Note on a special determinant

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Suppose a polynomial or convergent power series

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots . \tag{1}$$

is raised to powers $j = 0, 1, 2, 3, \dots$. The coefficients of x^k in $[f(x)]^j$, $k = 0, 1, 2, \dots$, may be entered as elements in positions (j, k) in an array or matrix F, thus:

$$F = \begin{bmatrix} 1 & . & . & . & \dots \\ a_0 & a_1 & a_2 & a_3 & \dots \\ a_0^2 & 2a_0 a_1 & a_1^2 + 2a_0 a_2 & 2(a_0 a_3 + a_1 a_2) & \dots \\ a_0^3 & 3a_0^2 a_1 & 3(a_0^2 a_2 + a_0 a_1^2) & a_1^3 + 3a_0^2 a_3 + 6a_0 a_1 a_2 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix}. \tag{2}$$

By construction all elements in column (k) have weight (sum of suffixes) equal to k.

The array F has interesting properties, which have been considered in detail by H. W. Turnbull, *Proc. London Math. Soc.* **37** (1934), 106-146. The reciprocal array F^{-1} corresponds to the reversion of the series (1). One of the theorems proved (p. 121) is that the determinant $|F|_n$ obtained by taking the first n rows and columns of F has the value $a_1^{\frac{1}{2}n(n-1)}$.

The following simple proof of this may be put on record. Consider $[f(x) - a_0]^j$; it does not contain a_0. Translating this operation on $f(x)$, for $j = 0, 1, 2, \dots$, into an operation on F, we have at once

$$\begin{bmatrix} 1 & . & . & . & \dots \\ -a_0 & 1 & . & . & \dots \\ a_0^2 & -2a_0 & 1 & . & \dots \\ -a_0^3 & 3a_0^2 & -3a_0 & 1 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix} F = \begin{bmatrix} 1 & . & . & . & \dots \\ . & a_1 & a_2 & a_3 & \dots \\ . & . & a_1^2 & 2a_1 a_2 & \dots \\ . & . & . & a_1^3 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix}, \tag{3}$$

the right-hand array being F with a_0 obliterated. Taking now determinants of both sides, we have

$$|F|_n = a_1^{0+1+2+\dots+(n-1)} = a_1^{\frac{1}{2}n(n-1)}.$$

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We may prove in the same way that if F_m is the array formed from the rows of F beginning at $j = m$ instead of $j = 0$, then the determinant formed from the first n rows and columns of F_m has the value

$$|F_m|_n = a_0^{mn}\, a_1^{\frac{1}{2}n(n-1)}.$$

To prove this we observe that $[f(x)]^m\,[f(x) - a_0]^j$, with m fixed, $j = 0, 1, 2, \ldots.$, can possess no power of a_0 higher than a_0^m. Obliterating from F_m such higher powers, we have

$$\begin{bmatrix} 1 & . & . & . & \ldots \\ -a_0 & 1 & . & . & \ldots \\ a_0^2 & -2a_0 & 1 & . & \ldots \\ -a_0^3 & 3a_0^2 & -3a_0 & 1 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{bmatrix} F_m = \begin{bmatrix} a_0^m & ma_0^{m-1} & & & \ldots \\ . & a_0^m a_1 & & & \ldots \\ . & . & a_0^m a_1^2 & & \ldots \\ . & . & . & a_0^m a_1^3 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{bmatrix}.$$

It follows as before that

$$|F_m|_n = a_0^{mn}\, a_1^{\frac{1}{2}n(n-1)}.$$

The substitution of various special functions such as e^{ax}, $(1+ax)^p$, and so on for $f(x)$ gives nothing very new, mostly variations on the old theme, that the difference-product of the numbers $0, 1, 2, \ldots., n$ is $n!\,(n-1)!\,\ldots.\,3!\,2!\,1!$ or $1^n\,2^{n-1}\,3^{n-2}\,\ldots.\,n^1$, or the equally old theme, that the difference-product of $0,\ 1,\ \frac{1}{2},\ \frac{1}{3},\ \ldots.,\ \frac{1}{n}$ is $(1^1\,2^2\,3^3\,\ldots.\,n^n)^{-1}$.