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ASSOCIATED PRIME IDEALS IN NON-NOETHERIAN RINGS

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The theory of associated prime ideals is one of the most basic notions in the study of modules over commutative Noetherian rings. For modules over non-Noetherian rings however, the classical associated primes are not so useful and in fact do not exist for some modules M. In [**4**] [**22**] a prime ideal P of a ring R is said to be attached to an R-module M if for each finite subset I of P there exists $m \in M$ such that $I \subseteq \text{ann}_R(m) \subseteq P$. In [**4**] the attached primes were compared to the associated primes and the results of [**4**], [**22**], [**23**], [**24**] show that the attached primes are a useful alternative in non-Noetherian rings to associated primes. Several other methods of associating a set $\mathscr{A}(M)$ of prime ideals to a module M over a non-Noetherian ring have proven very useful in the past. The most common of these is the set $\text{Ass}_f(M)$ of weak Bourbaki primes of M [**2**, pp. 289-290]. Another method, which was used by Krull in 1929 [**8**] and later studied by Banaschewski [**1**] and Kuntz [**9**], is the following. Call a prime ideal P a Krull prime of M if for each $a \in P$ there exists an $m \in M$ such that $a \in \text{ann}_R(m) \subseteq P$. In his study of pseudo-Noetherian rings [**14**] [**15**], K. McDowell was led in [**16**, pp. 36-37] to consider the set of attached primes of M, which he called the strong Krull primes of M due to their relationship to the Krull primes. We prefer McDowell's terminology because of this connection with the Krull primes and also because the "attached prime" terminology has previously been used to mean something entirely different. (See for example the papers [**12**, **13**, **25**, **27**, **29**, **30**, **31**].) McDowell developed many of the basic properties of strong Krull primes in [**16**] and in [**17**] where he compared the strong Krull primes to other well-known types of associated primes that have been used in non-Noetherian rings.

After defining and summarizing in Section 1 some of the relationships among seven different notions of associated primes which appear in the literature, we show, in Section 2, that unlike the Nagata primes and weak Bourbaki primes, the Krull primes and strong Krull primes behave well with respect to flat ring extensions. Some further results on the behavior of these primes in polynomial rings are also given. After some remarks in Section 3 on when the strong Krull primes and weak Bourbaki primes are the same, we give in Section 4 a consequence of this behavior for

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seminormality in abelian group rings. In Section 5 we show that the main result of [**23**] can be strengthened by replacing the strong Krull primes by weak Bourbaki primes. In the final section we give a brief discussion of the notion of "associated prime" in general.

The authors were fortunate to have had access to the unpublished manuscript [**17**] and gratefully acknowledge the debt that the present work owes to it.

1. Definitions and notation. We begin by giving the definitions of seven different notions of associated prime ideals that appear in the literature. For this we will need some notation. All rings will be commutative with identity. If M is an R-module we let

$$Z_R(M) = \{r \in R | rm = 0 \text{ for some } m \in M, m \neq 0\}.$$

If S is a multiplicative subset of R, $M(S)$ denotes

$$\{m \in M | sm = 0 \text{ for some } s \in S\}.$$

If $S = R - P$ where P is a prime ideal of R we write $M(P)$ instead of $M(S)$. If $N \subseteq M$, $\text{ann}_R(N)$ denotes the annihilator of N $(= \{x \in R | xN = 0\})$. We will write $\text{ann}_R(m)$ instead of $\text{ann}_R(\{m\})$ if $m \in M$.

Definition. Let P be a prime ideal of R, and M an R-module.

(a) If $P = \text{ann}_R(m)$ for some $m \in M$ we call P a *Bourbaki prime of* M. We let Ass(M) denote the set of Bourbaki primes of M. See for instance [**2**, Chapter IV] for a discussion of these primes.

(b) If $P \in \text{Ass}(M/M(P))$ then P is called a *Noether prime of* M. The set of these primes will be denoted Ne(M). They are discussed in [**10**].

(c) P is called a *Zariski-Samuel prime of* M if $\text{ann}_R(m)$ is P-primary for some $m \in M$. These primes will be denoted $Z-S(M)$ and are discussed in [**33**].

(d) If P is minimal over $\text{ann}_R(m)$ for some $m \in M$, then P is called a *weak Bourbaki prime of* M. We will denote the set of weak Bourbaki primes of M by $\text{Ass}_f(M)$. Many of the basic properties of these primes are found in [**2**, Chapter IV, Section 1, exercises 17-19] and [**11**].

(e) P is called a *strong Krull prime of* M if for every finitely generated ideal I contained in P there exists an $m \in M$ such that $I \subseteq \text{ann}(m) \subseteq P$. We will let sK($M$) denote the set of strong Krull primes of M. They are discussed in [**17**] and also in [**4**] [**22**] [**23**] [**24**] where they are called attached primes.

(f) P is called a *Krull prime of* M if for each $p \in P$ there exists $m \in M$ such that $p \in \text{ann}(m) \subseteq P$. We will denote the set of these primes by $K(M)$. See [**1**], [**9**] for the basic properties of Krull primes.

(g) If there exists a multiplicative subset S of R such that $S^{-1}P$ is maximal in $Z_{S^{-1}R}(S^{-1}M)$, then P is called a *Nagata prime of* M. Many of the basic properties of these primes can be found in [**20**] [**33**]. We will denote the set of Nagata primes of M by $N(M)$.

