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ON A PROBLEM OF TURÁN ABOUT POLYNOMIALS III

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Let

$$T_n(x) := \cos n\theta,$$

$$U_n(x) := \frac{\sin (n+1)\theta}{\sin \theta},$$

where $x := \cos \theta$, denote the nth degree Chebyshev polynomials of the first and second kind, respectively. Further, let

$$Q_n(x) := \frac{1}{\sqrt{2}} \frac{\cos \{(2n+1)\theta/2\}}{\cos (\theta/2)},$$

$$R_n(x) := \frac{1}{\sqrt{2}} \frac{\sin \{(2n+1)\theta/2\}}{\cos (\theta/2)},$$

$$x := \cos \theta.$$

Given non-negative integers λ and μ we define

$$\nu(n) := n - \left(\left[\frac{\lambda+1}{2}\right] + \left[\frac{\mu+1}{2}\right]\right) + 1$$

and

$$(1) \quad P_{n,\lambda,\mu}(x) := \begin{cases} (1-x)^{\lambda/2}(1+x)^{\mu/2}T_{\nu(n)-1}(x) & \text{if } \lambda \text{ and } \mu \text{ are both even} \\ (1-x)^{(\lambda+1)/2}(1+x)^{(\mu+1)/2}U_{\nu(n)-1}(x) & \text{if } \lambda \text{ and } \mu \text{ are both odd} \\ (1-x)^{\lambda/2}(1+x)^{(\mu+1)/2}Q_{\nu(n)-1}(x) & \text{if } \lambda \text{ is even and } \mu \text{ is odd} \\ (1-x)^{(\lambda+1)/2}(1+x)^{\mu/2}R_{\nu(n)-1}(x) & \text{if } \lambda \text{ is odd and } \mu \text{ is even.} \end{cases}$$

Let

$$(2) \quad x_1 \leqq x_2 \leqq \ldots \leqq x_{\nu(n)}$$

be the roots of the equation

$$(3) \quad 1 - \frac{P^2_{n,\lambda,\mu}(x)}{(1-x)^{\lambda}(1+x)^{\mu}} = 0,$$

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and put

$$F(x):=(1-x)^{[(\lambda+1)/2]}(1+x)^{[(\mu+1)/2]}\prod_{l=1}^{\nu(n)}(x-x_l).$$

Now set

$$(4)\qquad F_l(x):=F(x)/(x-x_l)$$

and denote by

$$\xi_1\leqq\xi_2\leqq\ldots\leqq\xi_{n-j},\ \eta_1\leqq\eta_2\leqq\ldots\leqq\eta_{n-j}$$

the roots of

$$F_{\nu(n)}^{(j)}(x)=0,\quad F_1^{(j)}(x)=0,$$

respectively. We have proved [**4**] that if $p_n(x)$ is a polynomial of degree n such that

$$(5)\qquad |p_n(x)|\leqq(1-x)^{\lambda/2}(1+x)^{\mu/2}\quad\text{for } -1<x<1,$$

then

$$(6)\qquad |p_n^{(j)}(z)|\leqq|P_{n,\lambda,\mu}^{(j)}(z)|$$

for all real values of z lying outside the interval (ξ_1,η_{n-j}).

Here we shall show (see Theorem 1′) that (6) holds everywhere outside the open disk D^0 with (ξ_1,η_{n-j}) as diameter, and that too under a weaker assumption. The idea of such an extension was suggested by a result of Erdös [**2**, Theorem 7].

The proof of our main result depends on the following

LEMMA. *Let*

$$-1=y_0<y_1<y_2<\ldots<y_N=1$$

and set

$$\omega(x):=(1+x)^{n_1}(1-x)^{n_2}\prod_{m=0}^{N}(x-y_m)$$

where n_1, n_2 are non-negative integers. Further, let

$$\omega_m(x):=\omega(x)/(x-y_m),\quad m=0,1,2,\ldots,N.$$

If we put $n:=N+n_1+n_2$ and denote by

$$\alpha_{m,1}\leqq\alpha_{m,2}\leqq\ldots\leqq\alpha_{m,n-j},\quad m=0,1,2,\ldots,N$$

the zeros of $\omega_m^{(j)}(x)$, then for all z lying outside the closed disk $\bar{D}$ with $[\alpha_{N,1},\alpha_{0,n-j}]$ as diameter, the angle between any two of the vectors $\omega_m^{(j)}(z)$ is less than $\pi/2$.

Proof. First we show that if $0 < h < k \leqq N$, then the zeros of $\omega_h^{(j)}(x)$ and $\omega_k^{(j)}(x)$ interlace. To be precise

(7) $\alpha_{k,1} \leqq \alpha_{h,1} \leqq \alpha_{k,2} \leqq \alpha_{h,2} \leqq \ldots \leqq \alpha_{k,n-j} \leqq \alpha_{h,n-j}$.

