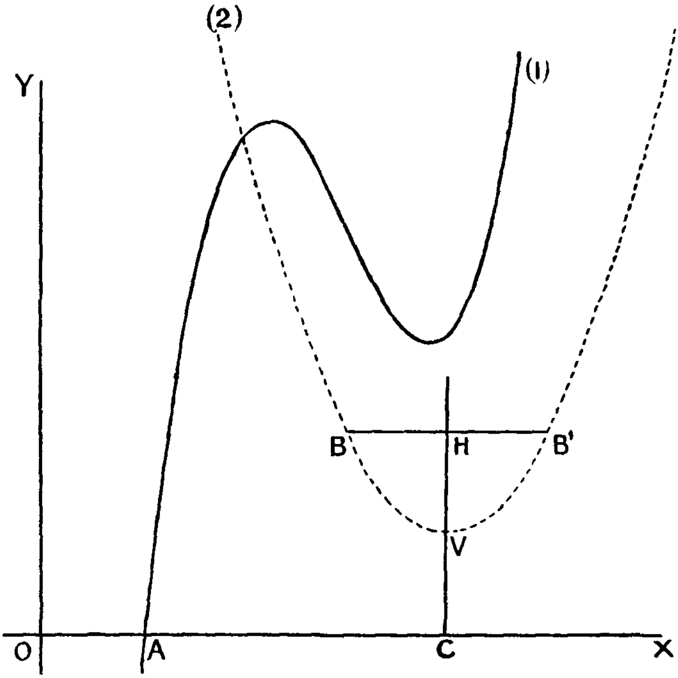

The figure shows the curves

$$y=(x-1)(x^2-8x+17),$$

and $$y= \qquad x^2-8x+17,$$

and the roots are

$$1,\ 4+\sqrt{-1},\ 4-\sqrt{-1}.$$

T. D. Salmon.

The Arithmetic Mean of a number of real positive numbers is not less than their Geometric Mean.— The subjoined proof is not given in the current text books, but is handed on by oral tradition.

The usual proof requires *in general* the assumption of an infinite series of operations (with the consequent limit theorems involved), as all the n given numbers tend to equality. Let us take an arithmetical example, and let us tabulate the sequences involved in the way suggested in my paper on "The Teaching of Limits and Convergence to Scholarship Candidates" in the May (1911) issue of the *Mathematical Gazette.*

Suppose we take the numbers 6, 7, 13, and replace them step by step by the arithmetic means of the greatest and least, and let them at each step be arranged in ascending order.

Chart to illustrate usual proof in actual operation :—

Step.	Least Quantity.	Middle Quantity.	Greatest Quantity.	Gap between Greatest and Least Quantity.
1	6	7	13	7
2	7	9·5	9·5	2·5
3	8·25	8·25	9·5	1·25
4	8·25	8·875	8·875	·625
5	8·5625	8·5625	8·875	·3125
6	8·5625	8·71875	8·71875	·15625
7	8·640625	8·640625	8·71875	·078125
8	8·640625	8·6796875	8·6796875	·0390625
etc.	etc.	etc.	etc.	etc.

We can thus see on the chart the sequences whereby the sum of the three numbers involved remains equal, while the three numbers themselves are replaced by three others in ascending order of magnitude and getting more and more nearly equal. This is a very good method of exhibiting the process to the boy, and it is a very excellent problem for him in the exercise of mathematical induction to show that in the above case the number of steps is infinite.

The subjoined proof requires only a finite number of steps, and needs no theory of limits. I do not know who is the author.

Let the given numbers be $a_1, a_2, a_3, \ldots, a_n$, supposed arranged in ascending order of magnitude.

Let $$a_1 a_2 a_3 \ldots a_n = G^n \tag{1}$$

where G is thus their Geometric Mean.

