9

The properties of the Lund model fragmentation formulas; the external-part formulas

9.1 Introduction

In the previous chapter we derived a stochastical process for string fragmentation. The result is a unique process which is at the basis of the Lund model for the fragmentation of quark and gluon jets. We used only some general properties of a kinematical nature together with the necessary requirements of causality and relativistic covariance. The whole discussion is based on (semi-)classical arguments (quantum mechanics does of course enter into our assumptions on $q\bar{q}$-pair production).

In particular the process led to precise formulas for the production properties (we called these the external-part formulas) and the decay properties (the corresponding internal-part formulas, see chapter 10) of a finite-energy cluster of rank-connected hadrons.

The term 'external-part' is used to imply that the cluster is in general part of a larger-energy (possibly infinitely-large-energy) cluster. Two independent Lorentz invariants are necessary to specify the external properties of the cluster; these may be taken as the squared mass s and the lightcone fraction z used up by the cluster. They describe how the cluster starts and ends on some (space-time or energy-momentum-space) points that are inside (or on the border of) the larger external cluster.

In this chapter we will consider the external-part formulas in detail and in particular show the following.

E1 In the Lund model the cluster will be produced in accordance with the same formula as for a single particle (but with the squared hadronic mass $m^2 \to s$).

E2 The finite-energy version, H_s, of the space-time distribution of vertices, H in Eq. (8.14), approaches H very fast when s is larger than a few squared hadronic masses m^2.

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We next consider the two functions H and f in the Lund fragmentation model in detail in order to understand some of their properties. After that we will exhibit some general properties of all iterative cascade fragmentation models of the Feynman-Field kind, [13]. We end the chapter with a discussion of an interesting analogy (first pointed out by Artru, [25]) between the proper time of a vertex in space-time and the momentum transfer between the group of particles produced to the left and those produced to the right of that vertex in energy-momentum space.

9.2 The production properties of a cluster

We start with the results in Eqs. (8.34), (8.40) and (8.42):

$$\Gamma = s\frac{1-z}{z}, \; dP_{ext} = ds\frac{dz}{z}z^{a_0}\left(\frac{1-z}{z}\right)^{a_n}\exp(-b\Gamma) \tag{9.1}$$

Here Γ corresponds to the squared proper time of the last vertex, which has parameter a_n, and s is the squared mass of the particle cluster stretching between the vertex with parameter a_0 to the vertex with a_n. (Note that the expressions do not contain any relation to the decay of the cluster; in particular, the index n in this case does not indicate the multiplicity!)

These formulas can be rewritten in several different ways, each of which exhibits some particular feature of the Lund model fragmentation process.

1 The vertex distribution in proper time for a finite energy

If we use the first equation in (9.1) to solve for z in terms of Γ and then change the second equation into a distribution in s and Γ we obtain

$$dP_{ext} = dsd\Gamma\frac{\Gamma^{a_n}s^{a_0-a_n}}{(s+\Gamma)^{a_0+1}}\exp(-b\Gamma) \tag{9.2}$$

For a fixed and finite value of s we can read off an expression for the correspondence to the distribution $H(\Gamma)$ in Eq. (8.14):

$$H_{sn}(\Gamma) \sim \frac{s^{a_0}\Gamma^{a_n}}{(s+\Gamma)^{a_0+1}}\exp(-b\Gamma) = \frac{s^{a_0}H_{s\to\infty,n}}{C(s+\Gamma)^{a_0+1}} \tag{9.3}$$

In this way we have obtained the result we expected but multiplied by a factor in s and $\Gamma + s$, the power depending upon the starting vertex. (The indices on H in the final expression are meant to show that it is s-independent and has the correct power a_n.)