If we set

$$\omega_{h,k}(x) := \omega_h(x)/(x - y_k) = \omega_k(x)/(x - y_h)$$

then by Leibnitz's rule

(8) $\omega_h^{(j)}(x) = (x - y_k)\omega_{h,k}^{(j)}(x) + j\omega_{h,k}^{(j-1)}(x).$

Thus, if β is a zero of $\omega_{h,k}^{(j)}(x)$, then

(9) $\omega_h^{(j)}(\beta) = j\omega_{h,k}^{(j-1)}(\beta).$

It follows from Rolle's theorem that the zeros of $\omega_h^{(j)}(x)$, $\omega_k^{(j)}(x)$ and $\omega_{h,k}^{(j)}(x)$ lying in $(-1, 1)$ must all be simple.

Now we distinguish three different cases.

Case (i). $0 < h < k < N$. If $\beta_1 < \beta_2 < \ldots < \beta_Q$ are the zeros of $\omega_{h,k}^{(j)}(x)$ in $(-1, 1)$ then from (9) it follows that in each of the intervals $(\beta_1, \beta_2), (\beta_2, \beta_3), \ldots, (\beta_{Q-1}, \beta_Q)$ there is at least one zero of $\omega_h^{(j)}(x)$. Further, for sufficiently small and positive values of ϵ

$$\left.\begin{array}{l}\operatorname{sgn} \omega_h^{(j)}(-1 + \epsilon) = (-1)^{n-n_1-1} \\ \operatorname{sgn} \omega_h^{(j)}(\beta_1) = \operatorname{sgn} \omega_{h,k}^{(j-1)}(\beta_1) = (-1)^{n-n_1-2}\end{array}\right\} \text{ if } j \leqq n_1 + 1,$$

whereas

$$\left.\begin{array}{l}\operatorname{sgn} \omega_h^{(j)}(-1 + \epsilon) = (-1)^{n-j} \\ \operatorname{sgn} \omega_h^{(j)}(\beta_1) = \operatorname{sgn} \omega_{h,k}^{(j-1)}(\beta_1) = (-1)^{n-j-1}\end{array}\right\} \text{ if } j > n_1 + 1.$$

Hence $\omega_h^{(j)}(x)$ must also have a zero in $(-1, \beta_1)$. Again, for sufficiently small and positive values of ϵ

$$\left.\begin{array}{l}\operatorname{sgn} \omega_h^{(j)}(1 - \epsilon) = (-1)^{n_2+1-j} \\ \operatorname{sgn} \omega_h^{(j)}(\beta_Q) = \operatorname{sgn} \omega_{h,k}^{(j-1)}(\beta_Q) = (-1)^{n_2+2-j}\end{array}\right\} \text{ if } j \leqq n_2 + 1,$$

whereas

$$\left.\begin{array}{l}\operatorname{sgn} \omega_h^{(j)}(1 - \epsilon) = +1 \\ \operatorname{sgn} \omega_h^{(j)}(\beta_Q) = \operatorname{sgn} \omega_{h,k}^{(j-1)}(\beta_Q) = -1\end{array}\right\} \text{ if } j > n_2 + 1,$$

and so $\omega_h^{(j)}(x)$ must have a zero in $(\beta_Q, 1)$ as well.

Since $\omega_h^{(j)}(x)$ has exactly $Q + 1$ zeros in $(-1, 1)$ it must have one and only one zero in each of the intervals $(-1, \beta_1), (\beta_1, \beta_2), \ldots, (\beta_{Q-1}, \beta_Q), (\beta_Q, 1)$. Thus, if $\alpha_1 < \alpha_2 < \ldots < \alpha_Q < \alpha_{Q+1}$ be the zeros of $\omega_h^{(j)}(x)$ in $(-1, 1)$, then

(10) $\alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \ldots < \alpha_Q < \beta_Q < \alpha_{Q+1}$.

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From (8) and

(8′) $\omega_k^{(j)}(x) = (x - y_h)\omega_{h,k}^{(j)}(x) + j\omega_{h,k}^{(j-1)}(x)$

it follows that

(11) $\omega_k^{(j)}(\alpha_q) = (y_k - y_h)\omega_{h,k}^{(j)}(\alpha_q), \quad q = 1, 2, \ldots, Q + 1.$

Hence, in view of (10), the sign of $\omega_k^{(j)}(\alpha_q)$ alternates as q increases from 1 to $Q + 1$. Consequently, $\omega_k^{(j)}(x)$ must vanish at least once in each of the intervals

(12) $(\alpha_1, \alpha_2), (\alpha_2, \alpha_3), \ldots, (\alpha_Q, \alpha_{Q+1})$.

