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POSITIVE p-SUMMING OPERATORS, VECTOR MEASURES AND TENSOR PRODUCTS

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Introduction

In this paper we shall introduce a certain class of operators from a Banach lattice X into a Banach space B (see Definition 1) which is closely related to p-absolutely summing operators defined by Pietsch [**8**].

These operators, called positive p-summing, have already been considered in [**9**] in the case $p=1$ (there they are called cone absolutely summing, c.a.s.) and in [**1**] by the author who found this space to be the space of boundary values of harmonic B-valued functions in $h_B^p(D)$.

Here we shall use these spaces and the space of majorizing operators to characterize the space of bounded p-variation measures V_B^p and to endow the tensor product $L^p \otimes B$ with a norm in order to get $L^p(B)$ as its completion in this norm.

Some definitions and previous results

Throughout this paper X will denote a Banach lattice and B a Banach space. Given $1 \leqq p \leqq \infty$ we shall always write p' for such a number that $(1/p)+(1/p')=1$.

Definition 1. An operator T belonging to $L(X,B)$ is called *positive p-summing* $(1 \leqq p < \infty)$ if there exists a constant $C>0$ such that for all *positive* elements $x_1, x_2, \ldots, x_n$ in X we have

$$\left(\sum_{i=1}^{n} \|Tx_i\|_B^p\right)^{1/p} \leqq C \cdot \sup_{\|\xi\|_{X^*} \leqq 1} \left(\sum_{i=1}^{n} |\langle \xi, x_i \rangle|^p\right)^{1/p}. \tag{1}$$

We shall denote by $\Lambda_p(X,B)$ the space of such operators and the infimum of the constants will be the norm on it.

A duality argument allows us to write the following equivalent formulation of (1):

$$\left(\sum_{i=1}^{n} \|Tx_i\|_B^p\right)^{1/p} \leqq C \cdot \sup\left\{\left\|\sum_{i=1}^{n} \alpha_i \cdot x_i\right\|_X : \sum_{i=1}^{n} \alpha_i^{p'} \leqq 1, \alpha_i \geqq 0\right\}. \tag{1'}$$

Obviously the space of p-absolutely summing operators $\Pi_p(X,B)$ is included in $\Lambda_p(X,B)$ and the same techniques as for p-absolutely summing operators lead us to see that for $p \leqq q$, $\Lambda_p(X,B) \subseteq \Lambda_q(X,B)$ and

$$|T|_{\Lambda_q} \leqq |T|_{\Lambda_p} \quad \text{for all } T \text{ in } \Lambda_p(X,B). \tag{2}$$

Definition 2 (see [9]). An operator T belonging to $L(B,X)$ is called *majorizing* if there exists a constant $C>0$ such that for every $x_1, x_2, \dots, x_n$ in B

$$\left\| \sup_{1 \leqq i \leqq n} |Tx_i| \right\|_X \leqq C \cdot \sup_{1 \leqq i \leqq n} \|x_i\|_B. \tag{3}$$

We shall denote by $M(B,X)$ the space of such operators and we shall set the following norm on it:

$$|T|_m = \sup \left\{ \left\| \sup_{1 \leqq i \leqq n} |Tx_i| \right\|_X : \{x_i\} \in B, \|x_i\|_B \leqq 1 \right\}.$$

If we consider $A \otimes B$ as a subspace of $L(A^*, B)$, that is $u = \sum_{i=1}^n a_i \otimes b_i$ represents the operator T_u defined by $T_u(\xi) = \sum_{i=1}^n \langle \xi, a_i \rangle \cdot b_i$, then it is easy to see that $A \otimes B$ is included in $\Lambda_p(A^*, B)$ and $M(A^*, B)$. Let us denote by $A \hat{\otimes}_p B$ and $A \check{\otimes}_m B$ the completion of the space $A \otimes B$ endowed with the norms induced by $\Lambda_p(A^*, B)$ and $M(A^*, B)$ respectively.

Applications to tensor products and vector measures

Let $(\Omega, \mathscr{B}, \mu)$ be a finite measure space and $1 \leqq p < \infty$. We shall denote by $L^p(\mu, B)$ the space of measurable functions such that $\|f\|_p = (\int_\Omega \|f(t)\|^p \, d\mu)^{1/p} < +\infty$.