For any fixed value of s the function H_{sn} in Eq. (9.3) approaches 0 fast for large values of Γ owing to the exponential decrease. This feature is independent of s. A simple estimate implies for $\Gamma > \Gamma_0 \simeq (a_n + 1)/b$ that the exponential falloff dominates the distribution H_{sn}. Therefore for

$s \gg \Gamma_0$ (from phenomenological investigations Γ_0 correponds to a few GeV^2) a proper normalisation of H_{sn} will lead to an s-independent result. Then it is a good approximation, when $s \gg \Gamma_0$, that

$$dP_{ext} \simeq ds s^{-(a_n+1)} d\Gamma \frac{H_n(\Gamma)}{C_n} \tag{9.4}$$

The constant $C \equiv C_n$ in Eq. (9.3) is, of course, the normalisation constant for H_n. In this way dP_{ext} depends only upon the flavor n of the final vertex. Actually this is just what we started with when we derived the distributions H_α and $f_{\alpha\beta}$: after many steps along a lightcone there is a certain probability of finding a vertex of a particular kind independently of where we started. We will come back to this *saturation property* later when we consider the internal-part formulas for the decay of a cluster.

This serves as a confirmation for the consistency of the assumption J, at the beginning of the last chapter, that there is, even in the limit $s \to \infty$, a finite number of vertices at the centre of phase space.

2 *The energy-momentum distribution of a finite-mass cluster*

Another obvious way to use the formula (9.1) is to exhibit the probability of obtaining a cluster with a given mass $\sqrt{s}$, thereby taking a fraction z of the positive lightcone component of the original system:

$$dP_{ext} = ds \exp(bs) \frac{dz}{z} z^{a_0} \left(\frac{1-z}{z}\right)^{a_n} \exp\left(\frac{-bs}{z}\right) \tag{9.5}$$

We have then divided the expression for Γ from Eq. (9.1) into one z-dependent and one z-independent part in the exponential.

The remarkable feature of this result is that (besides the purely s-dependent parts and the normalisation) we evidently recover the 'old' formula, which was derived for a single particle, with the mass m exchanged for the mass of the cluster $\sqrt{s}$. Consequently, whether a single particle or a large-mass cluster arises in going between two vertices with a-parameters a_0 and a_n the (mass-dependent) probability distribution for picking a particular fraction of the energy-momentum is the same.

9.3 The properties of the distributions H and f

At this point it is worthwhile to consider the shape and the properties of the unique Lund model distributions in more detail.

1 *The properties of the proper time distribution H*

The distribution in proper time H is the mathematically well-known Γ-distribution (this is not a misguided pun!) which occurs e.g. in connection

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with radiative processes. Depending upon the values of the parameters it has a maximum at $\Gamma = a/b$, a mean value $\langle\Gamma\rangle = (a+1)/b$ and a variation around the mean $\left\langle(\Gamma - \langle\Gamma\rangle)^2\right\rangle = (a+1)/b^2$.

Typical phenomenological parametrisations for longitudinal jets (note the dependence of a and b upon the gluon radiation to be discussed in Chapter 17) would be $a \sim 0.5, b \sim 0.75$ GeV^{-2}. We conclude that for these values the typical proper time 'before' the string will break is somewhat more than 1 fm/c but that the fluctuations around this value is of the same order of magnitude.

2 *The properties of the fragmentation distribution f*

The distribution f is a more complex kind of function. We note that it vanishes exponentially fast close to the origin (it has an essential singularity there, considered as an analytical function) and that it vanishes according to a power law for $z \to 1$. In between there is evidently a maximum.

In order to investigate this maximum in more detail we rewrite the distribution f as an exponential (considering only the case when all the a-parameters are equal):

$$f \sim \exp\Phi \quad \text{with} \quad \Phi = -\frac{bm^2}{z} - \ln z + a\ln(1-z) \tag{9.6}$$

It is easy to prove that for $a = 0.5$ the quantity Φ has a maximum for

$$z = 1 + bm^2 - \sqrt{1 + (bm^2)^2} \simeq bm^2 - (bm^2)^2/2 \tag{9.7}$$

We conclude that the typical z-values will increase with bm^2 and that the maximum of f will occur for a z-value around 0.3 using the value of b quoted above and a mass-value close to the centre of the mesonic mass spectrum, the ρ-mass $m \simeq 0.77$ GeV/c^2.