Further, for sufficiently small and positive values of ϵ

$$\left.\begin{array}{l}\operatorname{sgn} \omega_k^{(j)}(-1 + \epsilon) = (-1)^{n-n_1-1} \\ \operatorname{sgn} \omega_k^{(j)}(\alpha_1) = \operatorname{sgn} \omega_{h,k}^{(j)}(\alpha_1) = (-1)^{n-n_1-2}\end{array}\right\} \text{if } j \leqq n_1 + 1,$$

whereas

$$\left.\begin{array}{l}\operatorname{sgn} \omega_k^{(j)}(-1 + \epsilon) = (-1)^{n-j} \\ \operatorname{sgn} \omega_k^{(j)}(\alpha_1) = \operatorname{sgn} \omega_{h,k}^{(j)}(\alpha_1) = (-1)^{n-j-1}\end{array}\right\} \text{if } j > n_1 + 1.$$

Hence $\omega_k^{(j)}(x)$ must have a zero in $(-1, \alpha_1)$ as well. Thus, if $\gamma_1 < \gamma_2 < \ldots < \gamma_{Q+1}$ are the zeros of $\omega_k^{(j)}(x)$ in $(-1, 1)$, then

$$\gamma_1 < \alpha_1 < \gamma_2 < \alpha_2 < \ldots < \gamma_Q < \alpha_Q < \gamma_{Q+1} < \alpha_{Q+1}.$$

At the point -1 the polynomials $\omega_h^{(j)}(x)$ and $\omega_k^{(j)}(x)$ have a zero of the same mulitplicity $m_1 \geqq 0$, where

$$m_1 := \begin{cases} n_1 + 1 - j & \text{if } j < n_1 + 1 \\ 0 & \text{if } j \geqq n_1 + 1. \end{cases}$$

Similarly at $+1$, the polynomials $\omega_h^{(j)}(x)$ and $\omega_k^{(j)}(x)$ have a zero of the same multiplicity $m_2 \geqq 0$, where

$$m_2 := \begin{cases} n_2 + 1 - j & \text{if } j < n_2 + 1 \\ 0 & \text{if } j \geqq n_2 + 1. \end{cases}$$

With this we see that (7) does hold in the case $0 < h < k < N$.

Case (ii). $0 = h < k < N$. The above proof with very little modification shows that if $\omega_{h,k}^{(j)}(x)$ has Q zeros $\beta_1 < \beta_2 < \ldots < \beta_Q$ in $(-1, 1)$, then $\omega_h^{(j)}(x)$ must have $Q + 1$ zeros $\alpha_1 < \alpha_2 < \ldots < \alpha_{Q+1}$ in $(-1, 1)$ such that (10) holds. Again, $\omega_k^{(j)}(x)$ must vanish at least once in each of the intervals (12). Besides, it must have a zero of multiplicity $m_1 + 1$ at -1 if $\omega_h^{(j)}(x)$ has a zero of multiplicity m_1 ($\geqq 1$) there. But if $\omega_h^{(j)}(-1) \neq 0$ then $\omega_k^{(j)}(x)$ must have a zero in $[-1, \alpha_1)$. This follows from the fact that

$$\operatorname{sgn} \omega_k^{(j)}(-\infty) = (-1)^{n-j},$$
$$\operatorname{sgn} \omega_k^{(j)}(\alpha_1) = \operatorname{sgn} \omega_{h,k}^{(j)}(\alpha_1) = (-1)^{n-j-1}.$$

At the point $+1$ the polynomials $\omega_h^{(j)}(x)$ and $\omega_k^{(j)}(x)$ have a zero of the same multiplicity $m_2 \geqq 0$. These observations show that (7) holds in this case also.

Case (iii). $0 = h < k = N$. Let $\beta_1 < \beta_2 < \ldots < \beta_Q$ be the zeros of $\omega_{h,k}^{(j)}(x)$ in $(-1, 1)$. As before it can be shown that $\omega_h^{(j)}(x)$ must vanish at least once in each of the intervals $(\beta_1, \beta_2), (\beta_2, \beta_3), \ldots, (\beta_{Q-1}, \beta_Q)$ as well as in $(-1, \beta_1)$.

