SINGULAR PERTURBATIONS OF NON-LINEAR ELLIPTIC AND PARABOLIC VARIATIONAL BOUNDARY-VALUE PROBLEMS

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Introduction. Singular perturbations of linear elliptic and parabolic boundary-value problems have been studied extensively by Visik and Lyusternik **(7)**, Huet **(5)**, and others. It is the purpose of this paper to extend the results of **(5)** to the non-linear elliptic and parabolic variational boundary-value problems considered during the last few years by Browder **(2, 4)**.

In §1, we give the notations and state the main assumptions on the non-linearity of the elliptic operators. In §2 we study the singular perturbations of non-linear elliptic variational boundary problems. In §3, we consider the case of non-linear parabolic variational boundary problems with a small parameter.

1. Let Ω be a bounded, open set of E^n with a C^∞ embedding mapping of its boundary $\partial\Omega$ into E^n. The points of Ω will be denoted by $x = (x_1, \ldots, x_n)$ and derivatives with respect to the x-variables by:

$$D_j = i^{-1}\partial/\partial x_j, \qquad 1 \leqslant j \leqslant n; \qquad D^\alpha = D_1^{\alpha_1} \ldots D_n^{\alpha_n},$$

$$\alpha = (\alpha_1, \ldots, \alpha_n) \qquad \text{with } |\alpha| = \sum_{j=1}^{n} \alpha_j.$$

The points of R^1 will be denoted by t and differentiation in t by $\partial/\partial t$. If u, v are functions on Ω, we denote by $\langle u, v\rangle$ their inner product in $L^2(\Omega)$.

Let $W^{m,2}(\Omega)$ be the Hilbert space defined by:

$$W^{m,2}(\Omega) = \{u\colon u \in L^2(\Omega), D^\alpha u \in L^2(\Omega) \text{ for } |\alpha| \leqslant m\}$$

(the derivatives are taken in the sense of the theory of distributions) with the norm:

$$||u||_{m,2} = \{\textstyle\sum_{|\alpha|\leqslant m} ||D^\alpha u||_2{}^2\}^{\frac{1}{2}}$$

and inner product:

$$(u, v)_m = \textstyle\sum_{|\alpha|\leqslant m} \langle D^\alpha u, D^\alpha v\rangle, \qquad u, v \text{ in } W^{m,2}(\Omega).$$

We denote by $C_c^\infty(\Omega)$ the family of infinitely differentiable functions with compact support in Ω. We consider differential operators of the form:

$$(1) \quad A_k u = \textstyle\sum_{|\alpha|,|\beta|\leqslant m_k} D^\alpha(a_{k\alpha\beta}(x, u, \ldots, D^{m_k-1}u)D^\beta u), \quad u \text{ in } W^{m_k,2}(\Omega).$$

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We assume the following conditions on the coefficients $a_{k\alpha\beta}$:

ASSUMPTION (I). *The coefficients $a_{k\alpha\beta}$ are continuous functions of all their arguments. There exists a continuous function $g_k(r)$ of the real variable r such that:*

$$|a_{k\alpha\beta}(x, u, \ldots, D^{m_k-1}u)| \leqslant g_k(\|u\|_{m_k-1,2})$$

for u in $W^{m_k,2}(\Omega)$.

Let ϵ be a small positive parameter. To two non-linear differential operators of the form (1) with $m_2 > m_1$, we associate the non-linear differential operator of the form (2):

$$(2) \quad A_\epsilon u = \sum_{|\alpha|,|\beta|\leqslant m_2} D^\alpha(a_{2\alpha\beta}(x, \epsilon u, \ldots, \epsilon D^{m_2-1}u)D^\beta \epsilon u) + \sum_{|\alpha|,|\beta|\leqslant m_1} D^\alpha(a_{1\alpha\beta}(x, u, \ldots, D^{m_1-1}u)D^\beta u)$$

for u in $W^{m_2,2}(\Omega)$.

Let V_k be a closed subspace of $W^{m_k,2}(\Omega)$ with $C_c^\infty(\Omega) \subset V_k$. We consider the Hilbert space $L^2([0, T], V_k)$ of equivalence classes of functions u from $[0, T]$ to V_k with

$$\int_0^T \|u(t)\|^2_{m_k,2}\, dt < +\infty.$$

The norm is given by

$$|||u|||_k = \left\{ \int_0^T \|u(t)\|^2_{W^{m_k,2}(\Omega)}\, dt \right\}^{\frac{1}{2}}.$$

$L^2([0, T], V_k)$ is a separable reflexive Hilbert space; $k = 1, 2$.

Let $(L^2([0, T], V_k))^*$ be the conjugate space of $L^2([0, T], V_k)$, i.e. the space of bounded conjugate linear functionals on $L^2([0, T], V_k)$. For $u \in L^2([0, T], V_k)$ and $w \in (L^2([0, T], V_k))^*$, we denote by (w, u) the pairing of w with u. For $u \in L^2([0, T], L^2(\Omega))$ and $v \in (L^2([0, T], L^2(\Omega))^*$, we denote by $((\ ,\ ))$ the pairing of v with u.

Let $A_k(t)$ be differential operators of the form:

$$(1') \quad A_k(t)u = \sum_{|\alpha|,|\beta|\leqslant m_k} D^\alpha(a_{k\alpha\beta}(x, t, u, \ldots, D^{m_k-1}u)D^\beta u)$$

for $u \in L^2(R^1, W^{m_k,2}(\Omega))$; $k = 1, 2$.