3 *The typical hyperbola breakup*

A useful exercise is to consider the relationship between the Γ-parameters of two adjacent vertices in the case where a hadron of mass m, taking a fraction z of the remaining lightcone energy-momentum, is produced in going from vertex 1 to vertex 2. It is left to the reader to prove that

$$\Gamma_2 = (1-z)\left(\Gamma_1 + \frac{m^2}{z}\right) \tag{9.8}$$

From Eq. (9.8) we deduce that if there is a fluctuation in the value of z taken by the hadron the result will be a value of Γ_2 that is much larger (for $z \ll z_t$, where z_t is a typical value of z) or much smaller (for

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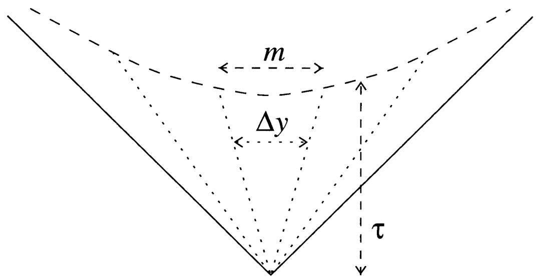

Fig. 9.1. The typical breakup hyperbola divided into particle-mass pieces.

$z \gg z_t$) than Γ_1. The first possibility is suppressed due to the area law (the exponential area suppression) while the second one is power suppressed.

The final result is that the Lund model fragmentation functions tend to produce vertices around a hyperbola (i.e. the locus of the points with a fixed value of $\Gamma \equiv \tau_t^2$), albeit with some fluctuations. The distance from the origin to the hyperbola, τ_t, is related to the typical mass of the hadrons in the cascade decay.

If we place all the vertices along this hyperbola the energy-momentum fractions taken by the hadrons form a geometrical series:

$$z_t, z_t(1-z_t), \ldots, z_t(1-z_t)^n, \ldots \tag{9.9}$$

(note that the remainder fraction is given by $(1-z_t)^n$ after n steps).

When we move along the hyperbola the remainder fraction cannot be too small. It must necessarily be larger than s_0/s with s_0 of order $\langle\Gamma\rangle$. Therefore we obtain a formula for the typical multiplicity, n_t, in a Lund model fragmentation event:

$$(1-z_t)^{n_t} \sim s_0/s \Rightarrow n_t \sim \frac{\log(s/s_0)}{\log[1/(1-z_t)]} \tag{9.10}$$

This result can be interpreted geometrically; see Fig. 9.1. The length of the hyperbola is $\sim \tau_t \log(s/s_0)$ with τ_t the hyperbola parameter defined above. (Note that the notion of length, of course, corresponds to the invariant length in Minkovski space.)

If the hyperbola is cut up into pieces corresponding to particle masses then each piece will cover a typical rapidity gap Δy. In Fig. 9.1 one hadron at rest is shown. According to our findings in Chapter 7 the space size of such a yoyo-state is given by its mass.

We conclude that with a hadron density $1/\Delta y$ along the hyperbola we will obtain the same multiplicity formula as in Eq. (9.10) if we put

$$\Delta y = \log[1/(1-z_t)] \tag{9.11}$$

9.4 The particle density in a general iterative cascade model

In order to understand the significance of the result in Eq. (9.11) we will consider some properties of the iterative cascade fragmentation models, which were mentioned in Chapter 7. For simplicity we consider the situation when there is only a single flavor and a single kind of meson. The probability of obtaining the first-rank particle with a given energy-momentum fraction z is $f(z)dz$. We note that f must be normalised to unity:

$$\int_0^1 dz f(z) = 1 \tag{9.12}$$

We now define the *totally inclusive single-particle distribution* $F(z)dz$ as the number of hadrons (irrespective of rank) with fractional energy-momentum z. This function is not normalised to unity as is f in Eq. (9.12) but, instead, to the total number of hadrons produced.