Now let $j \leqq n_2$. Then at $+1$ the polynomials $\omega_h^{(j)}(x)$, $\omega_{h,k}^{(j)}(x)$ have a zero of multiplicities $n_2 + 1 - j$, $n_2 - j \geqq 0$ respectively, whereas at -1 they have a zero of the same multiplicity m_1 $(\geqq 0)$. Hence $\omega_h^{(j)}(x)$ has precisely Q zeros $\alpha_1 < \alpha_2 < \ldots < \alpha_Q$ in $(-1, 1)$ which satisfy

$$(13) \quad \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \ldots < \alpha_Q < \beta_Q.$$

In each of the intervals $(\alpha_1, \alpha_2), (\alpha_2, \alpha_3), \ldots, (\alpha_{Q-1}, \alpha_Q)$ the polynomial $\omega_k^{(j)}(x)$ must have at least one zero. Besides, it has a zero of multiplicity $m_1 + 1$ at -1 if $\omega_h^{(j)}(x)$ has a zero of multiplicity $m_1 \geqq 1$ there. But if $\omega_h^{(j)}(-1) \neq 0$ then $\omega_k^{(j)}(x)$ must have a zero in $[-1, \alpha_1)$ since

$$\operatorname{sgn} \omega_k^{(j)}(-\infty) = (-1)^{n-j},$$

$$\operatorname{sgn} \omega_k^{(j)}(\alpha_1) = \operatorname{sgn} \omega_{h,k}^{(j)}(\alpha_1) = (-1)^{n-j-1}.$$

Further, in view of (13) we have

$$\operatorname{sgn} \omega_k^{(j)}(\alpha_Q) = \operatorname{sgn} \omega_{h,k}^{(j)}(\alpha_Q) = (-1)^{n_2-j+1},$$

$$\operatorname{sgn} \omega_k^{(j)}(1 - \epsilon) = (-1)^{n_2-j} \text{ if } \epsilon > 0 \text{ is small,}$$

and so $\omega_k^{(j)}(x)$ has at least one zero in $(\alpha_Q, 1)$ as well. Thus we see that (7) holds if $j \leqq n_2$.

If $j \geqq n_2 + 1$ then $\omega_h^{(j)}(x)$ must have a zero in $(\beta_Q, 1)$ since

$$\operatorname{sgn} \omega_h^{(j)}(\beta_Q) = \operatorname{sgn} \omega_{h,k}^{(j-1)}(\beta_Q) = -1,$$

$$\operatorname{sgn} \omega_h^{(j)}(1 - \epsilon) = +1 \text{ if } \epsilon > 0 \text{ is small.}$$

Hence $\omega_h^{(j)}(x)$ has $Q + 1$ zeros $\alpha_1 < \alpha_2 < \ldots < \alpha_{Q+1}$ in $(-1, 1)$ such that (10) holds.

In each of the intervals (12), $\omega_k^{(j)}(x)$ must have at least one zero. At -1, it has a zero of multiplicity $m_1 + 1$ if $\omega_h^{(j)}(x)$ has a zero of multiplicity $m_1 \geqq 1$ there, whereas if $\omega_h^{(j)}(-1) \neq 0$ then it $(\omega_k^{(j)}(x))$ must have a zero in $[-1, \alpha_1)$. Hence again (7) holds.

Having established (7) we are ready to proceed with the proof of the lemma.

Now consider any two of the vectors $\omega_m^{(j)}(z)$, say $\omega_h^{(j)}(z)$ and $\omega_k^{(j)}(z)$ where $z \in \mathbf{C}\backslash\bar{D}$. Without loss of generality we may assume $h < k$ so that (7) holds. If $\operatorname{Im} z \geqq 0$ and the values of "arg" are all taken between 0 and

π, then

$$0 \leqq \arg \{\omega_h^{(j)}(z)/\omega_k^{(j)}(z)\}$$
$$= \arg \{(z - \alpha_{h,n-j})/(z - \alpha_{k,1})\}$$
$$- \sum_{\mu=2}^{n-j} \arg \{(z - \alpha_{k,\mu})/(z - \alpha_{h,\mu-1})\} < \pi/2$$

since

$$0 \leqq \sum_{\mu=2}^{n-j} \arg \{(z - \alpha_{k,\mu})/(z - \alpha_{h,\mu-1})\}$$
$$\leqq \arg \{(z - \alpha_{h,n-j})/(z - \alpha_{k,1})\} < \frac{\pi}{2}.$$

Hence the lemma holds in the case when $\operatorname{Im} z \geqq 0$. The proof is similar if $\operatorname{Im} z < 0$.

THEOREM 1. *Let*

$$-1 = y_0 < y_1 < y_2 < \ldots < y_N = 1,$$

and suppose that $P_n(z)$ is a polynomial of degree $n: = N + n_1 + n_2$ having the following properties:

(i) *it has zeros of multiplicities n_1 and n_2 at y_0 and y_N respectively, where either or both of the numbers n_1 and n_2 may be zero,*

(ii) *the polynomial*

$$\hat{P}_n(z): = P_n(z)/\{(1 + z)^{n_1}(1 - z)^{n_2}\}$$

has alternating signs at the points $y_0, y_1, y_2, \ldots, y_N$.

