *satisfies* (3.6). *Then* f *satisfies the functional equation*

$$\frac{1}{N}\sum_{k=1}^{N} f(z+\omega^k\zeta) = f(z) + f(\zeta)$$

for all $z, \zeta \in \mathbb{C}$.

PROOF. The proof is the same as that of Corollary 2.4. □

Secondly, we consider (3.2). In view of Corollary 2.4, it suffices to construct a set $\Omega \subset \mathbb{R}^2$ of measure zero satisfying (2.10).

THEOREM 3.6. *Let* $\Omega = e^{-\pi i/4}$ *be the rotation of* $K \times K$ *by* $-\pi/4$. *Then* Ω *satisfies* (2.10) *and has two-dimensional Lebesgue measure 0.*

PROOF. The proof is very similar to that of Theorem 3.2. However we give the proof for completeness. Let $x, y, t \in \mathbb{R}$ and let

$$P_{x,y,t} = \{(x+y,t), (x-y,t), (x+t,y), (x-t,y), (x,t)\}.$$

Then Ω satisfies (2.10) if and only if, for every $x, y \in \mathbb{R}$, there exists $t \in \mathbb{R}$ such that

$$e^{\pi i/4} P_{x,y,t} \subset K \times K. \tag{3.7}$$

The inclusion (3.7) is equivalent to

$$S_{x,y,t} := \left\{\frac{1}{\sqrt{2}}(u-v),\ \frac{1}{\sqrt{2}}(u+v) : (u,v) \in P_{x,y,t}\right\} \subset K.$$

It is easy to check that the set $S_{x,y,t}$ is the set of form $U + tV$, where

$$U = \left\{\frac{1}{\sqrt{2}}(x+y),\ \frac{1}{\sqrt{2}}(x-y),\ \frac{1}{\sqrt{2}}x\right\}, \quad V = \left\{\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right\}.$$

By Lemma 3.1, for given $x, y \in \mathbb{R}$ and $M > 0$ there exists $t \geq M$ such that

$$S_{x,y,t} = U + tV \subset K.$$

Thus, Ω satisfies (2.10). This completes the proof. □

By Corollary 2.7 and Theorem 3.5 we have the following (compare with [7, Theorem 2.1]).

THEOREM 3.7. *Let* $d, \delta \geq 0$. *There exists a set* $\Omega \subset \mathbb{R}^2$ *of Lebesgue measure zero such that if* $f : \mathbb{R} \to Y$ *satisfies the inequality*

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \delta$$

only for all $x, y \in \Omega$, *then there exists a unique quadratic mapping* $q : \mathbb{R} \to Y$ *such that*

$$\|f(x) - q(x)\| \leq \tfrac{3}{2}\delta$$

for all $x \in \mathbb{R}$.

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