Inside the scaling cascade scheme this number is in general divergent. In this subsection we will derive the behaviour of a general iterative cascade model and in the next we will specialise to one particular shape of f and perform some of the calculations in detail.

To investigate the properties of F we note that there is an integral equation which relates F and f:

$$F(z) = f(z) + \int_0^{1-z} \frac{d\zeta}{(1-\zeta)} f(\zeta) F\left(\frac{z}{(1-\zeta)}\right) \tag{9.13}$$

The interpretation of the equation is that a hadron with z may be the first-rank hadron in the jet (this is the first contribution $f(z)dz$ on the right-hand side of Eq. (9.13)). After the first-rank particle has left a fraction $1-\zeta$ (with probability $f(\zeta)d\zeta$) the number of hadrons with z that occur further down in the jet is $F(z/(1-\zeta))dz/(1-\zeta)$. This gives the integrand in the second term of Eq. (9.13) (after division dz). We must sum over all values of ζ compatible with the requirement that the argument of F is between 0 and 1.

The equation can be solved by means of the moments method. We obtain from Eq. (9.13)

$$\begin{gathered} M(r) = \int_0^1 z^r F(z)dz, \quad C(r) = \int_0^1 (1-z)^r f(z)dz, \\ m(r) = \int_0^1 z^r f(z)dz \Rightarrow M(r) = \frac{m(r)}{1-C(r)} \end{gathered} \tag{9.14}$$

which we leave to the reader to prove.

The normalisation condition in Eq. (9.12) implies that $C(0) = m(0) = 1$. This evidently means that $M(r)$ diverges when $r \to 0$, which corresponds to the normalisation equation for F. The reason for this divergence is that in Eq. (9.13) no provision is made for ending the cascade: there is no smallest value of z, or in other words the process is totally scaling. Instead one obtains a *rapidity plateau* (note that $y \simeq \log z$ implies $dz/z = dy$). After a rapidity region in the forward direction, the *fragmentation region* of the original quark, there will be a uniform distribution of hadrons in rapidity space. In practice this fragmentation region is about 1.5–2 rapidity units. This means that a large part of the energy is inside the fragmentation region. It is populated by the first few particles in rank but the density of particles is strongly fluctuating and dependent upon the flavor quantum number carried by the original color charge.

In the Lund model it is not sufficient to consider only the fractional energy-momentum along the jet, i.e. in one of the lightcone directions, as in iterative cascade models. There is also the energy-momentum along the opposite lightcone direction, for which we must account. This is the reason why in the last subsection we had to bring the plateau to an end by the request that we can use up the fractional energy-momentum only to the level s_0/s. In the integral equation in Eq. (9.13) the plateau will, however, go on forever.

The height of the plateau, i.e. the density of hadrons in the centre, can be calculated by simple means from Eq. (9.14). We may conclude, by expanding for small values of the moment parameter r, that $M(r) \to R/r$ where

$$\frac{1}{R} = -\left(\frac{dC(r)}{dr}\right)_{r=0} = \int_0^1 \log\left(\frac{1}{1-z}\right) f(z)dz \tag{9.15}$$

which we again leave for the reader to prove.

We conclude that $F(z)dz$ behaves as $Rdz/z \equiv Rdy$ for small values of z, i.e. for rapidities far from the 'tip' of the jet. Thus the result in Eq. (9.11) is very general with Δy identified with $1/R$, i.e. the mean value of $\log[1/(1-z)]$ as calculated from the fragmentation function f.

We may in an intuitive sense identify Δy with the *mean loss of rapidity per produced hadron*. It is interesting to note that we again find a similar scaling energy-momentum distribution as for the virtual quanta in the MVQ and the partons in the PM in Chapters 2 and 5. In particular the result obtained in the Schwinger model for excitations by means of an external charged pair $\pm g$ leads to the result $R = 1$; cf. Chapter 6.