For an R-module M we have

$$\mathrm{Ass}(M) \subseteq Z-S(M) \subseteq \mathrm{Ass}_f(M) \subseteq \mathrm{sK}(M) \subseteq K(M) \subseteq N(M)$$

and

$$\mathrm{Ass}(M) \subseteq \mathrm{Ne}(M) \subseteq \mathrm{Ass}_f(M).$$

Further, all of these containments can be proper [**33**] [**9**] [**10**] [**17**], and there are no inclusions relating $\mathrm{Ne}(M)$ and $Z-S(M)$. The equality $Z-S(M) = N(M)$ holds if the zero submodule of M has a primary decomposition [**20**], and if R is Noetherian then $\mathrm{Ass}(M) = N(M)$. Also, it can happen that $\mathrm{Ass}(M)$, $\mathrm{Ne}(M)$, and $Z-S(M)$ are empty for $M \neq 0$, whereas clearly

$$\mathrm{Ass}_f(M) = \emptyset \Leftrightarrow M = 0.$$

In [**9**] the Krull primes of M were studied only for cyclic modules. However, most of these results extend easily to arbitrary R-modules M. For example, if M is an R-module and P is a prime ideal, the following statements are equivalent (see [**9**, Proposition 1]):

(a) $P \in K(M)$,

(b) $P \subseteq Z_R(M/M(P))$,

(c) $P = Z_R(M/M(P))$.

Also, if S is a multiplicative subset of R it follows that

$$K_{S^{-1}R}(S^{-1}M) = \{S^{-1}P | P \in K_R(M), P \cap S = \phi\}.$$

the corresponding result also holds for $\mathrm{Ass}_f(M)$ [**2**] and for $\mathrm{sK}(M)$ [**4**] [**17**].

2. Associated primes and ring extensions. In this section $\phi:A \to B$ will be a ring homomorphism and

$${}^a\phi:\mathrm{Spec}(B) \to \mathrm{Spec}(A)$$

will denote the induced map. If M is a B-module the A-module obtained from M by restriction of scalars will be denoted ${}_\phi M$. If M is an A-module, then $M \otimes_A B$ is a B-module and when we consider it as an A-module we will write ${}_\phi(M \otimes_A B)$.

In [**11**] Lazard showed that if M is a B-module then

$$\mathrm{Ass}_f({}_\phi M) \subseteq {}^a\phi(\mathrm{Ass}_f(M))$$

and that equality holds if ϕ is flat. Thus if M is an A-module and ϕ is flat then

$$\mathrm{Ass}_f({}_\phi(M \otimes_A B) = {}^a\phi(\mathrm{Ass}_f(M \otimes_A B)).$$

Heinzer and Ohm [**5**, Example 4.4] gave an example of a flat A-algebra $\phi:A \to B$ and an A-module M having a prime

$$P \in {}^a\phi(\mathrm{Ass}_f(M \otimes_A B))$$

with $P \notin \mathrm{Ass}_f(M)$, i.e., weak Bourbaki primes of $M \otimes_A B$ do not necessarily contract to weak Bourbaki primes of M. More recently Heitmann has given an example showing that Nagata primes of $M \otimes_A B$ do not always contract to Nagata primes of M [**6**] (again where $\phi:A \to B$ is flat). To date the only positive result obtained without further restrictions on ϕ is [**5**, Proposition 4.5] which states that if $\phi:A \to B$ is flat, M is a cyclic A-module, and $P \in \mathrm{Ass}_f(M \otimes_A B)$, then ${}^a\phi(P) \in N(M)$. In this section we strengthen this by showing that if ϕ is flat and M an A-module, then strong Krull primes of $M \otimes_A B$ contract to strong Krull primes of M and Krull primes of $M \otimes_A B$ contract to Krull primes of M. Some consequences of this will be considered later. We also give a couple of results on the behavior of associated primes in polynomial extensions.

(2.1) PROPOSITION. *If M is a B-module, then*

$$\mathrm{sK}({}_\phi M) = {}^a\phi(\mathrm{sK}(M)).$$

Proof. Let $Q \in \mathrm{sK}(M)$, $P = {}^a\phi(Q)$ and let I be a finitely generated ideal of A contained in P. Then $IB \subseteq PB \subseteq Q$ so there exists $m \in M$ such that $IB \subseteq \mathrm{ann}_B(m) \subseteq Q$.

Thus

$$I \subseteq \phi^{-1}(\mathrm{ann}_B(m)) = \mathrm{ann}_A(m) \subseteq P \quad \text{and} \quad P \in \mathrm{sK}({}_\phi M).$$

For the opposite inclusion let $P \in \mathrm{sK}({}_\phi M)$. Since strong Krull primes localize well, we can assume that A is quasi-local with maximal ideal P. Then $PB \neq B$, for otherwise we would have

$$1 = \sum_{i=1}^{n} a_i b_i \quad \text{with } a_i \in P \text{ and } b_i \in B.$$

But then since $P \in \mathrm{sK}({}_\phi M)$ we would have $m \in M$ such that

$$\{a_1, \dots, a_n\} \subseteq \mathrm{ann}_A(m) \subseteq P \quad \text{and}$$

$$m = 1m = \sum_{i=1}^{n} a_i b_i m = 0$$

which is impossible. Let $Q \in \mathrm{Spec}(B)$ be minimal over PB. By localizing at 0 we may assume that (B, Q) is quasi-local and Q = radical of PB. Then if $I \neq 0$ is a finitely generated ideal of B contained in Q there exists an integer $n \geqq 1$ and a finitely generated ideal J of A contained in P with $I^n \subseteq JB \subseteq PB$. Since $P \in \mathrm{sK}({}_\phi M)$ there exists $m \in M$ such that $J \subseteq \mathrm{ann}_A (m) \subseteq P$. Then

$$I^n \subseteq JB \subseteq \mathrm{ann}_A (m)B \subseteq \mathrm{ann}_B (m) \subseteq Q.$$

Choose n such that $I^n m = 0$ but $I^{n-1}m \neq 0$ (where $I^o = A$). Then if $x \in I^{n-1}$ with $xm \neq 0$ we get

$$I \subseteq \mathrm{ann}_B (xm) \subseteq Q$$

and hence $Q \in \mathrm{sK}(M)$.

Remark. The above proof actually shows that any prime which is minimal over PB is in $\mathrm{sK}(M)$.

The next theorem was given by McDowell for the case that A is coherent and M is a finitely presented A-module. An application will be given in Section 4.