The following result can be found in [9].

$$L^p(\mu) \hat{\otimes}_1 B = L^p(\mu, B) \qquad 1 \leqq p < \infty. \tag{4}$$

This fact can be extended in the following way:

Theorem 1. *Let $1 \leqq p < \infty$, then for all $1 \leqq r \leqq p$*

$$L^p(\mu) \hat{\otimes}_r B = L^p(\mu, B).$$

Proof. Let $1 \leqq r \leqq p$. Since simple functions are dense in $L^p(\mu, B)$, it suffices to show that for each $s = \sum_{i=1}^n x_i \cdot \chi_{E_i}$ we have that the operator $T_s(\psi) = \int_\Omega s(t) \cdot \psi(t) \, d\mu(t)$ satisfies $|T_s|_{\Lambda_r} = \|s\|_p$.

Since $\|s\|_p = |T_s|_{\Lambda_1}$ and $|T_s|_{\Lambda_p} \leqq |T_s|_{\Lambda_r} \leqq |T_s|_{\Lambda_1}$ then it is enough to prove that $\|s\|_p \leqq |T_s|_{\Lambda_p}$.

$$\|s\|_p = \left(\sum_{i=1}^n \|x_i\|^p \cdot \mu(E_i) \right)^{1/p}$$

$$=\left(\sum_{i=1}^{n}\|T_s(\mu(E_i)^{-1}\cdot\chi_{E_i})\|^p\cdot\mu(E_i)\right)^{1/p}$$

$$=\left(\sum_{i=1}^{n}\|T_s(\mu(E_i)^{-1/p'}\cdot\chi_{E_i})\|^p\right)^{1/p}$$

$$\leqq|T_s|_{\Lambda_p}\cdot\sup\left\{\left\|\sum_{i=1}^{n}\alpha_i\cdot\mu(E_i)^{-1/p'}\cdot\chi_{E_i}\right\|_{L^{p'}}\sum_{i=1}^{n}\alpha_i^{p'}\leqq 1,\alpha_i\geqq 0\right\}$$

$$=|T_s|_{\Lambda_p}. \qquad \square$$

We can give another representation of $\Lambda_p(L^{p'}(\mu),B)$ in terms of vector measures.

Let us recall a space of B-valued measures, introduced by Bochner [2] in the scalar-valued case, which is a good substitute for $L^p(\mu,B)$ in several cases, for example for the duality $(L^p(\mu,B))^*=V^{p'}_{B^*}$ or for boundary values of functions in $h^p_B(D)$ [1].

Definition 3. A finitely additive vector measure $G:\mathscr{B}\to B$ is said to have bounded *p-variation* if

$$|G|_p=\sup_\pi\left\{\left(\sum_{E\in\pi}\frac{\|G(E)\|^p}{\mu(E)^{p-1}}\right)^{1/p}\right\}<+\infty \qquad (1<p<\infty) \tag{5}$$

where the "sup" is taken over all finite partitions of Ω and

$$|G|_\infty=\sup\left\{\frac{\|G(E)\|}{\mu(E)},E\in\mathscr{B}\right\}<+\infty \qquad (p=\infty). \tag{5'}$$

We shall denote by V^p_B the space of such measures and its norm is given by (5) or (5′) provided $1<p<\infty$ or $p=\infty$.

Let us recall some properties of this space.

(a) Every measure in V^p_B is countably additive, μ-continuous and with bounded variation.

(b) $L^p(\mu,B)$ is isometrically embedded in V^p_B.

Dinculeanu [4] characterized the space V^p_B in terms of $\mathscr{L}(L^{p'}(\mu),B)$, the space of operators in $L(L^{p'}(\mu),B)$ such that

$$|||T|||_p=\sup\left\{\sum_{i=1}^{n}|\alpha_i|\cdot\|T(\chi_{E_i})\|_B:\left\|\sum_{i=1}^{n}\alpha_i\cdot\chi_{E_i}\right\|_{L^{p'}}\leqq 1\right\}<+\infty.$$

The author proved in [1] that $\mathscr{L}(L^{p'}(\mu),B)=\Lambda_p(L^{p'}(\mu),B)$, hence we have the following:

Theorem 2. *For* $1<p\leqq\infty$, $\Lambda_p(L^{p'}(\mu),B)=V^p_B$.

Now we shall characterize V^p_B by means of the space of certain majorizing operators.

Theorem 3. *For* $1<p<\infty$, $M(B,L^p(\mu))=V^p_{B^*}$.

Proof. Let G be a measure of $V^p_{B^*}$ and take $x\in B$ with $\|x\|_B=1$. Consider now the measure $G_x(E)=\langle G(E),x\rangle$ for all measurable set E and the positive measure $|G|$. Both measures are countably additive, μ-continuous and with bounded variation. So, by the Radon–Nikodým theorem, there exist f_x and $g\geqq 0$ in $L^1(\mu)$ such that

$$G_x(E)=\int_E f_x(t)\,d\mu(t)\quad \text{for all } E\in\mathscr{B}, \tag{6}$$

$$|G|(E)=\int_E g(t)\,d\mu(t)\quad \text{for all } E\in\mathscr{B}. \tag{7}$$

It is not difficult to show, since G belongs to V^p_B, that f_x and g belong to $L^p(\mu)$ and moreover $\|g\|_p=|G|_p$ (see the argument in [1, Proposition 3]).