A detailed calculation of the inclusive distribution using a simple model

The method of moments is a very powerful mathematical technique but it may be difficult to understand the results on an intuitive level. We will therefore show by explicit calculation how the central plateau is built up by the contributions from the hadrons of different rank. The results of the calculation will also be useful further on, in Chapter 13.

We consider a very simple iterative cascade model with a constant fragmentation function, f, which then in order to be normalised as in Eq. (9.12) must equal unity. Although it is simple, it was used rather successfully at the beginning of the Lund model, [13], assuming then that all vector and pseudoscalar mesons were produced in accordance with their statistical weights. We now know that this is not the case. Further a constant distribution does not fulfil the requirements for a consistent fragmentation process listed at the beginning of Chapter 8.

A detailed study of the model is, however, instructive because it is straightforward to provide explicit results for the inclusive distributions of the nth-rank particles for all values of n. The first-rank particle is evidently distributed according to f. After it has taken z_1 (with the same probability for all z_1) the second-rank particle will take $z = z_2(1 - z_1)$, with a flat distribution for z_2 also.

This means that the inclusive distribution of the second-rank hadron is

$$\begin{aligned} D^{(2)}(z) &= \int dz_1 dz_2 \delta(z_2(1 - z_1) - z) \\ &= \int_0^{(1-z)} \frac{dz_1}{1 - z_1} = \int_z^1 \frac{dx_1}{x_1} = \log\left(\frac{1}{z}\right) \end{aligned} \tag{9.16}$$

Using the same method we obtain for the nth rank hadron

$$\begin{aligned} D^{(n)}(z) &= \int \left(\prod_{j=1}^{n} dz_j\right) \delta\left(z_n \prod_{j=1}^{n-1}(1 - z_j) - z\right) \\ &= \int \left(\prod_{j=1}^{n-1} \frac{dx_j}{x_j}\right) \Theta\left(\prod_{j=1}^{n-1} x_j - z\right) \end{aligned} \tag{9.17}$$

where Θ is the Heaviside function, equal to unity for a positive argument and vanishing elsewhere. We have also defined the obvious new variables $x_j = 1 - z_j$. In order to perform the integral we introduce $y_j = \log(1/x_j)$ and write, exchanging the product of the x_j for a sum of the rapidities y_j (the sum being introduced by means of a δ-distribution, $dy\delta(\sum y_j - y)$)

$$D^{(n)}(z) = \int dy\Theta[\exp(-y) - z] \int \left(\prod_{j=1}^{n-1} dy_j\right) \delta\left(\sum_{j=1}^{n-1} y_j - y\right) \tag{9.18}$$

We obtain a symmetrical integral (for $N = n - 1$), which is most easily solved by iteration:

$$I_N = \int \left(\prod_{j=1}^{N} dy_j\right) \delta\left(\sum_{j=1}^{N} y_j - y\right) = \frac{y^{N-1}}{(N-1)!} \tag{9.19}$$

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We finally obtain by direct integration over y

$$D^{(n)}(z) = \frac{[\log(1/z)]^{n-1}}{(n-1)!} \tag{9.20}$$

which is a nice and very satisfying result to derive! The following comments may be made.

- All but the first-rank hadron have a distribution in z which vanishes when $z \to 1$. Since $\log(1/z) = \log[1+(1-z)/z)] \simeq 1-z$ when $z \to 1$ we find that the nth rank distribution will vanish like

$$(1-z)^{n-1}. \tag{9.21}$$

The reason is evidently that there have already been $n-1$ earlier energy 'handouts'. The above result is a very general property of all physical systems, usually referred to as the *spectator relation*: if there are n basic constituents sharing a common energy and you require the inclusive distribution in energy for one of them its fraction usually behaves as in Eq. (9.21).