Further, let $\omega(x)$, $\omega_m(x)$ and $\alpha_{m,\mu}$ be as in the lemma.

Now, if $p(z)$ is a polynomial of degree n with real coefficients having zeros of multiplicities n_1^* $(\geqq n_1)$, $n_2^*(\geqq n_2)$ at -1, $+1$ respectively, and

(14) $\quad |p(y_m)| \leqq |P_n(y_m)|, \; m = 0, 1, 2, \ldots, N,$

then for z lying outside the open disk Δ^0 with $(\alpha_{N,1}, \alpha_{0,n-j})$ as diameter, we have

(15) $\quad |p^{(j)}(z)| \leqq |P_n^{(j)}(z)|.$

Proof. Let

$$\hat{p}(z): = p(z)/\{(1 + z)^{n_1}(1 - z)^{n_2}\},$$
$$\Omega(z): = \omega(z)/\{(1 + z)^{n_1}(1 - z)^{n_2}\}.$$

By Lagrange's interpolation formula

$$\hat{p}(z) = \sum_{m=0}^{N} \frac{\hat{p}(y_m)}{\Omega'(y_m)} \frac{\Omega(z)}{z - y_m}$$

and so

$$(16)\quad p(z) = \sum_{m=0}^{N} \frac{\hat{p}(y_m)}{\Omega'(y_m)}\, \omega_m(z).$$

Clearly

$$\Omega'(y_m) = (-1)^{N-m}|\Omega'(y_m)|,\ m = 0, 1, 2, \dots, N.$$

Hence, differentiating the two sides of (16) j times, we obtain

$$(17)\quad p^{(j)}(z) = (-1)^N \sum_{m=0}^{N} \frac{(-1)^m \hat{p}(y_m)}{|\Omega'(y_m)|}\, \omega_m{}^{(j)}(z).$$

In particular,

$$(17')\quad |P_n{}^{(j)}(z)| = \left| \sum_{m=0}^{N} \frac{|\hat{P}_n(y_m)|}{|\Omega'(y_m)|}\, \omega_m{}^{(j)}(z) \right|$$

since by hypothesis, the numbers

$$(-1)^m \hat{P}_n(y_m),\ m = 0, 1, 2, \dots, N$$

are all of the same sign.

If z lies outside the closed disk $\bar{\Delta}$ with $[\alpha_{N,1}, \alpha_{0,n-j}]$ as diameter then, according to the lemma, the angle between any two of the vectors $\omega_m{}^{(j)}(z)$ is less than $\pi/2$, and so

$$|p^{(j)}(z)| = \left| \sum_{m=0}^{N} \frac{(-1)^m \hat{p}(y_m)}{|\Omega'(y_m)|}\, \omega_m{}^{(j)}(z) \right| \leqq \left| \sum_{m=0}^{N} \frac{|\hat{p}(y_m)|}{|\Omega'(y_m)|}\, \omega_m^{(j)}(z) \right|$$

which, in conjunction with (14) and (17′), implies the desired inequality (15) for $z \in \mathbf{C}\backslash\bar{\Delta}$. By continuity, the inequality must also hold for $z \in \partial\Delta$.

The following result is an immediate consequence of Theorem 1.

THEOREM 1′. *Given non-negative integers* λ *and* μ *let* $P_{n,\lambda,\mu}(x)$ *and the points* $x_1, x_2, \dots, x_{\nu(n)}$ *be as in* (1) *and* (2) *respectively. Further, let* $F_l(x)$ *be defined as in* (4) *and denote by* ξ_1 *the smallest zero of* $F_{\nu(n)}{}^{(j)}(x)$ *and by* η_{n-j} *the largest zero of* $F_1{}^{(j)}(x)$. *If* $p_n(x)$ *is a polynomial of degree n with real coefficients having a zero of multiplicity at least* $[(\lambda + 1)/2]$ *at* 1 *and of multiplicity at least* $[(\mu + 1)/2]$ *at* -1 *such that* (5) *holds, or more generally*

$$|p_n(x_l)| \leqq |P_{n,\lambda,\mu}(x_l)|,\ l = 1, 2, \dots, \nu(n)$$

then

$$(18)\quad |p_n{}^{(j)}(z)| \leqq |P_{n,\lambda,\mu}^{(j)}(z)|$$

for all z lying outside the open disk D^0 *with* (ξ_1, η_{n-j}) *as diameter.*

The zeros of $P_{n,\lambda,\lambda}^{(j)}(z)$ are symmetric with respect to the imaginary

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axis and so for all $\rho > 0$

$$\max_{|z|\leqq\rho} |P^{(j)}_{n,\lambda,\lambda}(z)| = |P^{(j)}_{n,\lambda,\lambda}(\pm i\rho)|.$$

