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PRODUCT FRACTAL SETS DETERMINED BY STABLE PROCESSES

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Abstract

Let X_i be transient β_i-stable processes on $\mathbb{R}^{d_i}$, $i = 1, 2$. Assume further that X_1 and X_2 are independent. We shall find the exact Hausdorff measure function for the product sets $R_1(1) \times R_2(1)$, where $R_1(1) \times R_2(1) = \{(X_1(t_1), X_2(t_2)) \mid 0 \leq t_1, t_2 \leq 1\}$. The result of Hu generalizes [Some fractal sets determined by stable processes, *Probab. Theory Related Fields* **100** (1994), 205–225].

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1. Introduction

A Lévy process $\{X(t) \mid t \geq 0\}$ on $\mathbb{R}^d$ is called an α-stable process with $\alpha \in (0, 2]$ if the distribution of $X(1)$ is not degenerate (that is, it cannot be supported on any proper subspace of $\mathbb{R}^d$) and for any $t > 0$,

$$X(t) = t^{1/\alpha} X(1)$$

in law. An α-stable process on $\mathbb{R}^d$ is transient if and only if $\alpha < d$. It is well known that $X(1)$ has a bounded continuous density $p(1, x)$ (see [5]). An α-stable process on $\mathbb{R}^d$ is said to be of type A if $p(1, 0) > 0$; and type B otherwise. If an α-stable process with $\alpha \neq 1$ is of type B, then $0 < \alpha < 1$.

Before we give the main result, we recall briefly the definition of the Hausdorff measure function by referring to Falconer [1].

A function ϕ is said to belong to the class Φ if there exists a $\delta > 0$ such that ϕ is a right continuous and increasing function on $(0, \delta)$ with $\phi(0+) = 0$ and there exists a finite constant $K > 0$ such that

$$\frac{\phi(2s)}{\phi(s)} \leq K, \quad \text{for } 0 < s < \frac{1}{2}\delta.$$

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For $\phi \in \Phi$, the ϕ-Hausdorff measure of $E \subseteq \mathbb{R}^d$ is defined by

$$\phi\text{-}m(E) = \lim_{\varepsilon \to 0} \inf \left\{ \sum_{i=1}^{\infty} \phi(\mathrm{diam}(E_i)) \;\middle|\; E \subseteq \bigcup_{i=1}^{\infty} E_i,\ \mathrm{diam}(E_i) < \varepsilon \right\},$$

where $\mathrm{diam}(E_i)$ denotes the diameter of E_i. A function $\phi \in \Phi$ is called an exact Hausdorff measure function for E if $0 < \phi\text{-}m(E) < \infty$.

We recall some previous results concerning Hausdorff measure related to stable processes. It was proved in [5] that for a transient α-stable process X with $\alpha \neq 1, 2$, an exact Hausdorff measure function of $R(t)$ is $\phi(a) = a^{\alpha} \log\log \frac{1}{a}$ if $X(t)$ is of type A or $\phi(a) = a^{\alpha}(\log\log \frac{1}{a})^{1-\alpha}$ if X is of type B, where $R(t) = \{X(s) \mid s \in [0, t]\}$. Then, in 1994, the product of range of two independent stable subordinators (or one-sided stable processes) was considered in [2]. Specifically, it was shown that

$$\phi(a) = a^{\beta_1+\beta_2}\left(\log\log \frac{1}{a}\right)^{2-\beta_1-\beta_2}$$

is an exact Hausdorff measure function for the product set

$$R_1(1) \times R_2(1) = \{(X_1(t_1), X_2(t_2)) \mid 0 \le t_1, t_2 \le 1\},$$

where X_i are independent β_i-stable subordinators on $\mathbb{R}$ with $0 < \beta_i < 1$, $i = 1, 2$.

In this paper, we consider the more general case by a different method. We aim to find the exact Hausdorff measure function for $R_1(1) \times R_2(1)$, where X_i are independent transient β_i-stable processes on $\mathbb{R}^{d_i}$ with $\beta_i \in (0, 2)$ and $\beta_i \neq 1$, $i = 1, 2$. The main result is the following theorem.

THEOREM 1.1. *Let X_i be transient β_i-stable processes on $\mathbb{R}^{d_i}$ with $\beta_i \neq 1, 2$, $i = 1, 2$. Assume that X_1 and X_2 are independent and let $\phi_i(a) = a^{\beta_i} \log\log \frac{1}{a}$ if X_i is of type A or $\phi_i(a) = a^{\beta_i}(\log\log \frac{1}{a})^{1-\beta_i}$ if X_i is of type B. Then, with probability 1,*

$$0 < \phi\text{-}m(R_1(1) \times R_2(1)) < \infty,$$

where $\phi(a) = \phi_1(a)\phi_2(a)$.

