THE HAUSDORFF MEANS FOR DOUBLE SEQUENCES

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The basic theory of the Hausdorff means for double sequences was developed some thirty-three years ago by C.R. Adams [1], and independently by F. Hallenbach [3]. Yet today, many of the properties of these means remain largely uninvestigated. The calculations here, although clearly more complex, for the most part break down into obvious modifications of the calculations in the one dimensional case.

To bring out this very close analogy between the one dimensional case and the two dimensional case, we give here in summary form an elementary development of the theory of the Hausdorff means for double sequences. References to the proofs of the main results (Theorem 1 and 4) are given. The proof of Theorems 2, 3 and 5 may be found in the author's dissertation [7]. These proofs involve only the obvious generalizations of the proofs given by G.H. Hardy [5] or D.V. Widder [8] for the corresponding theorems in the one dimensional case. The interested reader may also refer to J. Copping [2] and H.J.Hamilton [4] for the development of a general theory of multiple sequence transformations.

DEFINITION 1. Let $A = (a_{mnk\ell})$ be a four dimensional matrix, and let $S = (s_{mn})$ be a two dimensional matrix whose elements are the elements of the double sequences $\{s_{mn}\}$. The two dimensional matrix

$$T = AS, \tag{1}$$

whose elements are the elements of the double sequence $\{t_{mn}\}$, where

$$(2) \qquad t_{mn} = \sum_{k,\ell=0}^{\infty} a_{mnk\ell} s_{k\ell}, \quad m, n = 0, 1, 2, \ldots,$$

is meaningful for every m, n, is a transformation of the double sequence $\{s_{mn}\}$. The matrix A is said to transform the double sequence $\{s_{mn}\}$ into the double sequence $\{t_{mn}\}$.

DEFINITION 2. The sequence $\{s_{mn}\}$ is said to be summable by the matrix A to the sum s if (2) is meaningful for every m and n, and if

$$\lim_{m,n \to \infty} t_{mn} = s < \infty,$$

where convergence is in the sense of Pringsheim.

DEFINITION 3. The transformation (1) is said to be regular if every convergent sequence $\{s_{mn}\}$ is transformed into a convergent sequence $\{t_{mn}\}$, if t_{mn} is meaningful for every m, n, and if

$$(3) \qquad \lim_{m,n \to \infty} t_{mn} = \lim_{k,\ell \to \infty} s_{k\ell}.$$

The transformation is said to be totally regular if, in addition, (3) holds whenever $\{s_{mn}\}$ diverges to positive or negative infinity.

THEOREM 1. In order that the summability of bounded sequences $\{s_{mn}\}$ by the matrix $A = (a_{mnk\ell})$ be regular, it is necessary and sufficient that

1. $\lim_{m,n \to \infty} a_{mnk\ell} = 0 \qquad k, \ell = 0, 1, 2, \ldots;$

2. $\lim_{m,n \to \infty} \sum_{k,\ell=0}^{\infty} a_{mnk\ell} = 1;$

3. $\lim_{m,n \to \infty} \sum_{k=0}^{\infty} |a_{mnk\ell}| = 0, \qquad \ell = 0, 1, 2, \ldots;$

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4. $\lim_{m,n\to\infty} \sum_{\ell=0}^{\infty} |a_{mnk\ell}| = 0$, $\quad k = 0, 1, 2, \ldots$;

5. $\sum_{k,\ell=0}^{\infty} |a_{mnk\ell}| < M < \infty$, $\quad m, n = 0, 1, 2, \ldots$.

Theorem 1 was first proved by Robison [6].

DEFINITION 4. The matrix $\rho = (\rho_{mnk\ell})$, whose elements are defined by

$$\rho_{mnk\ell} = (-1)^{k+\ell} \binom{m}{k}\binom{n}{\ell}, \qquad k \leq m, \ \ell \leq n,$$

$$= 0 \quad \text{otherwise},$$

is called a difference matrix.

THEOREM 2. (cf. Hardy [5], Theorem 196.) The difference matrix $\rho = (\rho_{mnk\ell})$ is its own inverse: $\rho = \rho^{-1}$.

DEFINITION 5. Let $\{\mu_{mn}\}$ be a given sequence and $\mu = (\mu_{mnk\ell})$ be a diagonal matrix whose only non-zero entries are $\mu_{mn} = \mu_{mnmn}$. The transformation matrix $H = \rho\mu\rho^{-1}$ is called a Hausdorff matrix corresponding to the sequence $\{\mu_{mn}\}$. The sequence $\{s_{mn}\}$ is said to be summable to s in the Hausdorff sense, corresponding to the sequence $\{\mu_{mn}\}$, if the sequence $\{t_{mn}\}$, where

$$T = HS,$$

approaches s as m, n tend to infinity, and t_{mn} is meaningful for all m, n.