(2.2) THEOREM. *If $\phi:A \rightarrow B$ is flat and M is an A-module, then*

$${}^a\phi(\mathrm{sK}(M \otimes_A B)) \subseteq \mathrm{sK}(M)$$

and equality holds if ϕ is faithfully flat.

Proof. Let $Q \in \mathrm{sK}(M \otimes_A B)$ and let $P = {}^a\phi(Q)$. Since strong Krull primes behave perfectly under localization we may pass to $A_P \rightarrow B_Q$ and thus to prove the first statement we may reduce to the case that (A, P) and (B, Q) are quasi-local and ϕ is faithfully flat. Let $I \subseteq P$ be a finitely generated ideal of A. Since $Q \in \mathrm{sK}(M \otimes_A B)$ we have

$$0 \neq \mathrm{Hom}_B (B/IB, M \otimes_A B) \cong \mathrm{Hom}_A (A/I, M) \otimes_A B;$$

the isomorphism from [**2**, p. 23, Proposition 11]. Thus

$$\mathrm{Hom}_A (A/I, M) \neq 0$$

since B is faithfully flat. For the last statement recall that if ϕ is faithfully flat, the map

$$M \rightarrow {}_\phi(M \otimes_A B)$$

is injective [**2**, p. 32, Proposition 8]. Thus

$$\mathrm{sK}(M) \subseteq \mathrm{sK}({}_\phi(M \otimes_A B)) = {}^a\phi(\mathrm{sK}(M \otimes_A B)),$$

the last equality by Proposition 2.1. Thus

$$\mathrm{sK}(M) = {}^a\phi(\mathrm{sK}(M \otimes_A B)).$$

We next consider the behavior of Krull primes under change of rings. A simple modification of the first part of the proof of Proposition 2.1 yields the following:

(2.3) PROPOSITION. *Let $\phi:A \to B$ be a ring homomorphism and M a B-module. Then*

$$^a\phi(K(M)) \subseteq K({}_\phi M).$$

(2.4) THEOREM. *If $\phi:A \to B$ is a flat A-algebra and M is an A-module, then*

$$^a\phi(K(M \otimes_A B)) \subseteq K(M).$$

Proof. Let $Q \in K(M \otimes_A B)$ and let $P = {}^a\phi(Q)$. Since Krull primes behave perfectly with respect to localization we may by passing to $A_P \to B_Q$ reduce to the case that (A, P) and (B, Q) are quasi-local and ϕ is faithfully flat. In this case we must show $P \subseteq Z_A(M)$. But if $x \in P$ the exact sequence

$$0 \to \operatorname{ann}_M(x) \to M \xrightarrow{x} M$$

gives

$$0 \to \operatorname{ann}_M(x) \otimes_A B \to M \otimes_A B \xrightarrow{x\otimes 1} M \otimes_A B.$$

Thus we get

$$\operatorname{ann}_M(x) \otimes_A B = \operatorname{ann}_{M\otimes B}(x \otimes 1) = \operatorname{ann}_{M\otimes B}(\phi(x)) \neq 0,$$

and hence $\operatorname{ann}_M(x) \neq 0$.

We conclude this section with a couple of results on associated primes in polynomial rings. The first shows a further relationship between Krull primes and strong Krull primes. The second will be useful in Section 5.

(2.5) THEOREM. *Let M be an R-module and X an indeterminate. Then*

$$\{PR[X] \mid P \in \mathrm{sK}(M)\} = \mathrm{sK}(M \otimes_R R[X]) = K(M \otimes_R R[X]).$$

Proof. We first show

$$\{PR[X] \mid P \in \mathrm{sK}(M)\} \subseteq \mathrm{sK}(M \otimes R[X]).$$

Let $f_1, \dots, f_n \in PR[X]$, $P \in \mathrm{sK}(M)$. Then there exists $m \in M$, such that m annihilates the coefficients of the f_i and $(0:_R m) \subseteq P$. Let $\phi:M \to M \otimes_R R[X]$ be the map $x \to x \otimes 1$. Then

$$\{f_1, \dots, f_n\} \subseteq (0:_{R[X]}\phi(m)) \subseteq PR[X].$$

Thus $PR[X] \in \mathrm{sK}(M \otimes_R R[X])$.

That $\mathrm{sK}(M \otimes_R R[X]) \subseteq K(M \otimes_R R[X])$ is clear. To show that

$$K(M \otimes_R R[X]) \subseteq \{PR[X] \mid P \in \mathrm{sK}(M)\}$$

we first show that if $Q \in K(M \otimes R[X])$ then Q is an extended prime; $Q = PR[X]$ where $P = Q \cap R$. For this we may assume $R = R_P$. If $Q \neq$

$PR[X]$, then $Q = (P, f)R[X]$ where $f \in R[X]$ is monic and irreducible modulo $PR[X]$. But then

$$f \in Q \in K(M \otimes_R R[X])$$

implies that f is a zero divisor on $M[X]$ and this is impossible since f is monic. Thus Q is an extended prime.

It remains only to show that if $PR[X] \in K(M \otimes_R R[X])$ then $P \in \mathrm{sK}(M)$, and for this we may again assume that $R = R_P$. Let $a_0, a_1, \ldots, a_n \in P$. To show that there exists $m \in M$ such that $a_0, \ldots, a_n \in (0:_R m) \subseteq P$ let

$$f = a_0 + a_1X + \ldots + a_nX^n.$$

Then $f \in PR[X]$ so there exists

$$g = m_0 + m_1X + \ldots + m_KX^k \in M[X] \cong M \otimes_R R[X]$$

such that

$$f \in (0:_{R[X]}g) \subseteq PR[X].$$

By [**21**, Theorem 1] there exists an integer j such that

$$c(f)^{j+1}c(g) = c(f)^jc(fg) = 0$$

where $c(h)$ denotes the R-submodule generated by the coefficients of h. If j is chosen minimal such that $c(f)^{j+1}c(g) = 0$, then for any non-zero $x \in c(f)^jc(g)$ we have

$$(a_0, \ldots, a_n) = c(f) \subseteq (0:_R x) \subseteq P.$$

Thus $P \in \mathrm{sK}(M)$.