We note that any stable subordinator on $\mathbb{R}$ with index $0 < \alpha < 1$ is of type B (see, for example, [5]). Therefore the result in [2] is a special case of Theorem 1.1. The proof of Theorem 1.1 is divided into two parts. In Section 2 we prove the lower bound and in Section 3 we prove the upper bound for $\phi\text{-}m(R_1(1) \times R_2(1))$. Though our result is stated for two independent stable processes, its method is valid for finitely many independent stable processes. Throughout this paper, we use $c_1, c_2, \ldots$ to denote positive finite constants whose values may or may not be known.

2. Lower bound for ϕ-$m(R_1(1) \times R_2(1))$

We start with the following lemma. It can be easily derived from the results in [4], which gives a way to get a lower bound for ϕ-$m(E)$. For any Borel measure μ on $\mathbb{R}^d$ and $\phi \in \Phi$, the upper ϕ-density of μ at $x \in \mathbb{R}^d$ is defined by

$$\overline{D}_{\mu}^{\phi}(x) = \limsup_{r\to 0} \frac{\mu(B(x, r))}{\phi(2r)},$$

where $B(x, r)$ denotes the closed ball with radius r and center x.

LEMMA 2.1. *For a given $\phi \in \Phi$, there exists a positive constant C_1 such that for any Borel measure μ on $\mathbb{R}^d$ and every Borel set $B \subseteq \mathbb{R}^d$,*

$$\phi\text{-}m(E) \geq C_1 \mu(E) \cdot \inf_{x\in E} \frac{1}{\overline{D}_{\mu}^{\phi}(x)}.$$

We now give the proof for the lower bound for ϕ-$m(R_1(1) \times R_2(1))$ in Theorem 1.1.

PROOF. Define the random Borel measure μ on $\mathbb{R}^{d_1+d_2}$ and μ_i on $\mathbb{R}^{d_i}$ with $i = 1, 2$ by

$$\mu(B) = \int_0^1 \int_0^1 \mathbb{I}_B(X_1(t_1), X_1(t_2))\, dt_1\, dt_2, \qquad B \subseteq \mathbb{R}^{d_1+d_2};$$
$$\mu_i(B_i) = \int_0^1 \mathbb{I}_{B_i}(X_i(t_i))\, dt_i, \qquad B_i \subseteq \mathbb{R}^{d_i}, i = 1, 2,$$

where $\mathbb{I}_B$ is the indicator function of the set B. For any fixed $(s_1, s_2) \in [0, 1]^2$,

$$\begin{aligned} \limsup_{r\to 0} & \frac{\mu(B((X_1(s_1), X_2(s_2)), r))}{\phi(r)} \\ & \leq \limsup_{r\to 0} \frac{\mu(B_1(X_1(s_1), r) \times B_2(X_2(s_2), r))}{\phi(r)} \\ & \leq \limsup_{r\to 0} \frac{\mu_1(B_1(X_1(s_1), r))}{\phi_1(r)} \limsup_{r\to 0} \frac{\mu_2(B_2(X_2(s_2), r))}{\phi_2(r)}, \end{aligned} \tag{2.1}$$

where $B((X_1(s_1), X_2(s_2)), r)$ denotes the closed ball of radius r and center $(X_1(s_1), X_2(s_2))$, while $B_i(X_i(s_i), r)$ denotes the closed ball of radius r and center $X_i(s_i)$, $i = 1, 2$. Define

$$\overline{Y}_i(t) = \begin{cases} X_i(s_i) - X_i(s_i - t) & \text{if } 0 \leq t < s_i, \\ X_i(t) & \text{if } t \geq s_i, \end{cases}$$