REMARK. It is easy to show that Hausdorff matrices are commutative.

<u>Example.</u> (cf. Hardy [5], § 11.2.) Let $A = (a_{mnk\ell})$, where

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$$a_{mnk\ell} = \frac{1}{m+1} \cdot \frac{1}{n+1}, \qquad k \leq m, \ \ell \leq n,$$

$$= 0 \quad \text{otherwise.}$$

To show that this is a Hausdorff matrix, let

$$\mu_{mn} = \frac{1}{m+1} \cdot \frac{1}{n+1} .$$

Then if $H = (h_{mnk\ell}) = \rho\mu\rho^{-1}$, we have

$$\sum_{k,\ell=0}^{\infty} h_{mnk\ell} x^k y^\ell = \sum_{k,\ell=0}^{\infty} \sum_{r,n=0}^{\infty} \rho_{mnrs} \mu_{rs} \rho_{rsk\ell} x^k y^\ell$$

$$= \sum_{r,s=0}^{\infty} \rho_{mnrs} \mu_{rs} \sum_{k,\ell=0}^{\infty} \rho_{rsk\ell} x^k y^\ell$$

$$= \sum_{r,s=0}^{\infty} \rho_{mnrs} \mu_{rs} (1-x)^r (1-y)^s$$

$$= \sum_{r,s=0}^{\infty} \rho_{mnrs} \int_{0,0}^{1,1} u^r (1-x)^r v^s (1-y)^s \, dudv$$

$$= \int_{0,0}^{1,1} (1-u+ux)^m (1-v+vy)^n \, dudv$$

$$= \sum_{k,\ell=0}^{m,n} \binom{m}{k}\binom{n}{\ell} x^k y^\ell \int_{0,0}^{1,1} u^k (1-u)^{m-k} v^\ell (1-v)^{n-\ell} \, dudv.$$

Hence

$$h_{mnk\ell} = \binom{m}{k}\binom{n}{\ell} \int_{0,0}^{1,1} u^k (1-u)^{m-k} v^\ell (1-v)^{n-\ell} \, dudv$$

$$= \begin{cases} \dfrac{1}{m+1} \cdot \dfrac{1}{n+1}, & k \leq m, \ \ell \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

Hence $A = (a_{mnk\ell})$ is a Hausdorff matrix.

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THEOREM 3. (cf. Hardy [5], Theorem 199.) A matrix $A = (a_{mnk\ell})$ is a Hausdorff matrix corresponding to the sequence $\{\mu_{mn}\}$ if and only if its elements have the form

$$a_{mnk\ell} = \binom{m}{k}\binom{n}{\ell} \sum_{r,s=0}^{m-k,\, n-\ell} (-1)^{r+s} \binom{m-k}{r}\binom{n-\ell}{s} \mu_{k+r,\ell+s} \quad .$$

THEOREM 4. (cf. Hardy [5], Theorem 208 (i).) The Hausdorff method of summability corresponding to the sequence $\{\mu_{mn}\}$ is regular if and only if

$$(3) \qquad \mu_{mn} = \int_{0,0}^{1,1} u^m v^n d^2 g(u,v) \, , \qquad m, n = 0, 1, 2, \ldots \, ,$$

where $g(u, v)$ is of bounded variation in the sense of Hardy-Krause in the unit square, and

$$(4) \qquad g(u,0) = g(u,0^+) = g(0^+,v) = g(0,v) = 0 \, , \qquad 0 \le u, v \le 1 \, ,$$

$$(5) \qquad g(1,1) - g(1,0) - g(0,1) + g(0,0) = 1 \, .$$

For a proof of this important result, the reader is referred to Hallenbach [3] and Adams [1].

REMARKS. Relative to the sequence $\{\mu_{mn}\}$, where the elements μ_{mn} are defined by (3), the Hausdorff method transforms bounded sequences into bounded sequences whenever $g(u, v)$ is of bounded variation. If it is also true that

$$g(u,0) = g(u,0^+) = g(0^+,v) = g(0,v) = k \, , \qquad 0 \le u, v \le 1 \, ,$$

then the method is regular for null sequences. If, in addition, (5) is satisfied, the method is regular. See Hallenbach [3].

THEOREM 5. Given a function $g(u, v)$, of bounded variation in the unit square, the corresponding Hausdorff transform $\{t_{mn}\}$, of a sequence $\{s_{mn}\}$, may be defined by

$$t_{mn} = \sum_{k,\ell=0}^{m,n} \binom{m}{k}\binom{n}{\ell} s_{k\ell} \int_{0,0}^{1,1} u^k (1-u)^{m-k} v^\ell (1-v)^{n-\ell} d^2 g(u,v) ,$$

and this transformation is convergence preserving.

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