- The result (9.20) can be described as a distribution in $y = \log(1/z)$:

$$D^{(n)}(z)dz = dy\frac{y^{n-1}}{(n-1)!}\exp(-y) \tag{9.22}$$

  i.e. a Poisson distribution in rapidity. The distributions are evidently all normalised. This is exactly what was obtained in the external excitation model, derived from the Schwinger model in [39]; cf. Chapter 6.

- From the sum over all ranks we obtain the totally inclusive distribution, which, according to the predictions from the integral equation (9.13), corresponds to the particle density $R = 1$:

$$D(z)dz = \sum_{n=1}^{\infty} D^{(n)}(z)dz = dy = \frac{dz}{z} \tag{9.23}$$

A useful exercise is to carry through the calculations above also for the case when f is exchanged for $f_a = (a+1)(1-z)^a$. Then one obtains $D_a(z) = (a+1)(1-z)^a/z$, i.e. a rapidity density equal to $R_a = a+1$.

In this way we can see in detail how the rapidity plateau occurring in the iterative cascade models is built up. From the properties of $D^{(n)}$ we conclude that the maximum of the distributions moves towards larger values in rapidity with increasing n; this is a useful exercise for the interested reader. Note, however, that an nth-rank hadron may very well have a larger z-value than the first-rank hadron (although with a small probability).

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- For the model with a constant f the rapidity plateau evidently goes all the way from the tip and there is no evidence of a fragmentation region (but for the model with f_a there is such a region).

The reason why the model with a constant f works rather well is that if three times more vector mesons than pseudoscalars are produced as direct particles then the decay products from the vector mesons will move towards smaller z-values. This effect means e.g. that while the prediction for inclusive π^+ mesons for $z \to 1$ is $zD^{\pi^+} \simeq 0.1$ it becomes around 0.4 for $z \simeq 0.4$ because of the many π^+'s from the decays.

It is possible to derive many other analytic expressions, e.g. for the two-particle correlation functions, by the same means as for Eq. (9.13) and we refer to the original literature, [13]. We will not do it here because there are many kinematical complications. If this kinematics is included in the analytical equations the results become so complicated that it is in general much easier to take recourse to computer simulations.

It is very satisfying and highly recommended at this point for the reader to obtain a set of simple but useful and understandable distributions by the use of a Monte Carlo simulation program such as JETSET or HERWIG, just in order to appreciate the effects of really introducing kinematics!

9.5 The relationship between the vertex proper time and the momentum transfer across the vertex

We will now use the external-part formulas in a way proposed by Artru, [25]. We will start by showing that the quantity Γ, which in space-time has been related to the proper time of a vertex, in energy-momentum space can be interpreted as the invariant squared momentum transfer between the two jets produced by the appearance of the vertex.

Thus a vertex appearing at the space-time point $V = (x_+, x_-)$ (with $\Gamma = \kappa^2 x_+ x_-$) will divide the total system (see Fig. 9.2) into a right-moving jet with energy-momentum $P_r = (p_{+0} - \kappa x_+,\ \kappa x_-)$ and a left-moving jet with $P_l = (\kappa x_+,\ p_{-0} - \kappa x_-)$.

The situation depicted in Fig. 9.2 can be interpreted as if the original q-particle with $P_+ = (p_{+0}, 0)$ is transformed into the right-movers and the original $\overline{q}$-particle with $P_- = (0, p_{-0})$ is transformed into the left-movers. There is then evidently a momentum transfer in this process equal to $q \equiv P_+ - P_r = -(P_- - P_l) = \kappa(x_+, -x_-)$.