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and

$$Y_i(t) = X_i(s_i + t) - X_i(s_i), \quad t \geq 0.$$

Then $\overline{Y}_i$ and Y_i are β_i-stable processes, $i = 1, 2$. By (2.1),

$$\begin{aligned}
&\limsup_{r\to 0} \frac{\mu(B((X_1(s_1), X_2(s_2)), r))}{\phi(r)} \\
&\quad \leq \left(\limsup_{r\to 0} \frac{\overline{T}_1(r)}{\phi_1(r)} + \limsup_{r\to 0} \frac{T_1(r)}{\phi_1(r)}\right) \cdot \left(\limsup_{r\to 0} \frac{\overline{T}_2(r)}{\phi_2(r)} + \limsup_{r\to 0} \frac{T_2(r)}{\phi_2(r)}\right),
\end{aligned}$$

where $\overline{T}_i(r)$ and $T_i(r)$ are the sojourn times of $\overline{Y}_i$ and Y_i in the closed ball $B_i(0, r) \subseteq \mathbb{R}^{d_i}$ respectively, $i = 1, 2$. Applying [5, Theorems 4 and 5], it follows that there exists a constant K_1 such that with probability 1,

$$\limsup_{r\to 0} \frac{\mu(B((X_1(s_1), X_2(s_2)), r))}{\phi(r)} \leq K_1. \tag{2.2}$$

Let

$$E = \{(X_1(s_1), X_2(s_2)) \mid s_1, s_2 \in [0, 1] \text{ and (2.2) holds}\}.$$

Then

$$\begin{aligned}
\mathbf{E}\mu(E) &= \mathbf{E} \int_0^1 \int_0^1 \mathbb{I}_E(X_1(s_1), X_2(s_2))\, ds_1\, ds_2 \\
&= \int_0^1 \int_0^1 \mathbf{P}\{(X_1(s_1), X_2(s_2)) \in E\}\, ds_1\, ds_2 \\
&= 1,
\end{aligned}$$

which implies that $\mu(E) = 1$ almost surely. By Lemma 2.1, $\phi\text{-}m(E) \geq C_1/K_1 > 0$ almost surely. Since $E \subseteq R_1(1) \times R_2(1)$, then with probability 1,

$$\phi\text{-}m(R_1(1) \times R_2(1)) \geq \phi\text{-}m(E) > 0.$$

That completes the proof for the lower bound. □

3. Upper bound for ϕ-$m(R_1(1) \times R_2(1))$

Before we give the proof for the upper bound, we prove an important lemma.

LEMMA 3.1. *Under the condition of Theorem 1.1, put* $P_i(a) = \inf\{t : \|X_i(t)\| \geq a\}$, *where* $i = 1, 2$. *Then there are positive constants* K_2, K_3, γ_0 *such that*

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$$\mathbf{P}\left(\sup_{\gamma\le a\le\delta}\frac{P_1(a)P_2(a)}{\phi(a)}\le K_2\right)<\exp\left\{-K_3\left(\log\frac{1}{\gamma}\right)^{1/8}\right\}$$

provided that $0<\gamma\le\gamma_0$ *and* $\delta\ge\gamma^{1/5}$.

PROOF. We only consider the case where $X_1(t)$ is of type A and $X_2(t)$ is of type B, the proofs for the other cases being similar. By [5, Lemmas 5 and 6] it can be seen directly that there exist positive constants c_3, c_4, λ_0 such that for $0<\lambda<\lambda_0$,

$$\mathbf{P}\left\{\sup_{0\le t\le\tau}|X_1(t)|\le\tau^{1/\beta_1}\lambda\right\}\ge\exp(-c_3\lambda^{-\beta_1}) \tag{3.1}$$

and

$$\mathbf{P}\left\{\sup_{0\le t\le\tau}|X_2(t)|\le\tau^{1/\beta_2}\lambda\right\}\ge\exp\{-c_4\lambda^{-\beta_2/(1-\beta_2)}\}. \tag{3.2}$$

We consider the sequence

$$a_k=\exp\,(-k^2),\qquad k=1,2,\ldots$$

which tends to zero very rapidly as $k\to\infty$. Put

$$t_{1,k}=\phi_1(a_k)=a_k^{\beta_1}\log\log\frac{1}{a_k}$$

and

$$t_{2,k}=\phi_2(a_k)=a_k^{\beta_2}\left(\log\log\frac{1}{a_k}\right)^{1-\beta_1}.$$

Let $c_1=(6c_3)^{1/\beta_1}$ and $c_2=(6c_4)^{(1-\beta_2)/\beta_2}$. For any $t\ge 0$, let $Y_i=2c_iX_i((2c_i)^{-\beta_i}t)$ with $i=1,2$. Then $Y_i(t)=X_i(t)$ in law for $i=1,2$. Therefore $\{Y_i(t),\,t\ge 0\}$ is still a β_i-stable process on $\mathbb{R}^{d_i}$, $\{Y_1(t)\}$ is of type A, $\{Y_2(t)\}$ is of type B, and they are also independent. Further, for $0<\lambda<\lambda_0$, (3.1) and (3.2) hold respectively for Y_1 and Y_2.