We make the following assumptions on $a_{k\alpha\beta}$:

ASSUMPTION (I'): *The coefficients $a_{k\alpha\beta}$ are functions defined on $\Omega \times R^1$ measurable in x, t and continuous in $(u, \ldots, D^{m_k-1}u)$. There exists a continuous function $g_k(r)$ of the real variable r such that*

$$|a_{k\alpha\beta}(x, t, u, \ldots, D^{m_k-1}u)|^2 \leqslant g_k\left(\int_{-\infty}^{\infty} \|u(t)\|^2_{m_k-1,2}\, dt\right)$$

for $u \in L^2(R^1, W^{m_k,2}(\Omega))$.

To $A_1(t)$ and $A_2(t)$ of the form (1′), we associate the non-linear differential operator of the form

$$(2') \quad A_\epsilon(t)u = \sum_{|\alpha|,|\beta|\leqslant m_2} D^\alpha(a_{2\alpha\beta}(x, t, \epsilon u, \ldots, \epsilon D^{m_2-1}u)D^\beta \epsilon u) + \sum_{|\alpha|,|\beta|\leqslant m_1} D^\alpha(a_{1\alpha\beta}(x, t, u, \ldots, D^{m_1-1}u)D^\beta u),$$

$m_2 > m_1$ and $u \in L^2(R^1, W^{m_2,2}(\Omega))$.

2. In this section, we study the singular perturbations of non-linear elliptic variational boundary-value problems.

Let V_1 be a closed subspace of $W^{m_1,2}(\Omega)$ with $C_c^\infty(\Omega) \subset V_1$. Corresponding to the non-linear elliptic operator A_1 of the form (1), we define the non-linear Dirichlet form:

$$a_1(u, v) = \sum_{|\alpha|,|\beta|\leqslant m_1} \langle a_{1\alpha\beta}(x, u, \ldots, D^{m_1-1}u)D^\alpha u, D^\beta v\rangle$$

for each pair u, v in $W^{m_1,2}(\Omega)$. With the Assumption (I) on the coefficients $a_{1\alpha\beta}$, the Dirichlet form is well defined.

Let V_1^* be the conjugate space of V_1. We now define the variational boundary-value problem corresponding to (A_1, V_1).

DEFINITION. *Let $f \in V_1^*$. Then u is said to be a solution of the variational boundary problem for $A_1 u = f$ satisfying the null boundary conditions corresponding to the space V_1 if:*
(1) *$a_1(u, v) = (f, v)$ for all $v \in V_1$,*
(2) *$u \in V_1$.*

THEOREM 2.1. *Let A_1 be a non-linear elliptic differential operator of the form (1), of order $2m_1$ and satisfying Assumption (I). Let V_1 be a closed subspace of $W^{m_1,2}(\Omega)$ such that $C_c^\infty(\Omega) \subset V_1$. Suppose that there exists a non-negative continuous function $c_1(r)$ on R^1 with $\lim_{r\to+\infty} c_1(r) = +\infty$ such that*

$$\operatorname{Re}\{a_1(u, u - v) - a_1(v, u - v)\} \geqslant c_1(||u - v||_{m_1,2})||u - v||_{m_1,2}.$$

Then for every $f \in V_1^*$, the variational boundary problem for $A_1 u = f$ with null V_1-boundary conditions has a unique solution.

The theorem is due to Browder **(2)**.

Let A_ϵ be the non-linear elliptic operator defined in Section 1. We have the following theorem for $A_\epsilon u_\epsilon = f$.

THEOREM 2.2. *Let A_k be two non-linear elliptic differential operators of the form (1), of order $2m_k$ with $m_2 > m_1$ and satisfying Assumption (I). Let V_k be two closed subspaces of $W^{m_k,2}(\Omega)$ such that $C_c^\infty(\Omega) \subset V_k$ with $V_2 \subset V_1$. Suppose that there exist two non-negative continuous functions $c_k(r)$ on R^1 and*

$$\lim_{r\to+\infty} c_k(r) = +\infty$$

such that

$$\operatorname{Re}\{a_k(u, u - v) - a_k(v, u - v)\} \geqslant c_k(||u - v||_{m_k,2})||u - v||_{m_k,2};$$

$u, v \in V_k$; $k = 1, 2$. Let ϵ be a small positive parameter and A_ϵ be the non-linear elliptic differential operator of the form (2). *Then for every f in V_2^*, there exists a unique solution u_ϵ of the variational boundary problem $A_\epsilon u_\epsilon = f$ with null V_2-boundary conditions.*

The proof of the theorem is essentially the same as that of Theorem 2.1; cf. **(2)**.

THEOREM 2.3. *Let A_ϵ be the non-linear elliptic differential operator of Theorem 2.2 Let u_ϵ be the solution of the variational boundary problem $A_\epsilon u_\epsilon = f_\epsilon$ with null V_2-boundary conditions. Let u_0 be the solution of $A_1 u_0 = f$ with null V_1-boundary conditions. Suppose that there is a set V dense in both V_1, V_2. Then, if $f_\epsilon \to f$ weakly in V_1^* as $\epsilon \to 0$:*

$$u_\epsilon \to u_0 \qquad \text{in } W^{m_1,2}(\Omega),\ \epsilon u_\epsilon \to 0 \qquad \text{in } W^{m_2,2}(\Omega).$$

Proof. We have:

$$a_\epsilon(u_\epsilon, u_\epsilon) = a_2(\epsilon u_\epsilon, u_\epsilon) + a_1(u_\epsilon, u_\epsilon).$$

Hence

$$c_2(\epsilon||u_\epsilon||_{m_2,2})||u_\epsilon||_{m_2,2} + c_1(||u_\epsilon||_{m_1,2})||u_\epsilon||_{m_1,2} \leqslant \operatorname{Re} a_\epsilon(u_\epsilon, u_\epsilon)$$
$$\leqslant \operatorname{Re}\{a_2(\epsilon u_\epsilon, u_\epsilon) + a_1(u_\epsilon, u_\epsilon)\}.$$