We note that in order to obtain positive masses for the two systems it is necessary that this momentum transfer is a spacelike vector, i.e. that the (Lorentz-)square of the vector is negative. The fact that it is the negative-lightcone component which becomes negative in our formula is due to

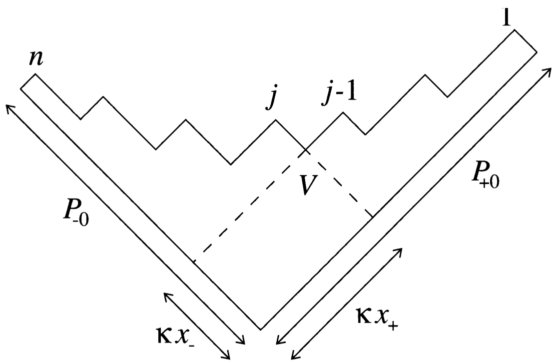

Fig. 9.2. The vertex $V = (x_+, x_-)$ divides the system into right-movers (hadrons $1, \ldots, j-1$) and left-movers (hadrons $j, \ldots, n$).

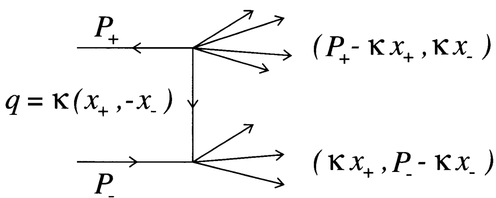

Fig. 9.3. The energy-momentum transfer between the right-movers produced by the original q and the left-movers produced by the original $\bar{q}$.

our choice to define the momentum transfer in the direction from right to left.

Pictorially we may describe the situation as in Fig. 9.3 in which there are two 'transformation points' where q becomes the rightmovers and $\bar{q}$ becomes the leftmovers and the momentum transfer q connects the two points. The result is evidently very similar to a simple Feynman diagram.

The system can be further partitioned. Each of the two new jets is naturally subdivided into smaller systems by means of internal vertices. This process can be continued until we reach the level of individual particles.

We may in this way relate the production process in the model, as shown in Fig. 9.4, to a particular *multiperipheral diagram* shown in Fig. 9.5 with the final-state hadrons coming out along a 'chain'.

There is a dual relationship between the picture we have had of producing the particles in space-time (cf. Fig. 9.4) and this kind of diagram.

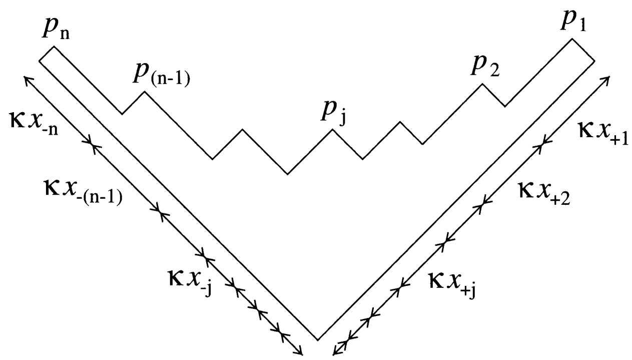

Fig. 9.4. The production of a multiparticle state in the Lund model. Note that the difference vectors between the space-time vertices correspond to the energy-momenta of the produced hadrons, while the positions of the vertices correspond to the energy-momentum transfers.

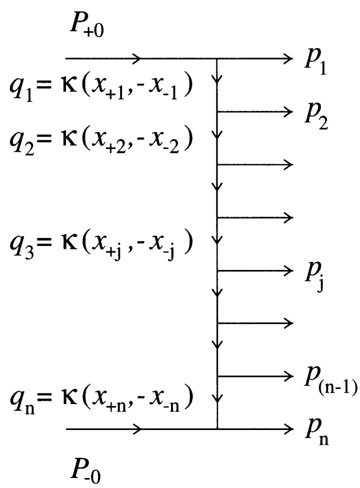

Fig. 9.5. The production process described in terms of momentum transfers in a chain.

The string production vertices (the space-time production points) are exchanged for a set of momentum transfers (the connecting *links* along the chain in the multiperipheral diagram). The invariant size of these momentum transfers (apart from the factor κ) corresponds in Figs. 9.2 and 9.5 to the distance from the origin to the space-time vertex points.