If I is an ideal of R we will let

$$\sqrt{I} = \{a \in R | a^n \in I \text{ for some } n \geqq 1\}.$$

(2.6) THEOREM. *Let M be an R-module and X an indeterminate. Then*

$$\mathrm{Ass}_f(M \otimes_R R[X]) = \{PR[X] | P \in \mathrm{Ass}_f(M)\}.$$

Proof. Let $Q \in \mathrm{Ass}_f(M \otimes_R R[X])$ and $P = Q \cap R$. Since $Q \in \mathrm{sK}(M \otimes R[X])$, then $Q = PR[X]$ by Theorem 2.5. To show that $P \in \mathrm{Ass}_f(M)$ we may assume that $R = R_P$. Assume that $PR[X] = Q$ is minimal over $(0:_{R[X]}g)$ where

$$g = m_0 + m_1X + \ldots + m_nX^n \in M[X] \cong M \otimes R[X], \quad m_i \in M.$$

First we show that $P \subseteq \sqrt{(0:_R c(g))}$. Let $a \in P$. Then $a \in PR[X]$ implies there exists $k \in R[X] - PR[X]$ such that

$$ka^n \in (0:_{R[X]}g) \quad \text{for some } n \geqq 1.$$

Thus $a^n kg = 0$ and since $c(k) = R$ we get

$$0 = c(a^n kg) = a^n c(g).$$

Thus $P \subseteq \sqrt{0 :_R c(g)}$. But then

$$P \subseteq \sqrt{(0 :_R c(g))} = \cap \sqrt{(0 :_R m_i)} \subseteq P,$$

and thus

$$P = \sqrt{(0 :_R m_i)} \quad \text{for some } i.$$

It remains to show that if $P \in \mathrm{Ass}_f(M)$, then

$$PR[X] \in \mathrm{Ass}_f(M \otimes_R R[X]).$$

Again it suffices to consider the case $R = R_P$. Then

$$P \in \mathrm{Ass}_f(M) \Rightarrow P = \sqrt{(0 :_R m)} \quad \text{for some } m \in M.$$

Claim. $PR[X] = \sqrt{0 :_{R[X]} m}$. Let $f \in PR[X]$. Since $c(f) \subseteq \sqrt{(0 :_R m)}$ we can choose $n \geqq 1$ such that $c(f)^n m = 0$. Then

$$c(f^n m) = c(f^n)m \subseteq c(f)^n m = 0$$

and hence

$$f \in \sqrt{(0 :_{R[X]} m)}.$$

Conversely, assume $f \notin PR[X]$. Then $c(f^n) = R$ for all $n > 0$. But then if $f^n m = 0$ we get

$$0 = c(f^n m) = c(f^n)m = m,$$

a contradiction. Thus

$$PR[X] = \sqrt{0 :_{R[X]} m}.$$

3. Ass_f **and** sK. Since for many purposes it is desirable to have $P \in \mathrm{Ass}_f(M)$ rather than just $P \in \mathrm{sK}(M)$ (e.g. as in the proof of Theorem 4.2) and since $\mathrm{sK}(M)$ is in some ways better behaved than $\mathrm{Ass}_f(M)$, it would be of interest to know when $\mathrm{Ass}_f(M) = \mathrm{sK}(M)$. Lacking a useful ideal-theoretical characterization of the rings R with the property that $\mathrm{Ass}_f(M) = \mathrm{sK}(M)$ for every R-module M we consider the following obviously sufficient condition for this to hold:

(MFG) Each prime ideal of R is minimal over a finitely generated ideal of R.

Perhaps the most obvious examples of non-Noetherian rings with this property are the rings with Noetherian spectrum. Recall that this means that R has the following equivalent properties [**5**]:

(3.1) (i) R has the ascending chain condition on radical ideals.

(ii) For each ideal A there exists a finitely generated ideal A_0 with $\sqrt{A} = \sqrt{A_0}$.

(iii) For each prime ideal P there exists a finitely generated ideal A such that $P = \sqrt{A}$.

(iv) R has the ascending chain condition on prime ideals and every ideal of R has finitely many minimal prime divisors:

We make a few simple observations on the rings satisfying MFG.

(3.2) (i) *A von-Neumann regular ring R satisfies* MFG, *but has Noetherian spectrum only if it is Noetherian.*

Proof. Let m be a non-finitely generated maximal ideal of R. Then m is minimal over the zero ideal, but is not the radical of any finitely generated ideal since any finitely generated ideal of R is contained in a principal maximal ideal.

(ii) *If R satisfies* MFG, *then R has the ascending chain condition on prime ideals. The converse holds if* Spec(R) *is a tree* (*i.e., if for each* $Q \in \operatorname{Spec}(R)$, $\{P \in \operatorname{Spec}(R) | P \subseteq Q\}$ *is a chain*).

Proof. Let $P_1 \subseteq P_2 \subseteq \ldots$ be a chain of prime ideals of R. Then $Q = \cup_{i=1}^{\infty} P_i$ is a prime ideal so Q is minimal over a finitely generated ideal I of R. But since I is finitely generated and $I \subseteq \cup_{i=1}^{\infty} P_i$, we must have $I \subseteq P_n$ for some n, and thus $P_n = P_{n+1} = \ldots = Q$ by the minimality of Q over I.

If $P \in \operatorname{Spec}(R)$ and $\operatorname{Spec}(R)$ is a tree, then the ascending chain condition on primes implies that P is not the union of primes properly contained in P. This clearly implies that P is minimal over a principal ideal of R.

(iii) *If P is a prime ideal of R and S is a multiplicative subset of R with* $S \cap P = \emptyset$, *then P is minimal over a finitely generated ideal if and only if* $S^{-1}P$ *is.*

(3.3) THEOREM. *Let* $\phi:R \to B$ *be a ring homomorphism. If R and all of the fibers* $B \otimes_R k(P)$, $P \in \operatorname{Spec}(R)$, *satisfy* MFG, *then B satisfies* MFG.