Due to (6) and (7) we have that

$$|G_x|(E)=\int_E |f_x(t)|\,d\mu(t)\leqq |G|(E)=\int_E g(t)\,d\mu(t)$$

and from this we obtain

$$|f_x(t)|\leqq |g(t)|\quad \mu\text{-a.e.} \tag{8}$$

Let us define $T:B\to L^p(\mu)$

$$y\mapsto T(y)=\|y\|_B\cdot f_{y/\|y\|_B}.$$

From (8) it is easy to show that $T\in M(B,L^p(\mu))$.

Indeed, if $x_1,x_2,\dots,x_n$ belong to B and $\|x_i\|_B=1$ then

$$\Big\|\sup_{1\leqq i\leqq n}|Tx_i|\Big\|_{L^p}\leqq \|g\|_p=|G|_p.$$

Conversely, given T in $M(B,L^p(\mu))$ and denoting by f_x the function Tx, we can define the measure $G:\mathscr{B}\to B^*$ by

$$\langle G(E),x\rangle=\int_E f_x(t)\,d\mu(t). \tag{9}$$

Now, let π be a partition of Ω. Given $\varepsilon>0$, for each $E\in\pi$ there exists $b_E\in B$ with $\|b_E\|_B=1$ such that

$$\mu(E)^{-1/p'}\cdot\|G(E)\|\leqq\langle\mu(E)^{-1/p'}\cdot G(E),b_E\rangle+\varepsilon/n^{1/p}. \tag{10}$$

From (10) the triangle inequality in ℓ^p implies

$$\left(\sum_{E\in\pi}(\mu(E)^{-1/p'}\cdot\|G(E)\|)^p\right)^{1/p}=\left(\sum_{E\in\pi}|\langle\mu(E)^{-1/p'}\cdot G(E),b_E\rangle|^p\right)^{1/p}+\varepsilon.$$

Now by using (9) we can write

$$\left(\sum_{E\in\pi}\frac{\|G(E)\|^p}{\mu(E)^{p-1}}\right)^{1/p}\leqq\left(\sum_{E\in\pi}\left(\mu(E)^{-1/p'}\cdot\left|\int_E f_{b_E}(t)\,d\mu(t)\right|\right)^p\right)^{1/p}+\varepsilon$$

$$=\sup_{\Sigma\alpha_E^{p'}=1}\left\{\sum_{E\in\pi}\int_E |f_{b_E}(t)|\cdot\alpha_E\cdot\mu(E)^{-1/p'}\cdot d\mu(t)\right\}+\varepsilon$$

$$\leqq\sup_{\Sigma\alpha_E^{p'}=1}\left\{\int_\Omega\left(\sup_{E\in\pi}|f_{b_E}(t)|\right)\left(\sum_{E\in\pi}\alpha_E\cdot\mu(E)^{-1/p'}\cdot\chi_E(t)\right)d\mu(t)\right\}+\varepsilon$$

$$\leqq\left\|\sup_{E\in\pi}|T(b_E)|\right\|_{L^p}\cdot\sup_{\Sigma\alpha_E^{p'}=1}\left\{\left\|\sum_{E\in\pi}\alpha_E\cdot\mu(E)^{-1/p'}\cdot\chi_E\right\|_{L^{p'}}\right\}+\varepsilon$$

$$\leqq|T|_m+\varepsilon.$$

Taking ε arbitrarily small and the "sup" over the partitions we obtain $|G|_p\leqq|T|_m$, completing the proof. □

This theorem allows us to prove the following result of [**5**].

Corollary. *$B\check{\otimes}_m L^p(\mu)=L^p(\mu,B)$ for each $1<p<\infty$.*

Proof. Given a simple function $s=\sum_{i=1}^n x_i\cdot\chi_{E_i}$ where x_i belongs to B, we notice that s clearly belongs to $L^p(\mu,B^{**})$ and therefore the measure $G_s(E)=\int_E s(t)\,d\mu(t)$ belongs to $V^p_{B^{**}}=M(B^*,L^p(\mu))$. So, denoting by T_s the operator associated with s we have $\|s\|_p=|G_s|_p=|T_s|_m$. Finally the density of simple functions in the space $L^p(\mu,B)$ gives us the corollary. □

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