Moreover, if $\lambda = \mu$ then $\xi_1 = -\eta_{n-j}$. Hence, as a corollary of Theorem 1′ we obtain

COROLLARY 1. *Let $p_n(x)$ be a polynomial of degree n with real coefficients satisfying the hypotheses of Theorem* 1′ *with $\lambda = \mu$. Then for all $\rho \geqq \eta_{n-j}$*

$$\max_{|z|\leqq\rho} |p_n{}^{(j)}(z)| \leqq |P^{(j)}_{n,\lambda,\lambda}(\pm i\rho)|.$$

As another consequence of Theorem 1′ we have

COROLLARY 2. *Let n be an odd integer. If $p_n(x) := \sum_{k=0}^{n} a_k x^k$ is a polynomial of degree n with real coefficients satisfying the hypotheses of Theorem* 1′ *with $\lambda = \mu$ and $\gamma_{n,\lambda,n}$ is the dominating coefficient of the polynomial $P_{n,\lambda,\lambda}(x)$, then*

(19) $|a_n| + |a_0| \leqq |\gamma_{n,\lambda,n}|.$

Proof. Since the polynomial $P_{n,\lambda,\lambda}(z)$ is clearly odd it must be of the form

$$\gamma_{n,\lambda,1}z + \gamma_{n,\lambda,3}z^3 + \ldots + \gamma_{n,\lambda,n}z^n.$$

According to Theorem 1′

(18′) $|p_n(z)| \leqq |P_{n,\lambda,\lambda}(z)| \quad \text{for } |z| \geqq 1,$

and so for all $\zeta \in \mathbf{C}$ such that $|\zeta| > 1$ the polynomial

$$p_n(z) - \zeta P_{n,\lambda,\lambda}(z) = a_0 + (a_1 - \zeta\gamma_{n,\lambda,1})z + a_2z^2 + \ldots + (a_n - \zeta\gamma_{n,\lambda,n})z^n$$

must have all its zeros in $|z| < 1$. Consequently

(20) $|a_0| < |a_n - \zeta\gamma_{n,\lambda,n}| \quad \text{for } |\zeta| > 1.$

This implies in particular that $|a_n| \leqq |\gamma_{n,\lambda,n}|$. So we can choose arg ζ such that

$$|a_n - \zeta\gamma_{n,\lambda,n}| = |\zeta|\,|\gamma_{n,\lambda,n}| - |a_n|.$$

Thus from (20) it follows that if $|\zeta| > 1$, then

$$|a_0| < |\zeta|\,|\gamma_{n,\lambda,n}| - |a_n|$$

and so (19) must hold.

Remark 1. The example $P_{n,\lambda,\lambda}(x)$ shows that (19) does not hold if n is even (note that $|P_{n,\lambda,\lambda}(0)| = 1$) but the above proof with a slight modification shows that in that case

(19′) $|a_n| \leqq |\gamma_{n,\lambda,n}| - (1 - |a_0|).$

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Inequalities (19), (19′) not only generalize but also strengthen the classical inequality of Chebyshev [**1**, page 63 (see Problem 8 (e))].

Earlier [**4**] we had proved the following

THEOREM A. *Let*

$$P_{n,\lambda,\lambda}(x) = \sum_{k=0}^{n} \gamma_{n,\lambda,k} x^k = \begin{cases} (1-x^2)^{\lambda/2} T_{n-\lambda}(x) & \text{if } \lambda \text{ is even} \\ (1-x^2)^{(\lambda+1)/2} U_{n-\lambda-1}(x) & \text{if } \lambda \text{ is odd.} \end{cases}$$

If $p_n(x) = \sum_{k=0}^{n} a_k x^k$ is a polynomial of degree at most n with real coefficients such that

(21) $|p_n(x)| \leqq (1 - x^2)^{\lambda/2}$

for $-1 < x < 1$, then

(22) $|a_{n-2j}| + |a_{n-2j-1}| \leqq |\gamma_{n,\lambda,n-2j}|, \quad \left(j = 0, 1, \ldots, \left[\frac{n-1}{2}\right]\right).$

It is natural to ask if (22) remains valid for polynomials with complex coefficients. The answer turns out to be negative. In fact, we shall prove that for every given $\epsilon > 0$ there exists a polynomial

$$p_{n,\lambda}(x) = \sum_{k=0}^{n} a_{\lambda,k} x^k$$

of degree n satisfying the conditions of Theorem A such that

(23) $|a_{\lambda,n-2j}| + \epsilon |a_{\lambda,n-2j-1}| > |\gamma_{n,\lambda,n-2j}|.$