But $a_\epsilon(u_\epsilon, u_\epsilon) = (f_\epsilon, u_\epsilon)$. So we obtain

$$c_1(||u_\epsilon||_{m_1,2})||u_\epsilon||_{m_1,2} \leqslant ||f_\epsilon||\ ||u_\epsilon||_{m_1,2}.$$

Since $f_\epsilon \to f$ weakly in V_1^*, f_ϵ is uniformly bounded in V_1^*. Hence

$$c_1(||u_\epsilon||_{m_1,2}) \leqslant M,$$

where M is a constant independent of ϵ. By hypothesis, the function $c_1(r)$ satisfies $\lim_{r\to+\infty} c_1(r) = +\infty$. Therefore there exists a constant M' independent of ϵ such that $||u_\epsilon||_{m_1,2} \leqslant M'$. A similar argument yields: $\epsilon||u_\epsilon||_{m_2,2} \leqslant M'$.

From the weak compactness of the unit ball in a Hilbert space it follows that there is a subsequence u_ϵ such that:

$$u_\epsilon \to v \text{ weakly in } W^{m_1,2}(\Omega) \qquad \text{as } \epsilon \to 0,$$
$$\epsilon u_\epsilon \to 0 \text{ weakly in } W^{m_2,2}(\Omega) \qquad \text{as } \epsilon \to 0.$$

We now show that $v = u_0$. First we note that v belongs to V_1. Indeed $||u_\epsilon||_{V_1} \leqslant M$, and V_1 with the topology induced by $W^{m_1,2}(\Omega)$ is a Hilbert space.

Consider $a_\epsilon(u_\epsilon, \phi)$; $\phi \in V$. We have

$$\sum_{|\alpha|,|\beta|\leqslant m_2} \epsilon\langle a_{2\alpha\beta}(x, \epsilon u_\epsilon, \ldots, \epsilon D^{m_2-1}u_\epsilon)D^\alpha u_\epsilon, D^\beta\phi\rangle$$
$$+ \sum_{|\alpha|,|\beta|\leqslant m_1}\langle a_{1\alpha\beta}(x, u_\epsilon, \ldots, D^{m_1-1}u_\epsilon)D^\alpha u_\epsilon, D^\beta\phi\rangle.$$

The last term is equal to

$$\sum_{|\alpha|,|\beta|\leqslant m_1}\langle D^\alpha u_\epsilon, \bar{a}_{1\alpha\beta}(x, u_\epsilon, \ldots, D^{m_1-1}u_\epsilon)D^\beta\phi\rangle.$$

Since $u_\epsilon \to v$ weakly in $W^{m_2,2}(\Omega)$ and Ω is a bounded domain with a C^∞ embedding mapping of its boundary $\partial\Omega$ into E^n, the Sobolev embedding mapping theorem yields that $u_\epsilon \to v$ in $W^{m_1-1,2}(\Omega)$. By taking a subsequence if necessary we may assume that

$$D^\alpha u_\epsilon \to D^\alpha v \text{ a.e. on } \Omega \text{ for } |\alpha| \leqslant m_1 - 1.$$

On the other hand, by hypothesis, $a_{1\alpha\beta}$ are continuous functions of their arguments; hence

$$\bar{a}_{1\alpha\beta}(x, u_\epsilon, \ldots, D^{m_1-1}u_\epsilon)D^\beta\phi \to \bar{a}_{1\alpha\beta}(x, v, \ldots, D^{m_1-1}v)D^\beta\phi \quad \text{a.e. on } \Omega \text{ as } \epsilon \to 0.$$

Moreover

$$|\bar{a}_{1\alpha\beta}(x, u_\epsilon, \ldots, D^{m_1-1}u_\epsilon)D^\beta\phi| \leqslant g_1(||u_\epsilon||_{m_1,2})|D^\beta\phi| \leqslant M|D^\beta\phi|$$

since $||u_\epsilon||_{m_1,2} \leqslant M$. By the Lebesgue bounded convergence theorem, it follows that

$$\bar{a}_{1\alpha\beta}(x, u_\epsilon, \ldots, D^{m_1-1}u_\epsilon)D^\beta\phi \to \bar{a}_{1\alpha\beta}(x, v, \ldots, D^{m_1-1}v)D^\beta\phi \text{ in } L^2(\Omega) \text{ as } \epsilon \to 0,$$

for $|\alpha|, |\beta| \leqslant m_1$.

Since $D^\alpha u_\epsilon \to D^\alpha v$ weakly in $L^2(\Omega)$, we have

$$\sum_{|\alpha|,|\beta|\leqslant m_1} \langle a_{1\alpha\beta}(x, u_\epsilon, \ldots, D^{m_1-1}u_\epsilon)D^\alpha u_\epsilon, D^\beta\phi\rangle \to a_1(v, \phi) \qquad \text{as } \epsilon \to 0.$$

Applying the same argument to

$$\sum_{|\alpha|,|\beta|\leqslant m_2} \epsilon\langle a_{2\alpha\beta}(x, \epsilon u_\epsilon, \ldots, \epsilon D^{m_2-1}u_\epsilon)D^\alpha u_\epsilon, D^\beta\phi\rangle,$$

we find that it goes to zero as $\epsilon \to 0$. Therefore $a_1(v, \phi) = (f, \phi)$ for all ϕ in V. So $a_1(v, w) = (f, w)$ for all w in V_1 since V is dense in V_1.