Models of a multiperipheral type have been under intense investigation

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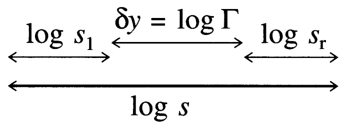

Fig. 9.6. The total range of rapidity $\sim \log s$ subdivided into the rapidity ranges of the right- and left-movers and the rapidity difference $\delta y \sim -\log \Gamma$.

since the 1950s and one notes the close resemblance to the Feynman diagrams for a two-body to many-body interaction. We will come back to such models in the next chapter after we have further developed our understanding of the Lund model fragmentation formulas.

We will meet the same kind of diagrams when we relate the lightcone singularities in deep inelastic scattering to the leading-log approximation formulas derived by the theorists working with Gribov, cf. Chapter 19.

We note that the momentum transfer divides the state so that the right-movers are, intuitively speaking, dragged apart from the left-movers. The proper measure for this effect is the rapidity difference between them.

We note that Γ can be expressed in terms of the total squared mass s of the original system and the squared masses s_l and s_r of the left- and right-moving systems through the equation (for the reader to prove):

$$s\Gamma = (s_l + \Gamma)(s_r + \Gamma) \approx s_l s_r \tag{9.24}$$

This means that for large-mass systems

$$-\ln \Gamma \simeq \ln s - \ln s_l - \ln s_r \tag{9.25}$$

The right-hand side of this expression is basically the rapidity difference δy between the right-movers and the left-movers. To see this we note that according to our results in the last subsection the available rapidity region for a system of mass $\sqrt{s}$ to deliver its particles is $\approx \log s$ (i.e. the length of the typical hyperbola). Then, as seen in Fig. 9.6, after having taken away the rapidity region inhabited by the left- and right-movers, we are left with a rather approximate measure of the rapidity difference δy. From the distribution in Γ, $H(\Gamma)$, the distribution in δy is

$$\sim d(\delta y) \exp[-(a+1)\delta y] \tag{9.26}$$

(we have neglected the slowly varying exponential). Thus we obtain a prediction for the (approximate) distribution of the rapidity gaps in our breakup process: when evaluated inclusively, it should be an exponentially decaying distribution with $a + 1$ as the exponential rate.

This is reminiscent of the formulas occurring in Regge-Mueller analysis, in which the parameter a plays the role of a Regge parameter. We will

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come back to this interpretation of a again later when we consider the internal decay properties of the clusters.

It is necessary to understand that there are several 'ifs and buts' in connection with the results on δy given above. We note that the rapidity difference between two neighbors in rank actually can be both positive and negative even if we know the rank-ordering. In order to see this consider the joint probability

$$P((\delta y)_n) = \int dz_1 f(z_1) \frac{dz_2}{1-z_1} f\left(\frac{z_2}{1-z_1}\right) \delta\left((\delta y)_n - \log\left(\frac{z_1}{z_2}\right)\right) \tag{9.27}$$

(the index n stands for rank neighbors). It is easy to manipulate the formula into

$$P((\delta y)_n) = N^2 \int_0^1 \frac{dz}{z}(1-z)^a \exp[bm^2 \exp(\delta y)_n] \times \exp\left(\frac{-4bm^2}{z}\cosh\frac{(\delta y)_n}{2}\right) \tag{9.28}$$

This integral must be calculated by numerical methods. One obtains a generally smooth distribution with a maximum around $(\delta y)_n = 0$ but with an appreciable tail on both sides.

Thus *a lower-rank particle may be faster than a higher-rank particle*. It may even happen (with admittedly a small probability) that two particles close in rapidity may be far away in rank-ordering.

In a real experiment it is very difficult to observe rank-ordering because many directly produced particles are resonances which will decay quickly (mostly into pions but also into some kaons etc). Furthermore most of the particles in the final state contain u- and d-flavors and antiflavors. It is nevertheless known in many experimental situations that if one orders the observed particles in rapidity then the distribution in the rapidity gaps will be basically exponential (for large values of δy). There are, however, many different contributions to this distribution and it is not a useful way of determining the parameter a from such measurements.

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