Proof. Let $Q \in \operatorname{Spec}(B)$ and let $P = Q \cap R$. Then P is minimal over some finitely generated ideal I of R since R satisfies MFG. Let $S = R - P$. Since

$$B \otimes_R k(P) = S^{-1}B/S^{-1}(PB)$$

satisfies MFG there exists a finitely generated ideal $J \subseteq Q$ such that $(S^{-1}Q)^*$ is minimal over $(S^{-1}J)^*$ where the asterisk denotes the image in $S^{-1}B/S^{-1}(PB)$. But then Q is minimal over $I + J$, for if $Q_1 \in \operatorname{Spec}(B)$ is such that $I + J \subseteq Q_1 \subseteq Q$, then

$$P_1 = Q_1 \cap R \subseteq P \Rightarrow P_1 = P.$$

Thus

$$(S^{-1}J)^* \subseteq (S^{-1}Q_1)^* \subseteq (S^{-1}Q)^*$$

and hence

$$S^{-1}Q_1 = S^{-1}Q \Rightarrow Q_1 = Q.$$

(3.4) COROLLARY. *Let R be a ring with the property* MFG, *and let $R \to B$ be a ring homomorphism. Then B has* MFG *in the following cases*:

(a) *B is a localization of a finitely generated R-algebra.*

(b) *B is an* INC *extension of R* [**3**], *i.e., no two comparable primes of B contract to the same prime of R.*

4. An application to seminormality. In this section we give an application of some of the previous results to an area not directly related to associated primes (or to the MFG condition). Recall that a ring R is said to be reduced if it has no nonzero nilpotents.

(4.1) *Definition.* A reduced ring R is *seminormal* if every rank 1 projective $R[X]$-module is of the form $M \otimes_R R[X]$ for some rank 1 projective R-module M. This is equivalent to the condition: if $a, b \in R$ with $a^2 = b^3$ then there exists $c \in R$ such that $a = c^3$, $b = c^2$ [**32**].

(4.2) THEOREM. *Let R be a reduced ring with Noetherian spectrum, and let π be a finite abelian group of order n. Then $R\pi$ is seminormal if and only if R is seminormal, n is regular on R, and nR, $n\bar{R}$ are radical ideals, where $\bar{R}$ is the integral closure of R.*

Proof. In [**28**, Proposition 2.4] one direction has been shown, namely the one assuming that $R\pi$ is seminormal. For the converse (see [**28**, Theorem 2.1]) the assumption that nR have no embedded primes was needed. By a remark following Proposition 2.4 in [**28**], this assumption can be removed if we know that weak-Bourbaki primes contract to weak Bourbaki primes under $R \to R\pi$. But this follows from Theorem 2.2 since R satisfies the MFG condition.

5. A result of Northcott. An important application of associated prime ideals is to the study of grade. Let Gr $(I; M)$ denote the polynomial grade of I on M [7]. This is also called the true grade in [**22**] [**23**]. The intimate connection of polynomial grade to Ass_f and sK can be inferred from the following well-known result [7] [**11**].

(5.1) *Remark.* The following are equivalent for an ideal I and an R-module M.

(i) Gr $(I, M) = 0$.

(ii) Each finitely generated ideal $J \subseteq I$ is contained in a member of $\mathrm{Ass}_f(M)$.

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(iii) Each finitely generated ideal $J \subseteq I$ is contained in a member of $\mathrm{sK}(M)$.

The connection between grade and strong Krull primes is further accented by the main result of [**23**] which states:

(5.2) THEOREM. *Let A be an R-ideal of positive (true) grade. Then A is projective if and only if*

(i) *A has a resolution of finite length by finitely generated projective modules, and*

(ii) *every $P \in \mathrm{sK}(R/A)$ has (true) grade one.*

In this section we show that this theorem remains valid if $\mathrm{sK}(R/A)$ is replaced by $\mathrm{Ass}_f(R/A)$. Since it is the if direction of the above theorem that requires the work, then this replacement can be regarded as a strengthening of Northcott's result. Our argument will procede along the same lines as Northcott's. However we can use Theorem 2.6 to reduce to the case that A contains a regular element. Thus one only needs the MacRae invariant for the case considered in [**26**], [**34**], or [**22**], where it was shown that for each torsion R-module M having a resolution of finite length by finitely generated projective R-modules, there is a smallest invertible ideal $G(M)$ containing the initial Fitting invariant $F(M)$ of M. Further, given a short exact sequence of such R-modules $0 \to M' \to M \to M'' \to 0$ one has

$$G(M) = G(M')G(M'').$$

We will need the following lemma which extends [**23**, p. 213, Lemma 4] by replacing $\mathrm{sK}(R/A)$ by $\mathrm{Ass}_f(R/A)$ and R by an arbitrary R-module M. The proof is essentially the same as that given in [**23**].

(5.3) LEMMA. *Let $A \neq R$ be an ideal and M an R-module. Then*

$$\mathrm{Gr}\,(A;\, M) = \inf\,\{\mathrm{Gr}\,(P;\, M) | P \in \mathrm{Ass}_f(R/A)\}.$$

Proof. By [**21**, p. 152, Theorem 16], there exists a prime ideal Q of R such that

$$A \subseteq Q \quad \text{and} \quad \mathrm{Gr}\,(A;\, M) = \mathrm{Gr}\,(Q;\, M).$$

Let P be a minimal prime ideal of A with $P \subseteq Q$. Then

$$P \in \mathrm{Ass}_f(R/A) \quad \text{and} \quad \mathrm{Gr}\,(A;\, M) = \mathrm{Gr}\,(P;\, M).$$

Note. The above proof actually shows that

$$\mathrm{Gr}\,(A;\, M) = \inf\,\{\mathrm{Gr}\,(P;\, M) | P \text{ is minimal over } a\}.$$

(5.4) THEOREM. *Let A be an ideal of R with* $\mathrm{Gr}\,(A;\, R) > 0$. *Then A is projective if (and only if) the following conditions hold:*