From the uniqueness of the solution of $A_1 u_0 = f$ with null V_1-boundary conditions, it follows that $v = u_0$.

It remains to show that $u_\epsilon \to u_0$ in $W^{m_1,2}(\Omega)$ and $\epsilon u_\epsilon \to 0$ in $W^{m_2,2}(\Omega)$ as $\epsilon \to 0$. Consider the expression $\mathrm{Re}\{a_1(u_\epsilon, u_\epsilon - u_0) - a_1(u_0, u_\epsilon - u_0)\}$. From the hypothesis, we obtain

$$\begin{aligned}||u_\epsilon - u_0||_{m_1,2}\, c_1(||u_\epsilon - u_0||_{m_1,2}) &\leqslant \mathrm{Re}\{a_1(u_\epsilon, u_\epsilon - u_0) - a_1(u_0, u_\epsilon - u_0)\} \\ &\leqslant \mathrm{Re}\{a_1(u_\epsilon, u_\epsilon - u_0) + (f, u_0) - (f, u_\epsilon)\}.\end{aligned}$$

Also

$$\epsilon||u_\epsilon||_{m_2,2}\, c_2(\epsilon||u_\epsilon||_{m_2,2}) \leqslant \mathrm{Re}\, a_2(\epsilon u_\epsilon, \epsilon u_\epsilon) = \epsilon\, \mathrm{Re}\, a_2(\epsilon u_\epsilon, u_\epsilon).$$

Therefore

$$\begin{aligned}||u_\epsilon||_{m_2,2}c_2(\epsilon||u_\epsilon||_{m_2,2}) + ||u_\epsilon - u_0||_{m_1,2}\, c_1(||u_\epsilon - u_0||_{m_1,2}) \leqslant \mathrm{Re}\{a_2(\epsilon u_\epsilon, u_\epsilon) \\ + a_1(u_\epsilon, u_\epsilon) - a_1(u_\epsilon, u_\epsilon) + a_1(u_\epsilon, u_\epsilon - u_0) - a_1(u_0, u_\epsilon - u_0)\}.\end{aligned}$$

The right-hand side of this inequality is equal to

$$\mathrm{Re}\{(f, u_\epsilon) - a_1(u_\epsilon, u_0) + (f, u_0) - (f, u)\} \to 0 \text{ as } \epsilon \to 0.$$

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So

$$\lim_{\epsilon\to 0}||u_\epsilon - u_0||_{m_1,2}\, c_1(||u_\epsilon - u_0||_{m_1,2}) = 0,$$
$$\lim_{\epsilon\to 0}||u_\epsilon||_{m_2,2}\, c_2(\epsilon||u_\epsilon||_{m_2,2}) = 0.$$

If $||u_\epsilon - u_0||_{m_1,2} \geqslant \eta > 0$ for $\epsilon \geqslant +0$, then $0 < c_1(\eta) \leqslant 0$, which is impossible. Similarly, suppose that $\epsilon||u_\epsilon||_{m_2,2} \geqslant \eta > 0$ for $\epsilon \geqslant +0$. Since $c_2(r)$ is positive for $r > 0$ by hypothesis, we would have $0 < c_2(\eta) \leqslant 0$, which is impossible. So

$$\lim_{\epsilon\to 0} u_\epsilon \to u_0 \quad \text{in } W^{m_1,2}(\Omega),$$
$$\lim_{\epsilon\to 0} \epsilon u_\epsilon \to 0 \quad \text{in } W^{m_2,2}(\Omega).$$

3. In this section, we study the singular perturbations of non-linear parabolic variational boundary problems.

Let V_k be a closed subspace of $W^{m_k,2}(\Omega)$, as before, and such that

$$C_c^\infty(\Omega) \subset V_k \ (k = 1, 2; m_2 > m_1)$$

and $V_2 \subset V_1$ algebraically and topologically. To the non-linear elliptic differential operator $A_k(t)$ of the form (1′) of order $2m_k$ corresponds the non-linear Dirichlet form

$$h_k(u, v) = \sum_{|\alpha|,|\beta|\leqslant m_k} \int_0^T \langle a_{k\alpha\beta}(x, t, u, \ldots, D^{m_k-1}u)D^\alpha u, D^\beta v\rangle dt;$$
$$u, v \in L^2([0, T], V_k); k = 1, 2.$$

With the assumption (I′) on the coefficients $a_{k\alpha\beta}$, the Dirichlet form is well defined.

Definition. *Let L_{0k} be the linear mapping of $F_k = \{v: v \in L^2([0, T], V_k)$, v is continuous from $[0, T]$ to V_k and is continuously differentiable from $[0, T]$ to $L^2(\Omega)$, $v(0) = 0\}$ into $(L^2([0, T], V_k))^*$ such that*

$$(L_{0k}\, u, v) = \int_0^T \left\langle \frac{du}{dt}, v\right\rangle dt$$

for all v in $L^2([0, T], V_k)$, $k = 1, 2$. Let L_k be the closure of L_{0k} as a linear operator with domain in $L^2([0, T], V_k)$ and range in $(L^2([0, T], V_k))^*$.

L_{0k} is preclosed and has a densely defined adjoint, so L_k is well defined.

Definition. *Let f be an element of $(L^2([0, T], V_k))^*$. Then an element $u \in D(L_k)$ and belonging to $L^2([0, T], V_k)$ is said to be a solution of the variational boundary-value problem*

$$\partial u/\partial t + A_k(t)u = f$$

if $(L_k\, u, v) + h_k(u, v) = (f, v)$ for all $v \in L^2([0, T], V_k)$, $k = 1, 2$.

We have the following theorem.