For any $k\ge 1$, let

$$\begin{aligned}
D_{i,k}&=\left\{\sup_{0\le t\le t_{i,k}}|Y_i(t)|\ge 2c_ia_k\right\},\\
G_{i,k}&=\left\{\sup_{t_{i,k+1}\le t\le t_{i,k}}|Y_i(t)-Y_i(t_{i,k+1})|\ge c_ia_k\right\},\\
H_{i,k}&=\left\{\sup_{0\le t\le t_{i,k+1}}|Y_i(t)|>c_ia_k\right\}.
\end{aligned}$$

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Then $D_{i,k} \subset G_{i,k} \cup H_{i,k}$ with $i = 1, 2$. Consequently,

$$\begin{aligned}
\mathbf{P}\bigg\{\bigcap_{k=m+1}^{2m}(D_{1,k} \cup D_{2,k})\bigg\} &\le \mathbf{P}\bigg\{\bigcap_{k=m+1}^{2m}[(G_{1,k} \cup H_{1,k}) \cup (G_{2,k} \cup H_{2,k})]\bigg\} \\
&= \mathbf{P}\bigg\{\bigcap_{k=m+1}^{2m}[(G_{1,k} \cup G_{2,k}) \cup (H_{1,k} \cup H_{2,k})]\bigg\} \\
&\le \mathbf{P}\bigg\{\bigcap_{k=m+1}^{2m}(G_{1,k} \cup G_{2,k}) \cup \bigg(\bigcup_{k=m+1}^{2m}(H_{1,k} \cup H_{2,k})\bigg)\bigg\} \\
&\le \prod_{k=m+1}^{2m}\mathbf{P}(G_{1,k} \cup G_{2,k}) + \sum_{k=m+1}^{2m}\mathbf{P}(H_{1,k} \cup H_{2,k}),
\end{aligned}$$

where the events $\{G_{1,k} \cup G_{2,k} \mid k \ge 1\}$ are independent.

Put $\mathbf{P}(G_{i,k}) = 1 - p_{i,k}$ and $\mathbf{P}(H_{i,k}) = q_{i,k}$ with $i = 1, 2$. Then by (3.1), for sufficiently large k,

$$\begin{aligned}
p_{1,k} &\ge \mathbf{P}\Big(\sup_{0 \le t \le t_{1,k}} |Y_1(t)| < c_1 a_k\Big) \\
&= \mathbf{P}\Big(\sup_{0 \le t \le t_{1,k}} |Y_1(t)| < t_{1,k}^{1/\beta_1} c_1 a_k t_{1,k}^{-1/\beta_1}\Big) \\
&\ge \exp\{-c_3[c_1 a_k t_{1,k}^{-1/\beta_1}]^{-\beta_1}\} \\
&= k^{-1/3}.
\end{aligned}$$

Simultaneously, by [3, Lemma 4.3] and [5, Lemma 7], for sufficiently large k,

$$\begin{aligned}
q_{1,k} &= \mathbf{P}\Big(\sup_{0 \le t \le t_{1,k+1}} |Y_1(t)| > c_1 a_k t_{1,k+1}^{-1/\beta_1} t_{1,k+1}^{1/\beta_1}\Big) \\
&\le 2d_1 \mathbf{P}\Big(|Y_1(t_{1,k+1})| > \frac{c_1}{2d_1} a_k t_{1,k+1}^{-1/\beta_1} t_{1,k+1}^{1/\beta_1}\Big) \\
&\le 2d_1 c_5 \bigg(\frac{c_1 a_k t_{1,k+1}^{-1/\beta_1}}{2d_1}\bigg)^{-\beta_1} \\
&= c_6 \phi_1(a_{k+1}) a_k{}^{-\beta_1} \\
&= c_6 \exp\{-(k+1)^2 \beta_1\} \log(k+1)^2 \exp(k^2 \beta_1) \\
&\le \exp(-k\beta_1).
\end{aligned}$$