(i) $\mathrm{Gr}\,(P;\, R) = 1$ *for all* $P \in \mathrm{Ass}_f(R/A)$,

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(ii) *A has a resolution of finite length by finitely generated projective R-modules.*

Proof. By adjoining an indeterminate we may assume that there exists a regular element $a \in A$. Then A is contained in some $P \in \mathrm{Ass}_f(R/aR)$. From [**26**, Lemma 2.5] we have $Pd_{R_P}(R_P/A_P) = 1$ and thus

$$A_P = F(R_P/A_P) = G(R_P/A_P) = G(R/A)R_P \quad [\mathbf{26}].$$

Therefore $G(R/A) \neq R$. Now to show that A is projective it suffices to show that $A = G(R/A)$. If $A \neq G(R/A)$ let $E = G(R/A)/A$. Then E is a torsion R-module having a resolution of finite length by finitely generated projective R-modules, and from the exact sequence

$$0 \to E \to R/A \to R/G(R/A) \to 0$$

we get

$$G(R/A) = G(E)G(R/A).$$

Therefore $G(E) = R$. By [**23**, Theorem 8] there exists a minimal prime P of Ann (E) with

$$\mathrm{Gr}\ (P, R) = \mathrm{Gr}\ (PR_p, R_p).$$

But then

$$P \in \mathrm{Ass}_f(E) \subseteq \mathrm{Ass}_f(R/A)$$

and thus we get

$$Pd_{R_P}(E_P) = Pd_{R_P}(E_P) + \mathrm{Gr}\ (PR_P;\ E_P) = \mathrm{Gr}_{R_P}\ (PR_P;\ R_P) = 1,$$

the second equality coming from [**22**, Theorem 2, p. 176] and the third by hypothesis. Thus $G(E_P) = F(E_P)$, [**22**], [**26**], and $F(E_P) = \mathrm{Ann}\ (E_P)$ since E_P is cyclic. Thus $G(E_P) \subseteq PR_P$, and this contradicts the previous conclusion that $G(E) = R$ since $G(E)R_P = G(E_P)$ [**26**].

6. Associated prime systems. In this section we add some remarks on associated prime systems in general which we hope will give some perspective to the various possible notions of associated prime ideal. For this discussion an axiomatic description of associated primes is convenient. Let R be a ring and let $\mathscr{A}$ be a mapping which associates to each R-module M a subset $\mathscr{A}(M)$ of Spec(R). (We will write $\mathscr{A}_R(M)$ if it is necessary to emphasize the ring R.) Examination of the notions of associated primes that have been used in the past shows the particular significance of the following properties for such a function:

(A1) $\cup\, \mathscr{A}(M) \subseteq Z_R(M)$.

(A2) If $0 \to M' \to M \to M'' \to 0$ is an exact sequence of R-modules, then

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$$\mathscr{A}(M') \subseteq \mathscr{A}(M) \subseteq \mathscr{A}(M') \cup \mathscr{A}(M'').$$

(A3) If S is a multiplicative subset of R and $P \in \mathscr{A}(M)$ with $P \cap S = \emptyset$, then $P \in \mathscr{A}(M/M(S))$.

(A4) If S is a multiplicative subset of R and $P \in \mathscr{A}(M/M(S))$, then $P \in \mathscr{A}(M)$.

(A5) $\mathscr{A}(M) = \emptyset$ if and only if $M = 0$.

Definition. If the function $\mathscr{A}$ satisfies the above properties we will call $\mathscr{A}$ an *associated prime system* for R. A function $\mathscr{A}$ which satisfies the first three of the above properties will be called a *formal associated prime system.*

Axiomatic definitions of associated primes have previously been considered by Merker [**18**] [**19**] and McDowell [**17**]. McDowell's notion of associating system is equivalent to our formal associated prime system. However, we feel that our definition is simpler and more natural since it is given in terms of a single ring R, whereas McDowell's definition requires the simultaneous consideration of all rings of quotients of R. It is well-known (e.g. [**2**, Chapter IV]) that Ass is a formal associated prime system and that Ass_f is an associated prime system. It is also straight-forward that Ne and $Z-S$ are formal associated prime systems and that sK is an associated prime system.

The following result shows that sK and Ass_f are the largest and smallest associated prime systems respectively. The first statement is due to McDowell [**17**]. We repeat the proof in our slightly different setting.

(6.1) THEOREM. *If $\mathscr{A}$ is a formal associated prime system for R and M is an R-module, then $\mathscr{A}(M) \subseteq$ sK(M). Each member of $\mathscr{A}(M)$ contains a member of $\mathrm{Ass}_f(M)$ and if $\mathscr{A}$ is an associated prime system, then $\mathrm{Ass}_f(M) \subseteq \mathscr{A}(M)$.*

Proof. Let $P \in \mathscr{A}(M)$. First assume that $P = Z_R(M)$. Let I be a finitely generated ideal contained in P. We want to show that there exists a non-zero $m \in M$ such that $I \subseteq \operatorname{ann}(m) \subseteq P$. We will proceed by induction on the number of generators n of I. For $n = 1$ this is clear so assume that it is true for n and let $I = (x_1, \ldots, x_{n+1})$, $I' = (x_1, \ldots, x_n)$. Let

$$S = \{x_{n+1}^k | k > 0\}.$$

Then

$$x_{n+1} \notin Z(M/M(S)) \quad \text{and} \quad P \notin \mathscr{A}(M/M(S)).$$

By (A2) $P \in \mathscr{A}(M(S))$ and $P = Z_R(M(S))$. By the induction hypothesis there is a non-zero $m \in M(S)$ such that $I' \subseteq \operatorname{ann}(m)$. Since $m \in M(S)$ there exists k such that $x_{n+1}^k m = 0$. Choose k minimal, so that $x_{n+1}^{k-1} m \neq 0$. Then

$$I \subseteq \operatorname{ann}(x_{n+1}^{k-1}m) \subseteq P \quad \text{and} \quad P \in \mathrm{sK}(M).$$

For the general case assume that $P \in \mathscr{A}(M)$ and let $S = R - P$. Then $P \cap S = \emptyset$ and by (A3)

$$P \in \mathscr{A}(M/M(P)) \quad \text{and} \quad P = Z_R(M/M(P)).$$

By the first part

$$P \in \mathrm{sK}(M/M(P)).$$

Therefore $P \in \mathrm{sK}(M)$ since strong Krull primes satisfy (A4).