THEOREM 3.1. *Let $A_1(t)$ be a non-linear elliptic differential operator of order $2m_1$, satisfying Assumption* (I′). *Suppose that there exists a non-negative continuous function $c_1(r)$ with $\lim_{r\to+\infty} c_1(r) = +\infty$, such that*

$$\operatorname{Re}\{h_1(u, u - v) - h_1(v, u - v)\} \geqslant c_1(|||u - v|||_1)|||u - v|||_1 \quad \text{for all } u, v \in L^2([0, T], V_1).$$

Then there exists a unique solution of the variational boundary-value problem for the parabolic equation

$$\partial u/\partial t + A_1(t)u = f \qquad \text{on } [0, T] \times \Omega$$

for given $f \in (L^2([0, T], V_1))^*$. This solution u is continuous from $[0, T]$ to $L^2(\Omega)$ and $u(0) = 0$.

This theorem has been proved by Browder **(4)** under weaker hypotheses on the ellipitic operator A_1 and for $L^p([0, T], V_1)$ where $1 < p < \infty$.

DEFINITION. *For each $u \in F_k$, let $L_{0k}^{\#}u$ be the element of $(L^2([0, T], L^2(\Omega))^*$ such that*

$$((L_{0k}^{\#}u, v)) = \int_0^T \left\langle \frac{du}{dt}, v\right\rangle dt \qquad \text{for all } v \in L^2([0, T], L^2(\Omega)).$$

Let $L_k^{\#}$ be the closure of $L_{0k}^{\#}$ as a linear operator with domain in $L^2([0, T], V_k)$ and range in $(L^2([0, T], L^2(\Omega))^*$, $k = 1, 2$.

First, we note that $L_{0k}^{\#}$ is preclosed and has a densely defined adjoint so that $L_k^{\#}$ is well defined.

Here ((,)) denotes the pairing of $L^2([0, T], L^2(\Omega))$ and of $(L^2([0, T], L^2(\Omega))^*$.

LEMMA 3.1. (1) $D(L_k) = D(L_k^{\#})$, $\quad k = 1, 2$.

(2) *$D(L_2^{\#}) \subset D(L_1^{\#})$ and $L_2^{\#}u = L_1^{\#}u$ if $u \in D(L_2^{\#})$.*

(3) *$((L_k^{\#}u, v)) = (L_k u, v)$ for all $u \in D(L_k)$ and $v \in L^2([0, T], L^2(\Omega))$.*

Proof. (i) Let $u \in D(L_k)$. Then there exists a sequence of elements $u_n \in D(L_{0k})$ such that $u_n \to u$ in $L^2([0, T], V_k)$, $L_{0k} u_n = L_k u_n \to v$ in $(L^2([0, T], V_k))^*$, and $v = L_k u$. But we have

$$((L_{0k}^{\#}u_n, w)) = \int_0^T \left\langle \frac{du_n}{dt}, w\right\rangle dt = (L_k u_n, w) \qquad \text{for all } w \in L^2([0, T], V_k).$$

Here ((,)) denotes the natural pairing between elements of $L^2([0, T], L^2(\Omega))$ and $(L^2([0, T], L^2(\Omega))^*$, and (,) is the natural pairing between $L^2(0, T, V_k)$ and $(L^2([0, T], V_k))^*$. Let J be the injection mapping from $(L^2([0, T], L^2(\Omega))^*$ into $(L^2([0, T], V_k))^*$ with

$$(J\phi, w) = ((\phi, w)) \qquad \text{for } \phi \in (L^2([0, T], L^2(\Omega))^*.$$

We have $JL_k^{\#}u_n = L_k u_n \to v$ in $(L^2([0, T], V_k))^*$. Since J is bounded and $L_k^{\#}$ is closed, $JL_k^{\#}$ is closed; hence $JL_k^{\#}u = v$ and $D(L_k) \subseteq D(L_k^{\#})$.

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Now let $u \in D(L_k^{\#})$. Then $u_n \to u$ in $L^2([0, T], V_k)$, $L_{0k}^{\#}u_n = L_k^{\#}u_n \to v$ in $(L^2([0, T], L^2(\Omega))^*$, $v = L_k^{\#}u$. So

$$((L_k^{\#}u_n, w)) = (L_k u_n, w) \to ((v, w)) \text{ as } n \to \infty.$$

Hence

$$L_k u_n \to g \text{ weakly in } (L^2([0, T], V_k))^*.$$

But L_k is closed, so weakly closed, and it follows that $g = L_k u$. Therefore $D(L_k) = D(L_k^{\#})$ and $((L_k^{\#}u, w)) = (L_k u, w)$.

(ii) We now show that $D(L_2^{\#}) \subset D(L_1^{\#})$.

Let $u \in D(L_2^{\#})$. Then there exists a sequence of elements u_n of $D(L_{02}^{\#})$ such that $u_n \to u$ in $L^2([0, T], V_2)$, $L_{02}^{\#}u_n = L_2^{\#}u_n \to v$ in $(L^2([0, T], L^2(\Omega))^*$, and $v = L_2^{\#}u$.

We have

$$((L_{02}^{\#}u_n, w)) = \int_0^T \left\langle \frac{du_n}{dt}, w \right\rangle dt$$

for all w in $L^2([0, T], L^2(\Omega))$. Since $u_n \in F_2$, u_n lies in $D(L_{01}^{\#})$ and

$$((L_{01}^{\#}u_n, w)) = ((L_{02}^{\#}u_n, w)) = \int_0^T \left\langle \frac{du_n}{dt}, w \right\rangle dt$$

for all $w \in L^2([0, T], L^2(\Omega))$. So $L_{01}^{\#}u_n = L_{02}^{\#}u_n$ for u_n in $D(L_{02}^{\#})$ and $L_{01}^{\#}u_n = L_1^{\#}u_n \to v$ in $(L^2([0, T], L^2(\Omega)))^*$. Since $L_1^{\#}$ is closed, it follows that $v = L_1^{\#}u = L_2^{\#}u$.