It is clearly enough to prove (23) for all sufficiently small $\epsilon > 0$. Now let

$$p_{n,\lambda}(x) := \{P_{n,\lambda,\lambda}(x) + i\epsilon^2 P_{n-1,\lambda,\lambda}(x)\}/\sqrt{1+\epsilon^4} = \sum_{k=0}^{n} a_{\lambda,k} x^k.$$

Then clearly

$$|p_{n,\lambda}(x)| \leqq (1 - x^2)^{\lambda/2} \quad \text{for } -1 \leqq x \leqq 1.$$

Further

$$a_{\lambda,n-2j} = \frac{1}{\sqrt{1+\epsilon^4}} \gamma_{n,\lambda,n-2j}, \quad a_{\lambda,n-2j-1} = \frac{i\epsilon^2}{\sqrt{1+\epsilon^4}} \gamma_{n-1,\lambda,n-2j-1}$$

and so

$$|a_{\lambda,n-2j}| + \epsilon |a_{\lambda,n-2j-1}| = \frac{1}{\sqrt{1+\epsilon^4}} \{ |\gamma_{n,\lambda,n-2j}| + \epsilon^3 |\gamma_{n-1,\lambda,n-2j-1}| \} > |\gamma_{n,\lambda,n-2j}|$$

if

$$\epsilon < 2|\gamma_{n-1,\lambda,n-2j-1}|/|\gamma_{n,\lambda,n-2j}|.$$

We take this opportunity to present a short proof of Theorem A. In fact, we shall prove the somewhat stronger

THEOREM A′. *Let $P_{n,\lambda,\lambda}(x)$ be as in Theorem A and denote by*

(24) $x_{n,1} < x_{n,2} < \ldots < x_{n,n-2[(\lambda+1)/2]+1}$

the roots of the equation

$$1 - \frac{P^2_{n,\lambda,\lambda}(x)}{(1-x^2)^\lambda} = 0.$$

Then for a polynomial $p_n(x) = \sum_{k=0}^n a_k x^k$ of degree at most n with real coefficients, inequality (22) *holds even if* (21) *is satisfied only at the points $x_{n,l}$ of* (24).

Proof. First we show that if (21) is satisfied at the points $x_{n,l}$, $(1 \leqq l \leqq n - 2[(\lambda+1)/2] + 1)$, then

(25) $$|a_{n-2j}| \leqq |\gamma_{n,\lambda,n-2j}|, \quad \left(j = 0, 1, \ldots, \left[\frac{n}{2}\right]\right).$$

It is clear that for $-1 < \theta < 1$ the polynomial

$$h_1(x, \theta) := \begin{cases} P_{n,\lambda,\lambda}(x) - (\theta/2)\{p_n(x) + p_n(-x)\} & \text{if } n \text{ is even} \\ P_{n,\lambda,\lambda}(x) - (\theta/2)\{p_n(x) - p_n(-x)\} & \text{if } n \text{ is odd,} \end{cases}$$

changes sign between two consecutive points $x_{n,l}$ and so must have at least $n - 2[(\lambda+1)/2]$ zeros in $(-1, 1)$. Besides, it has a zero of multiplicity $[(\lambda+1)/2]$ at each of the points -1, $+1$ and so all its zeros are real. The coefficients of x^{n-1}, x^{n-3}, ... being all zero, none of the other coefficients can vanish; for then by Descartes' rule of signs, the zeros of $h_1(x, \theta)$ could not all be real. This is possible only if (25) holds.

The preceding argument is based on an idea of O. D. Kellogg [**3**].

Next we show that if (21) is satisfied at the points $x = x_{n-1,1}, x_{n-1,2}, \ldots, x_{n-1,n-2[(\lambda+1)/2]}$, then

(26) $$|a_{n-2j+1}| \leqq |\gamma_{n-1,\lambda,n-2j+1}|, \quad \left(j = 1, 2, \ldots, \left[\frac{n+1}{2}\right]\right).$$

In fact, all we have to do is to apply the above reasoning to the function

$$h_2(x, \theta) := \begin{cases} P_{n-1,\lambda,\lambda}(x) - (\theta/2)\{p_n(x) + p_n(-x)\} & \text{if } n \text{ is odd} \\ P_{n-1,\lambda,\lambda}(x) - (\theta/2)\{p_n(x) - p_n(-x)\} & \text{if } n \text{ is even.} \end{cases}$$

Now let us consider the polynomial

$$f(x) := \tfrac{1}{2}\{(1+x)p_n(x) + (1-x)p_n(-x)\} = \sum_{k=0}^m b_k x^k \quad \text{(say).}$$