For the second statement let $P \in \mathscr{A}(M)$. Then since

$$\mathscr{A}(M) \subseteq \mathrm{sK}(M) \subseteq \operatorname{Supp}(M),$$

then P contains a minimal member Q of Supp (M). But then $Q \in \mathrm{Ass}_f(M)$.

For the final statement let $P \in \mathrm{Ass}_f(M)$, say P is minimal over the annihilator of $Rm = N$, $m \in M$. Then

$$\operatorname{Supp}(N/N(P)) = \{P\}$$

and so from A5 we must have $P \in \mathscr{A}(N/N(P))$. But then

$$P \in \mathscr{A}(N) \subseteq \mathscr{A}(M)$$

by A4 and A2.

The following methods were suggested by [**16**] and [**19**] respectively for obtaining other associated prime systems. (1) If $\mathscr{A}$ is an associated prime system for R and M is an R-module let $\mathscr{A}'(M)$ consist of those primes of R which are directed unions of elements of $\mathscr{A}(M)$. Is it known that the inclusions

$$\mathrm{Ass}_f(M) \subseteq \mathscr{A}'(M) \subseteq \mathrm{sK}(M)$$

may be strict where $\mathscr{A} = \mathrm{Ass}_f$ [**17**]. (2) Let

$$\mathscr{A}''(M) = \{P \cap R \mid P \in \mathrm{Ass}_f(M \otimes_R B) \text{ for some flat } R\text{-algebra } B\}.$$

It would be interesting to know if $\mathscr{A}'' = sK$. The next theorem shows how some of the standard results on Ass_f extend immediately to arbitrary associated prime systems.

(6.2) THEOREM. *Let $\mathscr{A}$ be an associated prime system for R and let M be an R-module. Then*

(i) $Z_R(M) = \cup \mathscr{A}(M)$

(ii) $\cap \mathscr{A}(M) = \{a \in R \mid$ *for each* $m \in M$ *there exists* $n > 0$ *such that* $a^n m = 0\}$ (= *the set of locally nilpotent elements on* M).

(iii) *If S is a multiplicative subset of R and*

$$X = \{P \in \mathscr{A}(M) \mid P \cap S = \emptyset\},$$

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then

$$\mathscr{A}(M/M(S)) = X \quad \text{and} \quad \mathscr{A}(M(S)) = \mathscr{A}(M) - X.$$

Proof. (i) follows since

$$Z_R(M) = \cup \operatorname{Ass}_f(M) \subseteq \cup \mathscr{A}(M) \subseteq Z_R(M).$$

To prove (ii) let I be the ideal of locally nilpotent elements on M. Then we have

$$\cap \mathscr{A}(M) \subseteq \cap \operatorname{Ass}_f(M) = I \quad [\mathbf{2}, \text{ p. } 289].$$

Further, if $a \notin P$ for some $P \in \mathscr{A}(M)$ let

$$S = \{a^n | n > 0\}.$$

Then $P \cap S = \emptyset$ so $P \in \mathscr{A}(M/M(S))$ by A3. Thus $M \neq M(S)$ and therefore a is not locally nilpotent on M. Thus $\cap \mathscr{A}(M) = I$.

To prove (iii) let $P \in \mathscr{A}(M/M(S))$. Then $P \in \mathscr{A}(M)$ by A4. Thus $P \in X$. Let $P \in \mathscr{A}(M(S))$. If $P \cap S = \emptyset$ we get the contradiction

$$P \in \mathscr{A}(M(S)/M(S)(S)) = \mathscr{A}(0) = \emptyset.$$

Thus $P \cap S \neq \emptyset$ and $P \in \mathscr{A}(M) - X$. The result now follows from A2.

We conclude this section with a brief discussion of the behavior of associated prime systems under localization. The reader will have noticed the absence of any need to localize thus far in this section since all that was needed was already built in. In fact an associated prime system on R determines in a natural way an associated prime system on each localization of R as follows:

Let S be a multiplicative subset of R and let $\phi: R \to S^{-1}R$ be the canonical map. If $\mathscr{A}_R$ is a formal associated prime system for R and M is an $S^{-1}R$-module define

$$\mathscr{A}_{S^{-1}R}(M) = \{S^{-1}P | P \in \mathscr{A}_R({}_\phi M)\}.$$

PROPOSITION. (a) *If $\mathscr{A}_R$ is a formal associated prime system for R, the induced map $\mathscr{A}_{S^{-1}R}$ is a formal associated prime system for $S^{-1}R$ and for each R-module M we have*

$$\mathscr{A}_{S^{-1}R}(S^{-1}M) \supseteq \{S^{-1}P | P \in \mathscr{A}_R(M) \text{ and } P \cap S = \emptyset\}.$$

(b) *If $\mathscr{A}_R$ is an associated prime system for R, then $\mathscr{A}_{S^{-1}R}$ is an associated prime system for $S^{-1}R$ and for each R-module M we have*

$$\mathscr{A}_{S^{-1}R}(S^{-1}M) = \{S^{-1}P | P \in \mathscr{A}_R(M) \text{ and } P \cap S = \emptyset\}.$$

Proof. Let $\mathscr{A}_R$ be a formal associated prime system for R. That the induced map $\mathscr{A}_{S^{-1}R}$ satisfies A1 and A2 is immediate. To show that $\mathscr{A}_{S^{-1}R}$ satisfies A3 we will use the following simple result.