The lemma is proved.

We define the global variational boundary-value problem.

DEFINITION. *For each $u \in F = \{u \in L^2(R^1, V_2)$, u is continuous from R^1 to V_2, u is in C^1 from R^1 to $L^2(\Omega)$, and u has compact support in $R^1\}$, let $L_0 u$ be the element of $L^2(R^1, V_2^*)$ such that*

$$(L_0 u, v) = \int_{-\infty}^{\infty} \left\langle \frac{du}{dt}, v \right\rangle dt$$

for all $v \in L^2(R^1, V_2)$. Let L be the closure of L_0 as a linear operator with domain in $L^2(R^1, V_2)$ and range in $L^2(R^1, V_2^*)$.

One can show that $L^* = -L$; cf. **(4)**.

DEFINITION. *Let $f \in L^2(R^1, V^*)$. Then u is said to be a global solution of the variational boundary-value problem for the equation*

$$\partial u/\partial t + A(t)u = f \qquad on\ \Omega \times R^1$$

if $(Lu, v) + h(u, v) = (f, v)$ for all $v \in L^2(R^1, V)$ and u is an element of $D(L) \cap L^2(R^1, V)$.

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THEOREM 3.2. *Let* $A_k(t)$ *be two non-linear elliptic differential operators of order* $2m_k$ *with* $m_2 > m_1$ *and satisfying Assumption (I′). Let* V_k *be two closed subspaces of* $W^{m_k,2}(\Omega)$ *and such that* $C_c^\infty(\Omega) \subset V_k$. *Suppose that there exist non-negative, continuous functions* $c_k(r)$ *with* $\lim_{r\to+\infty} c_k(r) = +\infty$ *such that*

$$\operatorname{Re}\{h_k(u, u - v) - h_k(v, u - v)\} \geqslant c_k(|||u - v|||_k)|||u - v|||_k$$

for all u, v *in* V_k, $k = 1, 2$. *Let* ϵ *be a small positive parameter. Let* f *be an element of* $(L^2[(0, T], V_2))^*$ *and* $\tilde{f}$ *be the element of* $L^2(R^1, V_2{}^*)$ *obtained by setting* $\tilde{f} = f$ *on* $[0, T]$ *and* 0 *outside of* $[0, T]$. *Then for each* $\epsilon > 0$, *there is a unique solution* u_ϵ *of the variational boundary-value problem*

$$\partial u_\epsilon/\partial t + A_\epsilon(t)u_\epsilon = \tilde{f} \qquad \text{on } R^1 \times \Omega.$$

Moreover the restriction of u_ϵ *to* $[0, T]$ *is the unique solution of the variational boundary-value problem for the equation*

$$\partial u_\epsilon/\partial t + A_\epsilon(t)u_\epsilon = f \qquad \text{on } [0, T] \times \Omega$$

with respect to V_2. u_ϵ *is continuous from* $[0, T]$ *to* $L^2(\Omega)$ *and* $u_\epsilon(0) = 0$.

The proof is essentially the same as in **(4)**. The uniqueness of the solution of the global variational boundary problem for $\partial u_\epsilon/\partial t + A_\epsilon(t)u_\epsilon = \tilde{f}$ follows from the assumptions that $c_1(r)$ and $c_2(r)$ are two non-negative functions.

THEOREM 3.3. *Let* $f \in L^2([0, T], L^2(\Omega))$. *With the hypotheses of Theorem* 3.2, *let* u_ϵ *be the solution of the variational boundary-value problem*

$$\partial u_\epsilon/\partial t + A_\epsilon(t)u_\epsilon = f$$

with respect to V_2 *on* $[0, T]$ *and with* $u_\epsilon(0) = 0$. *Let* u_0 *be the solution of the variational boundary-value problem*

$$\partial u_0/\partial t + A_1(t)u_0 = f$$

with respect to V_1 *on* $[0, T]$ *and with* $u_0(0) = 0$. *Suppose that there is a set* V *dense in both* V_1, V_2. *Then as* $\epsilon \to 0$, $u_\epsilon \to u_0$ *in* $L^2([0, T], W^{m_1,2}(\Omega))$ *and* $\epsilon u_\epsilon \to 0$ *in* $L^2([0, T], W^{m_2,2}(\Omega))$.

Proof. Set f to be equal to 0 outside of the interval $[0, T]$. Then f can be considered as an element of $L^2(R^1, L^2(\Omega))$. By Theorem 3.2, there exists a unique solution of the global variational boundary-value problem for the equation

$$\partial u_\epsilon/\partial t + A_\epsilon(t)u_\epsilon = f$$

with respect to V_2 on $R^1 \times \Omega$. Moreover the restriction of u_ϵ to $[0, T]$ is the unique solution of the equation on $[0, T] \times \Omega$. Hence

$$(Lu_\epsilon, w) + h_\epsilon(u_\epsilon, w) = ((f, w)) \text{ for all } w \text{ in } V_2.$$

In particular

$$(Lu_\epsilon, u_\epsilon) + h_\epsilon(u_\epsilon, u_\epsilon) = ((f, u_\epsilon)).$$

But since $L^* = -L$, we obtain

$$\operatorname{Re} h_\epsilon(u_\epsilon, u_\epsilon) = \operatorname{Re}((f, u_\epsilon)).$$