Similarly, for the β_2-stable type B process Y_2, by (3.2), [3, Lemma 4.3] and [5, Lemma 7], we obtain, for sufficiently large k,

$$p_{2,k} > k^{-1/3}, \quad q_{2,k} < \exp(-k\beta_2).$$

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Thus there exists m_0 such that for $m > m_0$,

$$\begin{aligned}
&\mathbf{P}\bigg\{\bigcap_{k=m+1}^{2m}(D_{1,k}\cup D_{2,k})\bigg\}\\
&\le \prod_{k=m+1}^{2m}\mathbf{P}(G_{1,k}\cup G_{2,k})+\sum_{k=m+1}^{2m}\mathbf{P}(H_{1,k}\cup H_{2,k})\\
&= \prod_{k=m+1}^{2m}(1-p_{1,k}p_{2,k})+\sum_{k=m+1}^{2m}(q_{1,k}+q_{2,k})\\
&\le \exp\bigg(-\sum_{k=m+1}^{2m}p_{1,k}p_{2,k}\bigg)+\sum_{k=m+1}^{\infty}\exp(-k\beta_1)+\sum_{k=m+1}^{\infty}\exp(-k\beta_2)\\
&\le \exp(-2^{-2/3}m^{1/3})+c_7\exp(-\beta_1 m)+c_8\exp(-\beta_2 m)\\
&\le \exp(-m^{1/4}).
\end{aligned}$$

Recall that $P_i(a)=\inf\{t\ge 0:|X_i(t)|\ge a\}$ with $i=1,2$. Then

$$\begin{aligned}
&\mathbf{P}\bigg\{\bigcap_{k=m+1}^{2m}(D_{1,k}\cup D_{2,k})\bigg\}\\
&=\mathbf{P}\bigg\{\bigcap_{k=m+1}^{2m}\bigg[\bigg(\sup_{0\le t\le t_{1,k}}|Y_1(t)|\ge 2c_1a_k\bigg)\cup\bigg(\sup_{0\le t\le t_{2,k}}|Y_2(t)|\ge 2c_2a_k\bigg)\bigg]\bigg\}\\
&=\mathbf{P}\bigg\{\bigcap_{k=m+1}^{2m}\bigg[\bigg(\sup_{0\le t\le (2c_1)^{-\beta_1}t_{1,k}}|X_1(t)|\ge a_k\bigg)\cup\bigg(\sup_{0\le t\le (2c_2)^{-\beta_2}t_{2,k}}|X_2(t)|\ge a_k\bigg)\bigg]\bigg\}\\
&=\mathbf{P}\bigg\{\bigcap_{k=m+1}^{2m}[(P_1(a_k)\le (2c_1)^{-\beta_1}t_{1,k})\cup(P_2(a_k)\le (2c_2)^{-\beta_2}t_{2,k})]\bigg\}\\
&\ge\mathbf{P}\bigg\{\bigcap_{k=m+1}^{2m}\bigg[\frac{P_1(a_k)P_2(a_k)}{\phi_1(a_k)\phi_2(a_k)}\le (2c_1)^{-\beta_1}(2c_2)^{-\beta_2}\bigg]\bigg\}\\
&\ge\mathbf{P}\bigg\{\sup_{a_{2m}\le a\le a_m}\frac{P_1(a)P_2(a)}{\phi(a)}\le (2c_1)^{-\beta_1}(2c_2)^{-\beta_2}\bigg\}.
\end{aligned}$$

Therefore

$$\mathbf{P}\bigg(\sup_{a_{2m}\le a\le a_m}\frac{P_1(a)P_2(a)}{\phi(a)}\le K_2\bigg)\le\exp(-m^{1/4}),$$

where $K_2=(2c_1)^{-\beta_1}(2c_2)^{-\beta_2}$. Choose $\gamma_0>0$ such that

$$\frac{1}{2}\sqrt{\log\frac{1}{\gamma_0}}-1>\frac{1}{\sqrt{5}}\sqrt{\log\frac{1}{\gamma_0}}>m_0.$$

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For any $0 < \gamma \le \gamma_0$ and $\delta \ge \gamma^{1/5}$,

$$\left(\frac{1}{2} - \frac{1}{\sqrt{5}}\right)\sqrt{\log \frac{1}{\gamma}} \ge \left(\frac{1}{2} - \frac{1}{\sqrt{5}}\right)\sqrt{\log \frac{1}{\gamma_0}} > 1,$$

and hence there is a positive integer

$$m \in \left(\frac{1}{\sqrt{5}}\sqrt{\log \frac{1}{\gamma}}, \frac{1}{2}\sqrt{\log \frac{1}{\gamma}}\right).$$