Let $$a_1 a_n = bG \tag{2}$$

and replace a_1 and a_n by b and G. We thus have the n numbers

$$b,\ a_2,\ a_3,\ \ldots,\ a_{n-1},\ G. \tag{3}$$

Suppose $b, a_2, \ldots, a_{n-1}$ arranged in ascending order of magnitude, and place G last. Let us obtain in this way

$$b_1,\ b_2,\ b_3 \ldots b_{n-1},\ G \tag{4}$$

It will be plain from (2), (3), (4) that

$$b_1 b_2 b_3 \ldots b_{n-1} = G^{n-1} \tag{5}$$

Replace now b_1 and b_{n-1} by c and G where

$$b_1 b_{n-1} = cG \tag{6}$$

and arrange $c, b_2, b_3, \ldots b_{n-2}$ in ascending order of magnitude, getting

$$c_1,\ c_2,\ \ldots c_{n-2},\ G,\ G \tag{7}$$

where, as before, it is plain that

$$c_1 c_2 \ldots c_{n-2} = G^{n-2} \tag{8}$$

We thus have the following n lines where the products of all the numbers in each line $= G^n$

$$a_1,\ a_2,\ a_3 \ldots,\ a_n \tag{9}$$

$$b_1,\ b_2,\ b_3 \ldots b_{n-1},\ G \tag{10}$$

$$c_1,\ c_2,\ c_3 \ldots c_{n-2},\ G,\ G \tag{11}$$

$$d_1,\ d_2,\ d_3,\ \ldots d_{n-3},\ G,\ G,\ G \tag{12}$$

....

G, G, G...G (n G's).

We wish to show that the sum of all the numbers in any of the above lines is greater than the sum of all the numbers in the one that follows it. It will be sufficient to show that

$$a_1 + a_2 + \ldots + a_n > b_1 + b_2 \ldots + b_{n-1} + G \tag{13}$$

as all the others are done in the same way.

Now $\qquad a_1+a_2+\ldots+a_n>b_1+b_2+\ldots+b_{n-1}+G$

if $\qquad a_1+a_2+\ldots+a_n>b+a_2+a_3+\ldots+a_{n-1}+G$ (by (3)).

i.e. if $\qquad a_1+a_n>b+G,$

i.e. if $\qquad G(a_1+a_n)>a_1a_n+G^2$ by (2),

i.e. if $\qquad 0>(G-a_1)(G-a_n). \qquad (14)$

But
$$G\equiv(a_1a_2\ldots a_n)^{\frac{1}{n}}$$
$$<(a_na_n\ldots a_n)^{\frac{1}{n}} \quad (i.e.\ a_n)$$
($\because$ a_n is the greatest),

and
$$>(a_1a_1\ldots a_1)^{\frac{1}{n}} \quad i.e.\ a_1$$
($\because$ a_1 is the least).

$\therefore$ $G-a_1$ is positive, and $G-a_n$ is negative, *i.e.* $(G-a_1)(G-a_n)$ is negative, which proves the theorem.

Thus
$$a_1+a_2+\ldots+a_n$$
$$>b_1+b_2+\ldots+b_{n-1}+G$$
$$>>c_1+c_2+\ldots+c_{n-2}+G+G$$
$$>>>d_1+d_2+\ldots+d_{n-3}+G+G+G$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

i.e. $>nG$.

WILLIAM P. MILNE.

Additional "Note on Right-Angled Triangles."—On pp. 95 and 96 of No. 8 (October 1911) of *Mathematical Notes* is given a numerical method of finding rational right-angled triangles.

It has been known for centuries that
$$p^2-q^2,\ 2pq,\ p^2+q^2 \qquad (A)$$
are the sides of a rational right-angled triangle whatever be the values of p and q; for
$$(p^2-q^2)^2+(2pq)^2=p^4-2p^2q^2+q^4+4p^2q^2=p^4+2p^2q^2+q^4=(p^2+q^2)^2.$$

Take $p=2$, $q=1$, and we have the "hackneyed" triangle whose sides are 3, 4, 5.

Take $p=3$, $q=2$; then the triangle is 5, 12, 13.

Take $p=4$, $q=1$, and we have 8, 15, and 17.

When $p=4$, $q=3$, the sides are 7, 24, 25.

The sides will have no common divisor when p and q are prime to each other, one odd and the other even, in which case the above