We also have

$$\operatorname{Re} h_\epsilon(u_\epsilon, u_\epsilon) = \operatorname{Re} h_2(\epsilon u_\epsilon, u_\epsilon) + h_1(u_\epsilon, u_\epsilon),$$
$$\operatorname{Re} h_\epsilon(u_\epsilon, u_\epsilon) \geqslant c_2(\epsilon|||u_\epsilon|||_2)|||u_\epsilon|||_2 + c_1(|||u_\epsilon|||_i)|||u_\epsilon|||_1.$$

Therefore, $c_1(|||u_\epsilon|||_i) \leqslant M$ and $c_2(\epsilon|||u_\epsilon|||_2) \leqslant M$ where M is a constant independent of ϵ. The functions $c_k(r)$ are such that $\lim_{r\to+\infty} c_k(r) = +\infty$. Hence there exists a constant M' such that $|||u_\epsilon|||_1 \leqslant M'$, $\epsilon|||u_\epsilon|||_2 \leqslant M'$.

From the weak compactness of the unit ball in a reflexive Banach space, it follows that there is a subsequence u_ϵ such that $u_\epsilon \to v$ weakly in

$$L^2(R^1, W^{m_1,2}(\Omega))$$

and $\epsilon u_\epsilon \to 0$ weakly in $L^2(R^1, W^{m_2,2}(\Omega))$ as $\epsilon \to 0$. Moreover v belongs to $L^2(R^1, V_1)$.

We now show that $v = u_0$. We have $(Lu_\epsilon, w) = ((f, w)) - h_\epsilon(u_\epsilon, w)$ for all $w \in L^2(R^1, V)$. Since $|||u_\epsilon|||_1$ and $\epsilon\, |||u_\epsilon|||_2$ are uniformly bounded, we obtain

$$|(Lu_\epsilon, w)| \leqslant M\left\{\int_{-\infty}^{\infty} \|w(t)\|_{V_2}^2\, dt\right\}^{\frac{1}{2}}.$$

Hence

$$\int_{-\infty}^{\infty} \|Lu_\epsilon\|_{V_2^*}^2\, dt \leqslant M.$$

$D(L)$ with the graph norm is a Banach space and the injection mapping of $D(L)$, considered as a Banach space with the graph norm into

$$L^2([0, T], W^{m_2-1,2}(\Omega)),$$

is compact; cf. **(1)**. So $\epsilon u_\epsilon \to 0$ in $L^2([0, T], W^{m_2-1,2}(\Omega))$ as $\epsilon \to 0$. Let

$$|||w|||_\epsilon = \left\{\int_{-\infty}^{\infty} \|w(t)\|_{V_1}^2\, dt + \epsilon^2 \int_{-\infty}^{\infty} \|w(t)\|_{V_2}^2\, dt\right\}^{\frac{1}{2}} + \left\{\int_{-\infty}^{\infty} \|Lw\|_{V_2^*}^2\, dt\right\}^{\frac{1}{2}}.$$

$D(L)$ with the $|||\ \ |||$-norm is a Banach space and the injection mapping of $D(L)$, considered as a Banach space with the $|||\ \ |||_\epsilon$-norm into

$$L^2([0, T], W^{m_1-1,2}(\Omega))$$

is compact. Thus $u_\epsilon \to v$ in $L^2([0, T], W^{m_1-1,2}(\Omega))$ as $\epsilon \to 0$. By taking a subsequence if necessary, we may assume that

$$D^\alpha u_\epsilon \to D^\alpha v \text{ a.e. on } [0, T] \times \Omega \qquad \text{for } |\alpha| \leqslant m_1 - 1,$$
$$\epsilon D^\alpha u_\epsilon \to 0 \quad \text{a.e. on } [0, T] \times \Omega \qquad \text{for } |\alpha| \leqslant m_2 - 1.$$

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Consider $h_\epsilon(u_\epsilon, w) = h_2(\epsilon u_\epsilon, w) + h_1(u_\epsilon, w)$. We have

$$h_1(u_\epsilon, w) = \sum_{|\alpha|,|\beta|\leqslant m_1} \int_0^T \langle a_{1\alpha\beta}(x, t, u_\epsilon, \ldots, D^{m_1-1}u_\epsilon)D^\alpha u_\epsilon, D^\beta w\rangle dt$$
$$= \sum_{|\alpha|,|\beta|\leqslant m_1} \int_0^T \langle D^\alpha u_\epsilon, \bar{a}_{1\alpha\beta}(x, t, u_\epsilon, \ldots, D^{m_1-1}u_\epsilon)D^\beta w\rangle dt.$$

From the Assumption (I') on the coefficients $a_{1\alpha\beta}$, we deduce that

$$\bar{a}_{1\alpha\beta}(x, t, u_\epsilon, \ldots, D^{m_1-1}u_\epsilon)D^\beta w \to \bar{a}_{1\alpha\beta}(x, t, v, \ldots, D^{m_1-1}v)D^\beta w$$

a.e. on $[0, T] \times \Omega$. Moreover

$$|\bar{a}_{1\alpha\beta}(x, t, u_\epsilon, \ldots, D^{m_1-1}u_\epsilon)D^\beta w| \leqslant g_1(|||u_\epsilon|||_1)|D^\beta w| \leqslant M|D^\beta w|.$$

From the Lebesgue bounded convergence theorem, it follows that

$$\bar{a}_{1\alpha\beta}(x, t, u_\epsilon, \ldots, D^{m_1-1}u_\epsilon)D^\beta w \to \bar{a}_{1\alpha\beta}(x, t, v, \ldots, D^{m_1-1}v)D^\beta w$$

in $L^2([0, T], L^2(\Omega))$ as $\epsilon \to 0$. Since $D^\alpha u_\epsilon \to D^\alpha v$ weakly in $L^2([0, T], L^2(\Omega))$, we obtain $h_1(u_\epsilon, w) \to h_1(v, w)$ for all w in $L^2([0, T], V)$. A similar argument holds for $h_2(\epsilon u_\epsilon, w)$, yielding $\lim_{\epsilon\to 0} h_2(\epsilon u_\epsilon, w) \to 0$. Hence $h_\epsilon(u_\epsilon, w) \to h_1(v, w)$ for w in $L^2([0, T], V)$ as $\epsilon \to 0$.