It follows that

$$m > \frac{1}{\sqrt{5}}\sqrt{\log \frac{1}{\gamma}} \ge \frac{1}{\sqrt{5}}\sqrt{\log \frac{1}{\gamma_0}} > m_0$$

and $\gamma < a_{2m} < a_m < \gamma^{1/5} \le \delta$. Thus

$$\begin{aligned}\mathbf{P}\left(\sup_{\gamma \le a \le \delta} \frac{P_1(a)P_2(a)}{\phi(a)} \le K_2\right) &\le \mathbf{P}\left(\sup_{a_{2m} \le a \le a_m} \frac{P_1(a)P_2(a)}{\phi(a)} \le K_2\right) \\ &\le \exp(-m^{1/4}) < \exp\left\{-K_3\left(\log \frac{1}{\gamma}\right)^{1/8}\right\},\end{aligned}$$

where $K_3 = \frac{1}{\sqrt[8]{5}}$. The lemma is proved. □

We may actually prove that Lemma 3.1 holds for finitely many independent transient stable processes. The proof of Lemma 3.1 has a direct consequence.

COROLLARY 3.1. *Under the conditions of Theorem 1.1, let*

$$T_i(a, 1) = \int_0^1 \mathbb{I}_{B_i(0,a)}(X_i(t))\, dt$$

be the sojourn time of X_i in the closed ball $B_i(0, a)(\subset \mathbb{R}^{d_i})$ up to time 1. Then there exist positive constants K_2, K_3, γ_0 such that

$$\mathbf{P}\left(\sup_{\gamma \le a \le \delta} \frac{T_1(a, 1)T_2(a, 1)}{\phi(a)} \le K_2\right) < \exp\left\{-K_3\left(\log \frac{1}{\gamma}\right)^{1/8}\right\}$$

provided that $0 < \gamma \le \gamma_0$ and $\delta \ge \gamma^{1/5}$.

Now we introduce another lemma, which is exactly [5, Lemma 9].

LEMMA 3.2. *If $E = \bigcup_{i=1}^m I_i$, where each I_i is an interval of Λ_k for some integer k. Here Λ_k is the collection of cubes of side 2^{1-k} and center at a lattice point $(j_1/2^k, j_2/2^k, \ldots, j_d/2^k)$, where j_l are integers, closed on the left and open on right. Then we can find a subset $\{j_r\}$ such that $E = \bigcup I_{j_r}$ and no point of E is in more than 2^d of the intervals I_{j_r}.*

We now come to the proof of the upper bound for ϕ-$m(R_1(1)\times R_2(1))$ in Theorem 1.1.

PROOF. Let $\Lambda_n^{(i)}$ be the collection of cubes closed on the left and open on right of side 2^{1-n} with centers at a lattice point $(j_1/2^n, j_2/2^n, \ldots, j_{d_i}/2^n)$ where the j_l are integers, $i=1, 2$. Consider $\overline{\Lambda}_n^{(i)}$, the collection of cubes of side 2^{-n} and centers the same as those of $\Lambda_n^{(i)}$, $i=1,2$. Put $\Lambda_n=\Lambda_n^{(1)}\times\Lambda_n^{(2)}$ and $\overline{\Lambda}_n=\overline{\Lambda}_n^{(1)}\times\overline{\Lambda}_n^{(2)}$. Suppose $\delta=2^{-r}$ is given where r is a positive integer, and $\gamma_n=2^{-n}\le\min\{\gamma_0, 2^{-5r}\}$. We say that a cube $\overline{I}_{i,n}=\overline{I}_{i,n}^{(1)}\times\overline{I}_{i,n}^{(2)}$ of $\overline{\Lambda}_n$ is bad if the following conditions hold.

(1) $R_1(1)\times R_2(1)$ meets $\overline{I}_{i,n}$, where $R_j(1)$ meets $\overline{I}_{i,n}^{(j)}$ at $\tau_j\le 1$ with $j=1, 2$. In detail, for $j=1,2$, $\tau_j=\inf\{t\ge 0\mid X_j(t)\in\overline{I}_{i,n}^{(j)}\}\le 1$.

(2) For all a satisfying $\gamma_n\le a\le\delta$,

$$\int_{\tau_1}^{\tau_1+1}\mathbb{I}_{B(X_1(\tau_1),a)}(X_1(t_1))\,dt_1\int_{\tau_2}^{\tau_2+1}\mathbb{I}_{B(X_2(\tau_2),a)}(X_2(t_2))\,dt_2\le K_2\phi(a),$$

where the closed ball $B(X_i(\tau_i), a)\in\mathbb{R}^{d_i}$ with $i=1,2$.