Note that m is equal to n or $n+1$ according as n is even or odd respectively. In view of the fact that

$$\tfrac{1}{2}(|1 + x| + |1 - x|) \equiv 1 \quad \text{for } -1 < x < 1$$

we have

$$|f(x)| \leqq (1 - x^2)^{\lambda/2} \quad \text{for } x = x_{n,1}, x_{n,2}, \ldots, x_{n,n-2[(\lambda+1)/2]+1}$$

and so from (25), (26) we obtain

$$(27) \quad |a_{n-2j} + a_{n-2j-1}| = |b_{-2j}| \leqq |\gamma_{n,\lambda,n-2j}|, \quad \left(j = 0, 1, \ldots, \left[\frac{n-1}{2}\right]\right).$$

On the other hand, considering

$$g(x) := \tfrac{1}{2}\{(1 - x)p_n(x) + (1 + x)p_n(-x)\}$$

we can prove in the same way that

$$(28) \quad |a_{n-2j} - a_{n-2j-1}| \leqq |\gamma_{n,\lambda,n-2j}|, \quad \left(j = 0, 1, \ldots, \left[\frac{n-1}{2}\right]\right).$$

Inequalities (27) and (28) together give us the desired result.

We observe that, at least in the case of odd n, the conclusion of Theorem A′ can be considerably strengthened if $p_n(x)$ happens to be non-negative at the points (24). In fact, we have

THEOREM A″. *Let n be odd. If the polynomial $p_n(x) = \sum_{k=0}^{n} a_k x^k$ satisfies the hypotheses of Theorem A′ and is, in addition, non-negative at the points $x_{n,l}$ of* (24), *then*

$$(29) \quad |a_{n-2j}| + |a_{n-2j-1}| \leqq \tfrac{1}{2}|\gamma_{n,\lambda,n-2j}|, \quad \left(j = 0, 1, \ldots, \left[\frac{n-1}{2}\right]\right).$$

In the case of even n we can prove

THEOREM A‴. *Let n be even. If $p_n(x) = \sum_{k=0}^{n} a_k x^k$ is a polynomial of degree n with real coefficients such that*

$$0 \leqq p_n(x) \leqq (1 - x^2)^{\lambda/2}$$

at the points $x = x_{n-1,1}, x_{-1,2}, \ldots, x_{n-1,n-2[(\lambda+1)/2]}$, then

$$(30) \quad |a_{n-2j+1}| \leqq \tfrac{1}{2}|\gamma_{n-1,\lambda,n-2j+1}|, \quad \left(j = 1, 2, \ldots, \left[\frac{n+1}{2}\right]\right).$$

Proof of Theorems A″, A‴. First of all we observe that if $f(x) = \sum_{k=0}^{m} b_k x^k$ is a polynomial of degree m (even) with real coefficients such that

$$(31) \quad 0 \leqq f(x) \leqq (1 - x^2)^{\lambda/2}$$

at the points $x = x_{m-1,1}, x_{m-1,2}, \ldots, x_{m-1,m-2[(\lambda+1)/2]}$, then

$$|f(x) - f(-x)| \leqq (1 - x^2)^{\lambda/2}$$

at these points. Since $f(x) - f(-x)$ is a polynomial of degree $m - 1$ it follows from (26) that

$$(32)\quad |b_{m-2j+1}| \leqq \tfrac{1}{2}|\gamma_{m-1,\lambda,m-2j+1}|, \quad \left(j = 1, 2, \ldots, \left[\frac{m+1}{2}\right]\right),$$

which proves Theorem A‴.

Now if $p_n(x) = \sum_{k=0}^{n} a_k x^k$ is a polynomial of degree n (odd) satisfying the hypotheses of Theorem A″, then

$$f(x) := \tfrac{1}{2}\{(1 + x)p_n(x) + (1 - x)p_n(-x)\}$$

is a polynomial of degree $n + 1$ (even) with real coefficients such that (31) is satisfied at the points $x = x_{n,1}, x_{n,2}, \ldots, x_{n,n+1-2[(\lambda+1)/2]}$ and so according to (32) we must have

$$(33)\quad |a_{n-2j} + a_{n-2j-1}| \leqq \tfrac{1}{2}|\gamma_{n,\lambda,n-2j}|, \quad \left(j = 0, 1, \ldots, \left[\frac{n-1}{2}\right]\right).$$

On the other hand, considering

$$g(x) := \tfrac{1}{2}\{(1 - x)p_n(x) + (1 + x)p_n(-x)\}$$

we can prove in the same way that

$$(34)\quad |a_{n-2j} - a_{n-2j-1}| \leqq \tfrac{1}{2}|\gamma_{n,\lambda,n-2j}|, \quad \left(j = 0, 1, \ldots, \left[\frac{n-1}{2}\right]\right),$$

and so Theorem A″ holds.

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