On the other hand, we have by definition

$$(L_2 u_\epsilon, w) + h_\epsilon(u_\epsilon, w) = ((f, w)).$$

From Lemma 3.1, we have

$$(L_2 u_\epsilon, w) = ((L_2^\# u_\epsilon, w)) = ((L_1^\# u_\epsilon, w)).$$

Therefore

$$((L_1^\# u_\epsilon, w)) \to ((f, w)) - h_1(v, w) \qquad \text{as } \epsilon \to 0$$

for $w \in L^2([0, T], V)$.

Since $L_1^\#$ is weakly closed, it follows that

$$((L_1^\# v, w)) + h_1(v, w) = ((f, w)) \qquad \text{for all } w \text{ in } L^2([0, T], V).$$

By hypothesis, V is dense in V_1; hence

$$((L_1^\# v, w)) + h_1(v, w) = ((f, w)) \qquad \text{for all } w \text{ in } L^2([0, T], V_1).$$

Now $v \in D(L_1)$, for $D(L_1) = D(L_1^\#)$, and moreover $(L_1 v, w) = ((L_1^\# v, w))$. Hence

$$(L_1 v, w) + h_1(v, w) = ((f, w)) \qquad \text{for all } w \text{ in } L^2([0, T], V_1).$$

We deduce from Theorem 3.1 that $v = u_0$.

It remains to show that $u_\epsilon \to u_0$ in $L^2([0, T], W^{m_1,2}(\Omega))$ and $\epsilon u_\epsilon \to 0$ in $L^2([0, T], W^{m_2,2}(\Omega))$ as $\epsilon \to 0$.

First, we note that

$$c_1(|||u_\epsilon - u_0|||_1)|||u_\epsilon - u_0|||_1 \leqslant \operatorname{Re}\{h_1(u_\epsilon, u_\epsilon - u_0) - h_1(u_0, u_\epsilon - u_0)\}$$
$$\leqslant \operatorname{Re}\{h_1(u_\epsilon, u_\epsilon) - h_1(u_\epsilon, u_0) + h_1(u_0, u_0) - h_1(u_0, u_\epsilon)\}$$

and

$$c_2(\epsilon|||u_\epsilon|||_2)|||u_\epsilon|||_2 \leqslant \operatorname{Re}\{h_2(\epsilon u_\epsilon, u_\epsilon) + h_1(u_\epsilon, u_\epsilon) - h_1(u_\epsilon, u_\epsilon)\}.$$

So $c_2(\epsilon|||u_\epsilon|||_2)|||u_\epsilon|||_2 + c_1(|||u_\epsilon - u_0|||_1)|||u_\epsilon - u_0|||_1$ is majorized by

$$\operatorname{Re}\{h_\epsilon(u_\epsilon, u_\epsilon) - h_1(u_\epsilon, u_0) - h_1(u_0, u_\epsilon) + h_1(u_0, u_0)\}.$$

The last expression is equal to

$$\operatorname{Re}\{((f, u_\epsilon)) - (Lu_\epsilon, u_\epsilon) + h_1(u_0, u_0) - h_1(u_\epsilon, u_0) - h_1(u_0, u_\epsilon)\}.$$

But $L^* = -L$, so that $\operatorname{Re}(Lu_\epsilon, u_\epsilon) = 0$. Hence

$$c_1(|||u_\epsilon - u_0|||_1)|||u_\epsilon - u_0|||_1 + c_2(\epsilon|||u_\epsilon|||_2)|||u_\epsilon|||_2 \leqslant \operatorname{Re}\{((f, u_\epsilon)) + h_1(u_0, u_0) - h_1(u_\epsilon, u_0) - h_1(u_0, u_\epsilon)\}.$$

As $\epsilon \to 0$, the right-hand side of the inequality tends to

$$\operatorname{Re}\{((f, u_0)) - h_1(u_0, u_0)\}.$$

Let L' be the operator corresponding to L, involved in the definition of the global variational boundary problem for the equation $\partial u/\partial t + A_1(t)u = f$ on $R^1 \times \Omega$. Then, as for L, we have $(L')^* = -L'$. We obtain

$$\operatorname{Re}\{((f, u_0)) - h_1(u_0, u_0)\} = \operatorname{Re}(L'u_0, u_0) = 0.$$

Since $c_1(r)$ and $c_2(r)$ are two non-negative functions, we obtain

$$\lim |||u_\epsilon - u_0|||_1 c_1(|||u_\epsilon - u_0|||_1) = 0, \qquad \lim |||u_\epsilon|||_2 c_2(\epsilon|||u_\epsilon|||_2) = 0.$$

If $|||u_\epsilon - u_0||| \geqslant \eta > 0$ for $\epsilon \geqslant +0$, then $0 < c_1(\eta) \leqslant 0$, which is impossible. Hence $u_\epsilon \to u_0$ in $L^2([0, T], W^{m_1,2}(\Omega))$ as $\epsilon \to 0$. A similar argument shows that $\epsilon u_\epsilon \to 0$ in $L^2([0, T], W^{m_2,2}(\Omega))$ as $\epsilon \to 0$.

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