If (1) holds and (2) does not, then we say that $\overline{I}_{i,n}$ is good. For any cube $\overline{I}_{1,n}$ of $\overline{\Lambda}_n$,

$$\begin{aligned}
&\mathbf{P}(\overline{I}_{i,n}\text{ is bad }\mid 0\le\tau_1,\tau_2\le 1)\\
&=\mathbf{P}\bigg\{\sup_{\gamma_n\le a\le\delta}\frac{\int_{\tau_1}^{\tau_1+1}\mathbb{I}_{B(X_1(\tau_1),a)}(X_1(t_1))\,dt_1\int_{\tau_2}^{\tau_2+1}\mathbb{I}_{B(X_2(\tau_2),a)}(X_2(t_2))\,dt_2}{\phi(a)}\\
&\qquad\le K_2\mid 0\le\tau_1,\tau_2\le 1\bigg\}\\
&=\mathbf{P}\bigg\{\sup_{\gamma_n\le a\le\delta}\frac{\int_{\tau_1}^{\tau_1+1}\mathbb{I}_{B_1(0,a)}(X_1(t_1)-X_1(\tau_1))\,dt_1\int_{\tau_2}^{\tau_2+1}\mathbb{I}_{B_2(0,a)}(X_2(t_2)-X_2(\tau_2))\,dt_2}{\phi(a)}\\
&\qquad\le K_2\mid 0\le\tau_1,\tau_2\le 1\bigg\}\\
&=\mathbf{P}\bigg\{\sup_{\gamma_n\le a\le\delta}\frac{\int_0^1\mathbb{I}_{B_1(0,a)}(X_1(t_1+\tau_1)-X_1(\tau_1))\,dt\int_0^1\mathbb{I}_{B_2(0,a)}(X_2(t_2+\tau_2)-X_2(\tau_2))\,dt_2}{\phi(a)}\\
&\qquad\le K_2\mid 0\le\tau_1,\tau_2\le 1\bigg\},
\end{aligned}$$

where the closed ball $B_1(0,a)\in\mathbb{R}^{d_1}$ and $B_2(0,a)\in\mathbb{R}^{d_2}$. Put

$$Y_1(t)=X_1(t+\tau_1)-X_1(\tau_1),\quad Y_2(t)=X_2(t+\tau_2)-X_2(\tau_2),\quad t\ge 0.$$

Then Y_1, Y_2 are independent and have exactly the same law as X_1 and X_2 respectively by the strong Markovian property. Hence we may apply Corollary 3.1 to Y_1, Y_2 to obtain

$$\mathbf{P}(\overline{I}_{i,n}\text{ is bad }\mid 0\le\tau_1,\tau_2\le 1)<\exp(-c_9 n^{1/8}).$$

Let $M_{i,n}$ denote the number of cubes in $\overline{\Lambda}_n^{(i)}$ hit by the path $X_i(t)$ in one unit of time, $i = 1, 2$. Then by [3, Lemma 6.1] and the fact that X_1 is transient,

$$\begin{aligned}\mathbf{E}M_{1,n} &\le c_{10}\left[\mathbf{E}T\left(\frac{2^{-n}}{3}, 1\right)\right]^{-1}\\ &= c_{10}\left[\int_0^1 \mathbf{P}\left(|X_1(t)| \le \frac{2^{-n}}{3}\right) dt\right]^{-1}\\ &\le c_{10}\left[\frac{2^{-n}}{3}\right]^{-\beta_1}\left[\int_0^\infty \mathbf{P}(|X_i(t)| \le 1)\, dt\right]^{-1}\\ &\le c_{11}2^{n\beta_1}.\end{aligned}$$

Similarly, $\mathbf{E}M_{2,n} \le c_{12}2^{n\beta_2}$. Now we can deduce that N_n, the number of bad cubes $\overline{I}_{i,n}$, has expectation

$$\begin{aligned}\mathbf{E}N_n &\le \mathbf{E}M_{1,n}\mathbf{E}M_{2,n} \exp(-c_9 n^{1/8})\\ &\le c_{13}2^{n(\beta_1+\beta_2)} \exp(-c_9 n^{1/8}).\end{aligned}$$

Then, by the Markov inequality,

$$\mathbf{P}\{N_n \ge 2^{n(\beta_1+\beta_2)} \exp\,(-n^{1/10})\} < c_{14} \exp(-n^{1/10}).$$

Furthermore, we obtain

$$\sum_n \mathbf{P}\{N_n \ge 2^{n(\beta_1+\beta_2)} \exp\,(-n^{1/10})\} < \infty.$$