LEMMA. *If T is a multiplicative subset of $S^{-1}R$ and*

$$T_1 = \{r \in R | r/s \in T \text{ for some } s \in S\},$$

then

$$(ST_1)^{-1}R = T^{-1}(S^{-1}R),$$

and for each $S^{-1}R$-module

$$_{\phi}[M(T)] = {}_{\phi}M(ST_1).$$

Now to show that A3 holds let T be a multiplicative subset of $S^{-1}R$ and let T_1 be as in the lemma. If $S^{-1}P \in \mathscr{A}_{S^{-1}R}(M)$, $P \in \text{Spec}(R)$, with $S^{-1}P \cap T = \emptyset$, then $P \cap (ST_1) = \emptyset$. Thus

$$P \in \mathscr{A}_R({}_{\phi}M/{}_{\phi}M(ST_1))$$

since $P \in \mathscr{A}_R({}_{\phi}M)$ and $\mathscr{A}_R$ satisfies A3. Thus

$$P \in \mathscr{A}_R({}_{\phi}M/{}_{\phi}[M(T)]) = \mathscr{A}_R({}_{\phi}[M/M(T)])$$

and therefore

$$S^{-1}P \in \mathscr{A}_{S^{-1}R}(M/M(T)).$$

For the last statement of part (a) let $P \in \mathscr{A}_R(M)$ with $P \cap S = \emptyset$. Then by A3

$$P \in \mathscr{A}_R(M/M(S))$$

and since $M/M(S) \subset S^{-1}M$ we get

$$P \in \mathscr{A}_R({}_{\phi}(S^{-1}M)).$$

Thus

$$S^{-1}P \in \mathscr{A}_{S^{-1}R}(S^{-1}M).$$

The arguments for part (b) are similar.

REFERENCES

1. B. Banaschewki, *On the components of ideals in commutative rings,* Arch. Math *12* (1961), 22-29.
2. N. Bourbaki, *Commutative algebra* (Addison Wesley, Reading, Mass., 1972).
3. D. Dobbs, *Lying-over pairs of commutative rings,* Can. J. Math. *33* (1981), 454-475.
4. P. Dutton, *Prime ideals attached to a module,* Quart. J. Math. Oxford *29* (1978), 403-413.
5. W. Heinzer, and J. Ohm, *Locally Noetherian commutative rings*, Trans. Amer. Math. Soc. *158* (1971), 273-284.
6. R. Heitmann, *Prime divisors and flat extensions,* J. of Alg. *74* (1982), 293-301.
7. M. Hochster, *Grade sensitive modules and perfect modules,* Proc. London Math. Soc. *29*(1979), 91-100.

8. W. Krull, *Idealtherie in ringen ohne Endlichkeitsbedingung*, Math. Ann. *101* (1929), 729-744.
9. R. A. Kuntz, *Associated prime divisors in the sense of Krull*, Can. J. Math. *24* (1972), 808-818.
10. ——— *Associated prime divisors in the sense of Noether*, Bol: Soc. Math. Mexicana *18* (1973), 82-88.
11. D. Lazard, *Antour de la platitude*, Bull. Soc. Math. France *97* (1969), 81-128.
12. I. G. MacDonald, *Secondary representation of modules over a commutative ring*, Symp. Math. *11* (1973), 23-43.
13. I. G. MacDonald and R. Sharp, *An elementary proof of the non-vanishing of certain local cohomology modules*, Quart. J. Math., Oxford *23* (1972), 197-204.
14. K. McDowell, *A codimension theorem for pseudo-Noetherian rings*, Trans. Amer. Math. Soc. *214* (1975), 179-185.
15. ——— *Pseudo-Noetherian rings*, Can. Math. Bull. *19* (1976), 77-84.
16. ——— *Commutative coherent rings*, Mathematical Report No. 66, Department of Mathematics, McMaster University, Hamilton, Ontario (1974).
17. ——— *Associated prime divisors and finitely presented modules*, unpublished manuscript, Wilfrid Laurier University, Waterloo, Ontario (1975).
18. J. Merker, *Etude axiomatic des ideaux premiers associes*, C. R. Acad. Sc. Paris *269* (1969), 52-54.
19. ——— *Etude axiomatic des ideaux premiers associes*, C.R. Acad. Sc. Paris *269* (1969), 117-119.
20. M. Nagata, *Local rings* (Robert E. Krieger, New York, New York, 1975).
21. D. G. Northcott, *A generalization of a theorem on the content of polynomials*, Proc. Cambridge Philos. Soc. *55* (1959), 282-288.
22. ——— *Finite free resolutions* (Cambridge University Press, Cambridge, 1976).
23. ——— *Projective ideals and MacRae's invariant*, J. London Math. Soc. *24* (1981), 211-226.
24. ——— *Remarks on the theory of attached prime ideals*, Quart. J. Math. Oxford *33* (1982), 239-245.
25. A. Ooishi, *Matlis duality and the width of a module*, Hiroshima Math. J. *6* (1976), 573-587.
26. D. E. Rush, *The G-function of MacRae*, Can. J. Math. *26* (1974), 854-865.
27. ——— *Big Cohen Macauley modules*, Ill. J. of Math. *24* (1980), 606-611.
28. ——— *Picard groups in abelian groups rings*, J. of Pure and Applied Alg. *26* (1982).
29. R. Y. Sharp, *Some results on the vanishing of local cohomology modules*, Proc. London Math. Soc. *30* (1975), 177-195.
30. ——— *Secondary representations for injective modules over commutative Noetherian rings*, Proc. Edinburgh Math. Soc. *20* (1976), 143-151.
31. ——— *On the attached prime ideals of certain Artinian local cohomology modules*, Proc. Edinburgh Math. Soc. *24* (1981), 9-14.
32. R. G. Swan, *On seminormality*, J. of Alg. *67* (1980), 210-229.
33. D. H. Underwood, *On some uniqueness questions in primary representations of ideals*, Kyoto Math. J. *35* (1969), 69-94.
34. W. V. Vasconcelos, *Divisor theory in module categories*, North Holland Mathematical Studies no. *14* (North-Holland, Amsterdam, 1974).

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