Applying the first Borel–Cantelli lemma, there exists Ω_0 such that $\mathbf{P}(\Omega_0) = 1$, and for all $\omega \in \Omega_0$ there exists an integer $n_1 = n_1(\omega)$ such that for $n \ge n_1$,

$$N_n \le 2^{n(\beta_1+\beta_2)} \exp(-n^{1/10}).$$

It is easy to obtain

$$\phi(d^{1/2}2^{-n}) \le c_{15}2^{-n(\beta_1+\beta_2)}(\log n)^2.$$

Thus for any $n \ge n_1$,

$$\sum_{\overline{I}_{i,n}:\text{bad}} \phi(\text{diam}(\overline{I}_{i,n})) = N_n\phi(d^{1/2}2^{-n}) \le c_{15} \exp\,(-n^{1/10})(\log n)^2. \tag{3.3}$$

Now consider the good squares $\overline{I}_{i,n} = \overline{I}_{i,n}^{(1)} \times \overline{I}_{i,n}^{(2)}$ of the mesh $\overline{\Lambda}_n^{(1)} \times \overline{\Lambda}_n^{(2)}$. We have to show that the set of all good squares can be covered economically. For each good square $\overline{I}_{i,n}$, there exist $a \in [\gamma_n, 2^{-r}]$ such that

$$\phi(a) < \frac{1}{K_2}\int_{\tau_1}^{\tau_1+1} \mathbb{I}_{B(X_1(\tau_1),a)}(X_1(t_1))\, dt_1 \int_{\tau_2}^{\tau_2+1} \mathbb{I}_{B(X_2(\tau_2),a)}(X_2(t_2))\, dt_2.$$

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Furthermore, we can find an integer k_i with $2^{-k_i} > 5a \geq 2^{-k_i-1}$ and a square $I_i = I_i^{(1)} \times I_i^{(2)}$ of Λ_{k_i} such that $I_i^{(1)}$ contains $\overline{I}_{i,n}^{(1)}$ and $B(X_1(\tau_1), a)$, while $I_i^{(2)}$ contains $\overline{I}_{i,n}^{(2)}$ and $B(X_2(\tau_2), a)$. Then $k_i > r - 4$ and

$$\begin{aligned}\phi(\operatorname{diam}(I_i)) &= \phi(\sqrt{d}2^{1-k_i}) \leq \phi(20\sqrt{d}a) \leq c_{16}\phi(a)\\ &\leq c_{17}\int_0^2 \mathbb{I}_{I_i^{(1)}}(X_1(t_1))\,dt_1 \int_0^2 \mathbb{I}_{I_i^{(2)}}(X_2(t_2))\,dt_2.\end{aligned}$$

Now $\bigcup_{\overline{I}_{i,n}:\text{good}} I_i$ is a finite collection of squares to which we can apply Lemma 3.2. Hence there is a subset, denoted by $\{I_i\}_{i\in\Lambda}$, which still covers all the good squares, but no point is covered more than 2^d times. For this subset, it must be the case that

$$\begin{aligned}\sum_{i\in\Lambda}\phi(\operatorname{diam}(I_i)) &\leq \sum_{i\in\Lambda} c_{17}\int_0^2 \mathbb{I}_{I_i^{(1)}}(X_1(t_1))\,dt_1 \int_0^2 \mathbb{I}_{I_i^{(2)}}(X_2(t_2))\,dt_2\\ &\leq c_{17}\int_0^2\int_0^2 \sum_{i\in\Lambda}\mathbb{I}_{I_i}((X_1(t_1)), X_2(t_2))\,dt_1\,dt_2\\ &\leq c_{17}2^{d+2}. \end{aligned} \tag{3.4}$$

Using all the bad squares together with this covering of the good squares, we obtain a covering of $R_1(1) \times R_2(1)$ by squares all of diameter less than $\sqrt{d}2^{-r+5}$, that is,

$$R_1(1) \times R_2(1) \subseteq \Big(\bigcup_{\overline{I}_{i,n}:\text{ bad}} \overline{I}_{i,n}\Big) \cup \Big(\bigcup_{i\in\Lambda} I_i\Big).$$

On the other hand, for sufficient large n, by (3.3) and (3.4),

$$\sum_{\overline{I}_{i,n}:\text{ bad}} \phi(\operatorname{diam}\overline{I}_{i,n}) + \sum_{i\in\Lambda}\phi(\operatorname{diam}(I_i)) < c_{17}2^{d+2} + 1.$$

Thus with probability 1,

$$\phi\text{-}m[R_1(1) \times R_2(1)] < \infty.$$

That completes the proof. □

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